Math 160a - Fall 2002
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Homework set 2 - Corrected version
Due: 17th October 2002

1. Show that $(x^2 - 17)(x^2 - 19)(x^2 - 323)$ has solutions in $\mathbf{Q}_p$ for all p, and in $\mathbf{R}$, but not in $\mathbf{Q}$. You may assume without proof that it has solutions in $\mathbf{Q}_2$.

2. Prove that, for $0 \neq x \in \mathbf{Q}$,

$$\prod_p |\, x|_p = 1.$$

   The product here is over all of the finite primes, and ∞. The absolute value $|\cdot|_\infty$ is the normal absolute value on $\mathbf{R}$.

3. Find a solution to $x^3 - 4x + 4$ modulo 13^3, by following the proof of Hensel's lemma or otherwise.

4. Let p be a prime, and for each positive integer n, let $a_n \in \mathbf{Q}_p$. Prove that $\sum_{n=1}^{\infty}$ is convergent in $\mathbf{Q}_p$ if and only if $\lim_{n\to\infty} a_n \to 0$.

5. Let a, b, c be pairwise relatively prime integers and let p be an odd prime. Show that, if $p \nmid abc$, then the equation

$$aX^2 + bY^2 + cZ^2 = 0$$

   has a nontrivial solution in $\mathbf{Q}_p$.

6. Let $n \in \mathbf{Z}$, and let p be a prime. Show that

$$v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor,$$

   where $\lfloor \cdot \rfloor$ is the floor function.