**Math 160a - Fall 2002**
**Lloyd Kilford**

**Final Examination**
**Due: 11th December 2002**

You may use the notes for this course, your solutions to course homeworks, and the course textbook for reference. Please provide references for any quoted theorems. You may not use the Internet or other books for reference. Please do not discuss this exam with other students or faculty without contacting me first.

Do not spend more than one hour on any one question. If you do, write a line across the page, and continue under that line.

Each question is worth an equal number of marks, and partial credit will be given for incomplete answers.

Hand this exam in to my office (Sloan 358) or to my pigeonhole in a sealed envelope before 5pm on Wednesday the 11th of December 2003.

1. Let $p$ and $q$ be primes such that $2 < p < q$ and there exist integers $x$ and $y$ satisfying

$$pq(pq-4)(p+q+1) = x^2 + y^2.$$

By considering $p$ and $q$ modulo 4, or otherwise, determine if there are integers $m$ and $z$ such that $z^2 + p = mq$.

2. What are the class groups of $\mathbf{Q}(\sqrt{23})$, $\mathbf{Q}(\sqrt{-37})$, $\mathbf{Q}(\sqrt{79})$, $\mathbf{Q}(\sqrt{-17})$ and $\mathbf{Q}(\sqrt{-47})$? Factorise the ideals (2), (3) and (5) in the rings of integers of each of these fields.

3. Show that $\mathbf{Q}(\sqrt{-2})$ has class number 1. Using this, find all of the rational integer points of $y^2 = x^3 - 2$.

4. Let $p$ be a prime. Show that $X^4 - 17 = 2Y^2$ has solutions in $\mathbf{Q}_p$ for every prime $p$, and in $\mathbf{R}$. This equation has no solutions in $\mathbf{Q}$ - outline how you might show this.

5. Let $x$ be a positive integer. If

$$\begin{aligned} x &\equiv 1 \mod 2 \\ x &\equiv 2 \mod 3 \\ x &\equiv 4 \mod 5 \\ x &\equiv 6 \mod 7 \\ x &\equiv 10 \mod 11 \\ x &\equiv 12 \mod 13 \end{aligned}$$

and $x$ is a multiple of 17, what is the smallest positive value of $x$ possible, and what is the general form of any $x$ which satisfies these congruences?

6. Show that

$$\left(x^2 + 3y^2 - 17z^2\right) \cdot \left(x^2 + 5y^2 - 7z^2\right) = 0$$

has solutions over every $\mathbf{Q}_p$ and over $\mathbf{R}$, but has no solutions in $\mathbf{Q}$. Construct another counterexample to the Hasse Principle of this form.