

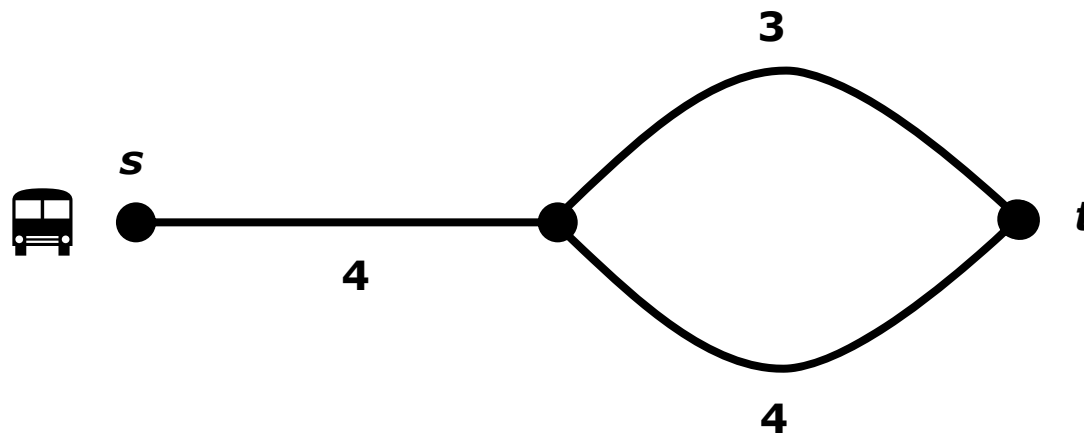
Optimal Paths in Networks with Random Edge Weights

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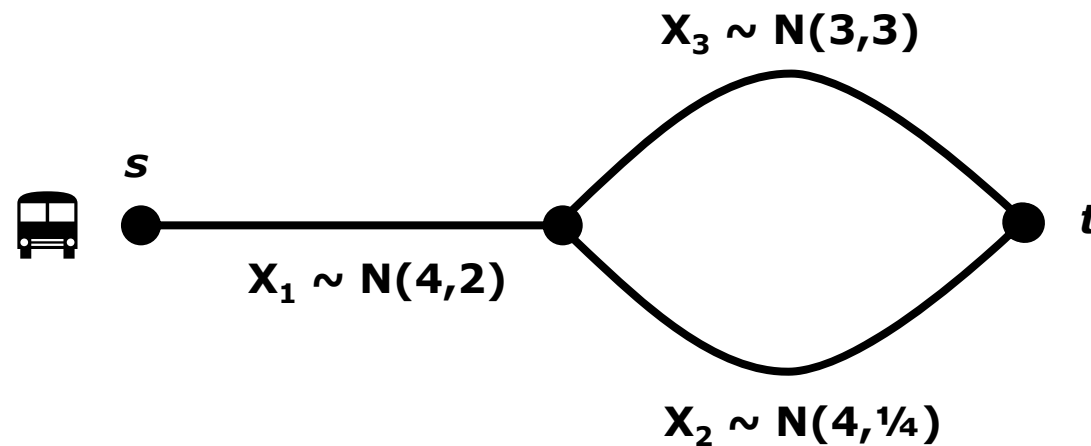
A simple network problem



Find the shortest path from s to t

- Low complexity algorithms are available
- Examples include Dijkstra's and Viterbi algorithms

Our problem



- Edge weights are independent Gaussian random variables X_i with $\mu_i > 0$ and σ_i^2
- A path weight from s to t is a sum S of Gaussian random variables

Find the **Best** path from s to t

What does the *best* path mean ?

Best path optimizes some feature of the path weight

- Tail probability

The path minimizes the value of $\Pr(\mathbf{S} > \boldsymbol{\Sigma})$

The path minimizes $\boldsymbol{\Sigma}$ such that $\Pr(\mathbf{S} > \boldsymbol{\Sigma}) = \epsilon$


- Intersection

The path minimizes $\hat{\mathbf{s}} = \max \{ \mathbf{s} \mid \Pr(\mathbf{S} = \mathbf{s}) \geq \epsilon \}$

In some sense, the best path minimizes the worst (maximum) \mathbf{S}

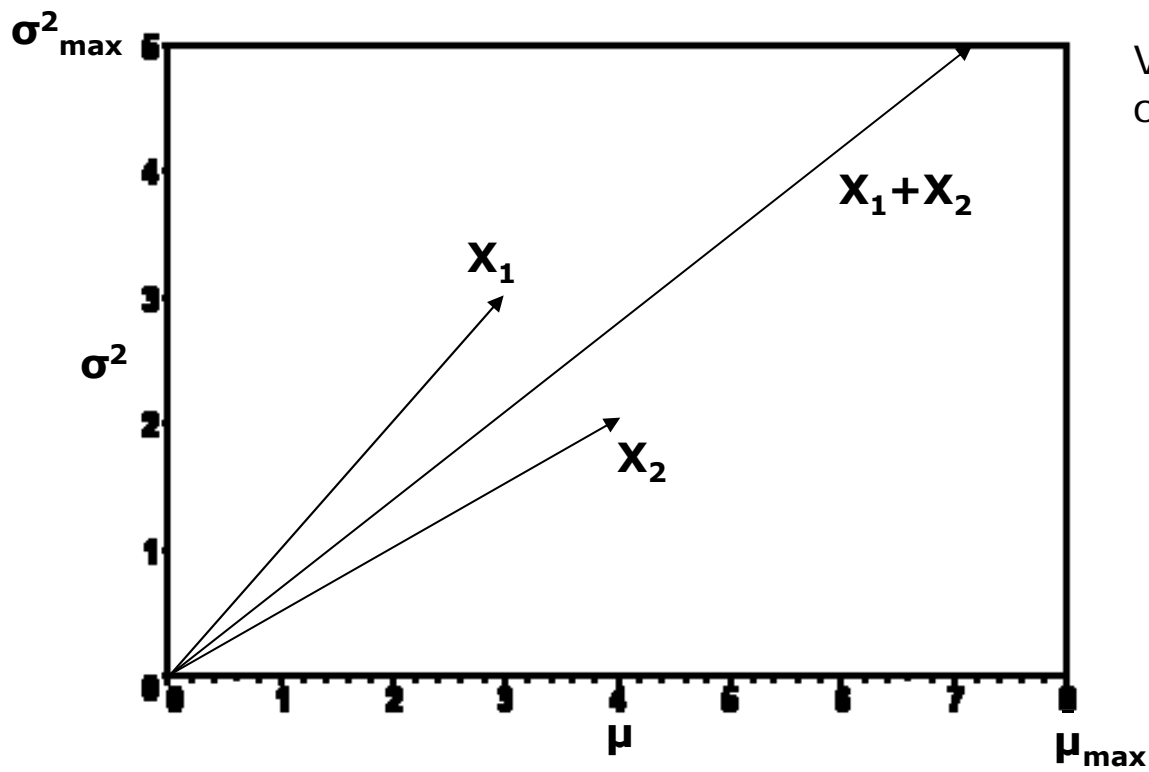
- For simplicity, we use the intersection definition of “best”

Review of Dijkstra's algorithm

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1: procedure DIJKSTRA ( $G, \mathbf{p}, s$ )
2:   for all  $v \in V$  do
3:      $l[v] \leftarrow \infty$ 
4:      $\pi[v] \leftarrow \text{NIL}$ 
5:    $Q \leftarrow V$ 
6:    $l[s] \leftarrow 0$ 
7:   while  $Q \neq \emptyset$  do
8:      $u \leftarrow \text{MIN}(Q)$ 
9:     for all node  $v \in N(u)$  do
10:      if  $l[v] \succ l[u] \oplus \mathbf{p}(u, v)$  then
11:          $l[v] \leftarrow l[u] \oplus \mathbf{p}(u, v)$ 
12:         $\pi[v] \leftarrow u$ 
```

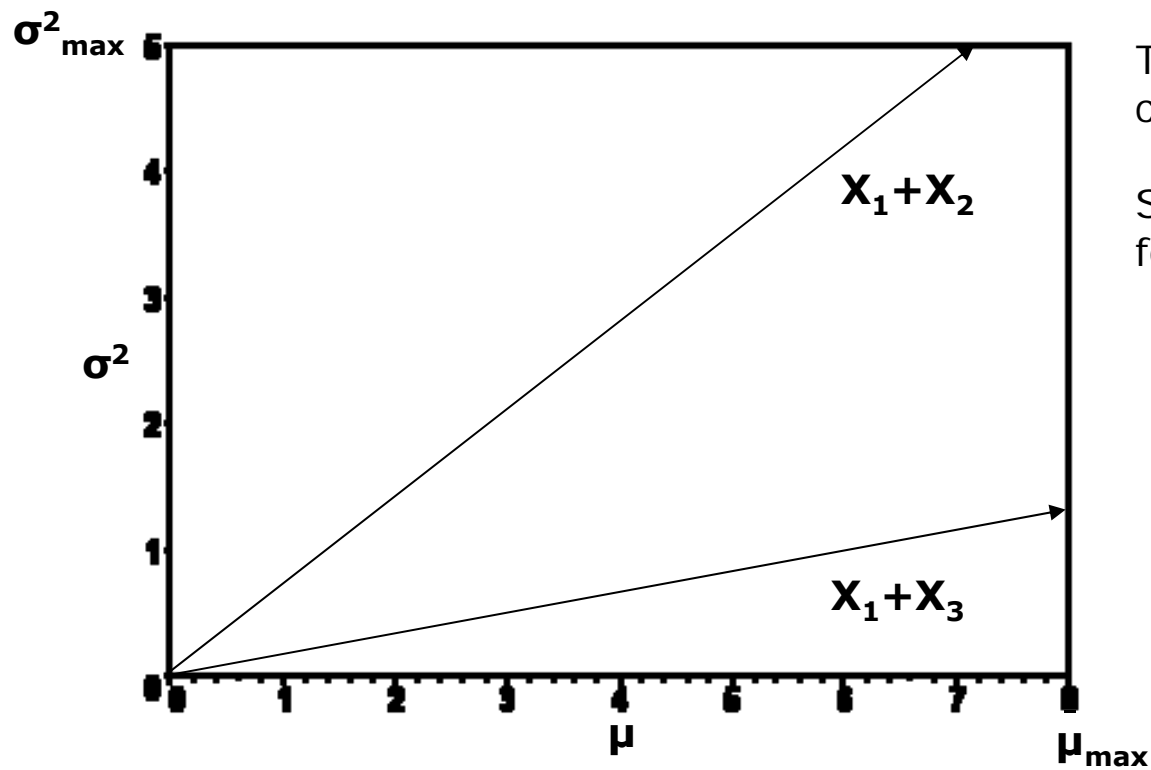
AJI AND McELIECE: THE GENERALIZED DISTRIBUTIVE LAW
IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 46, NO. 2, MARCH 2000

Addition



Vector addition within a box of μ_{\max} and σ^2_{\max}

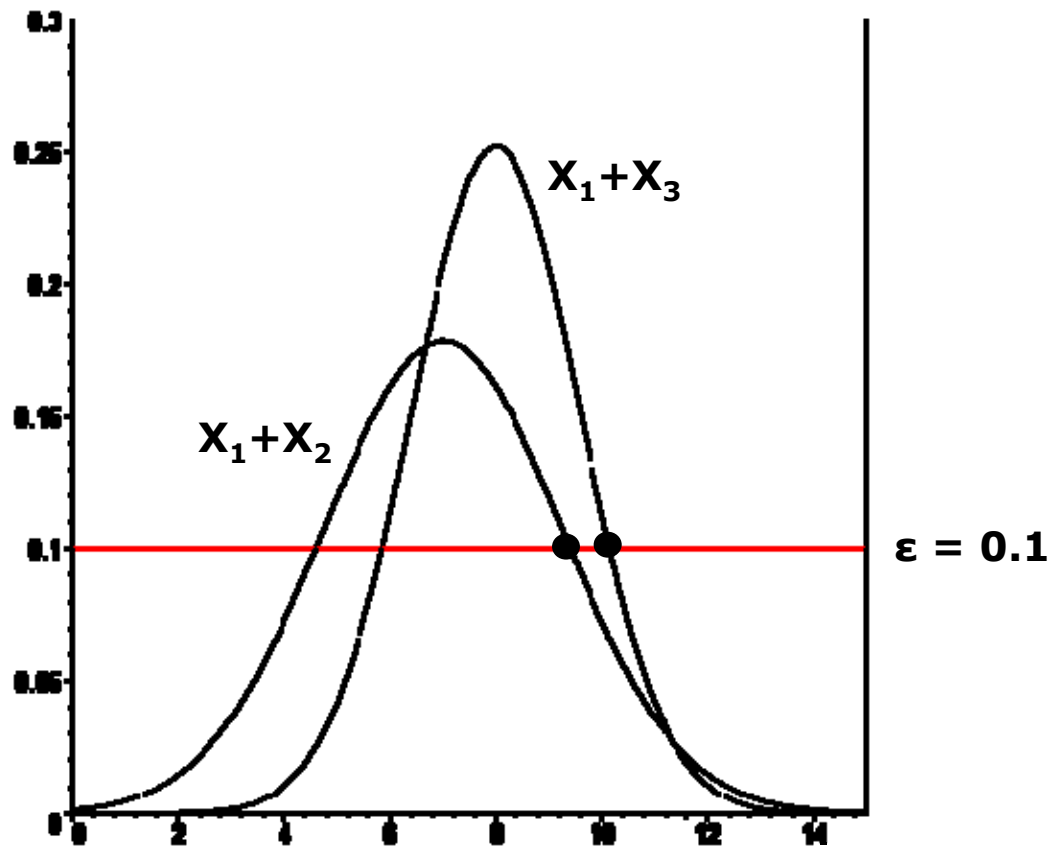
Comparison



The algorithm also needs to compare two paths

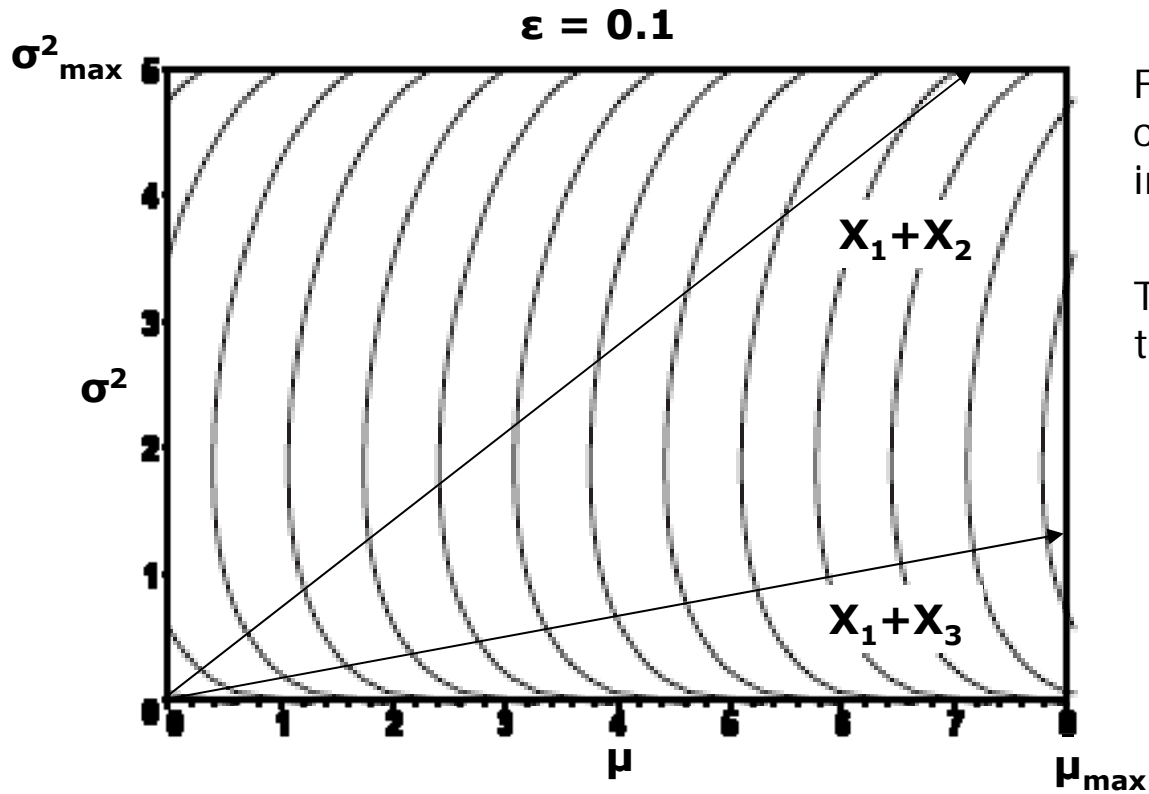
Solution: use the definition for the best path

Intersection



From the picture, at a threshold value of 0.1 the path **X1+X2** is best in the intersection, i.e., minimum worst-case interpretation

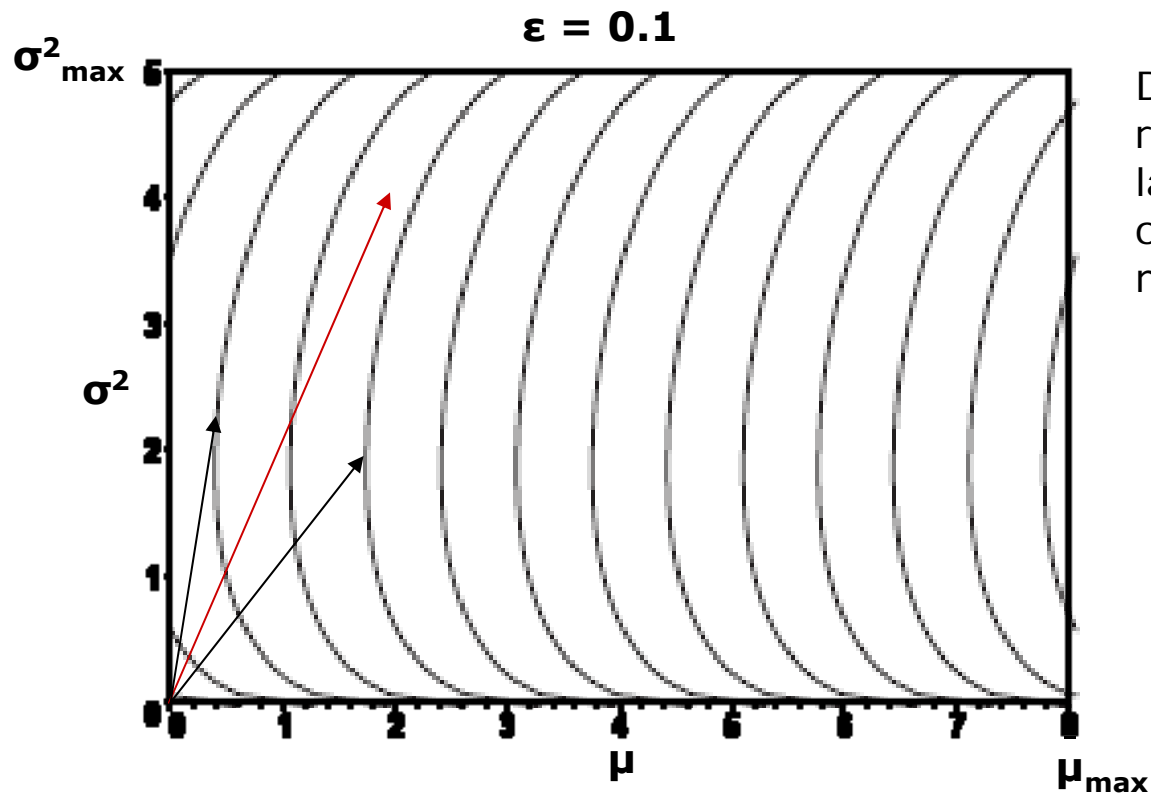
Isocontour



From left to right, we have contour of equal point of intersection $\hat{\mathbf{s}}$

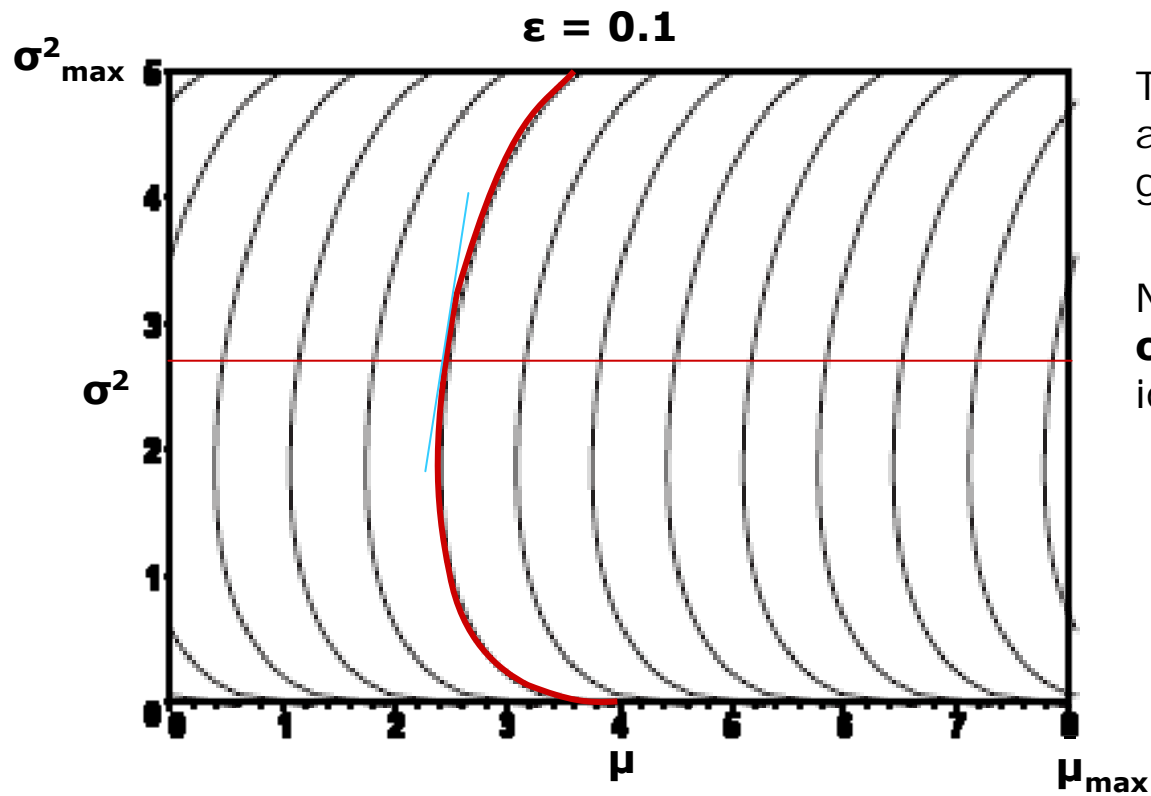
The path with minimum $\hat{\mathbf{s}}$ is the best path

Monotonicity



Dijkstra's Algorithm also requires that the sum be larger than the summands, otherwise the algorithm might produce loops

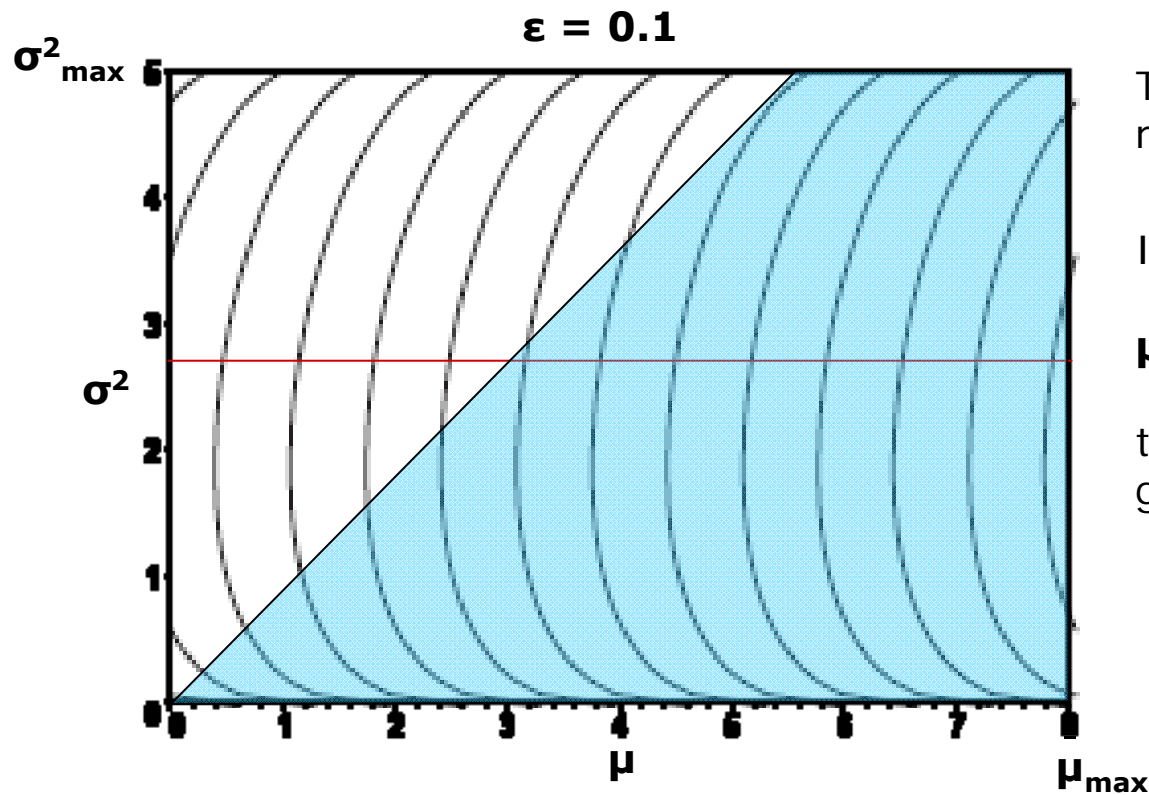
Domain



Think of each contour line as a function $\mu(\sigma^2)$ for a given value of ϵ and \hat{S}

Note that at each value of σ^2 , the slope $\mu'(\sigma^2)$ is identical for all values of \hat{S}

Domain (2)



This slope $\mu'(\sigma^2)$ is maximized at σ^2_{\max}

In the domain where

$$\mu/\sigma^2 > \mu'(\sigma^2_{\max})$$

the sum is guaranteed to be greater than the summands

Conclusion

- We have outlined a framework to apply Dijkstra's algorithm for network optimization problems where edge weights are random variables
- Problem is relevant in many areas of application, including finance, risk and performance analysis, mission critical network design, etc.
- Model is extensible to other types of random variables and definition of the best path (including tail probabilities)
- For rigorous proofs, refer to:

Soedarmadji E, McEliece R.J., "Optimal Worst-Case QoS Routing in Constrained AWGN Channel Network", to be presented at IEEE ICC 2007, Glasgow, Scotland.