Optimal Paths in Networks with Random Edge Weights

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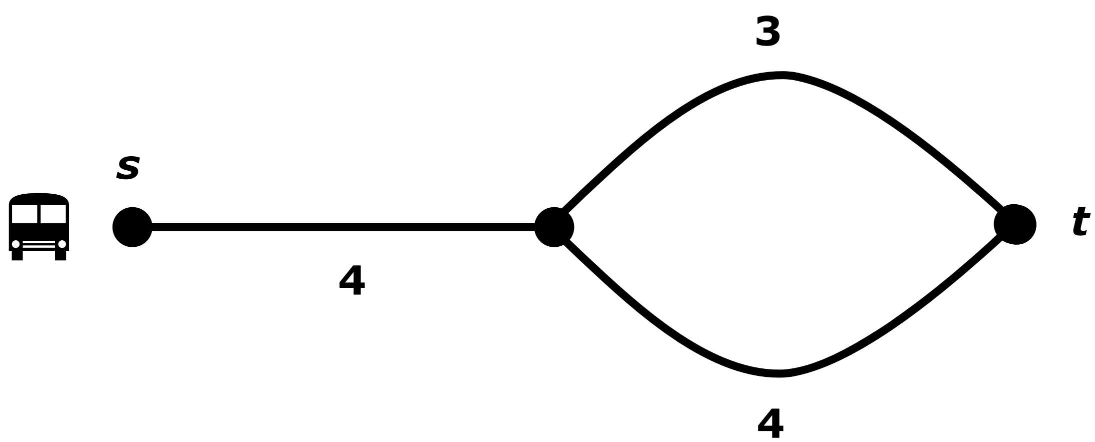
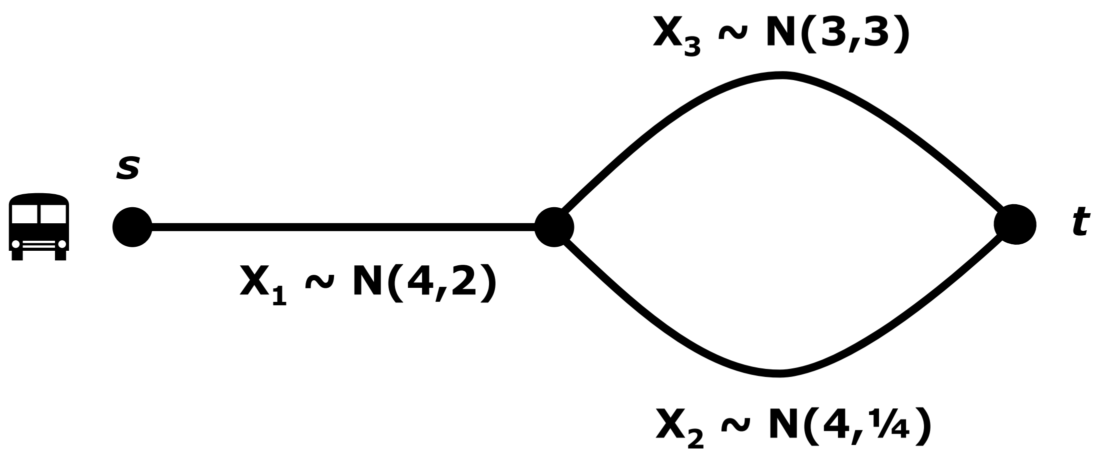
A simple network problem



Find the shortest path from *s* to *t*

- Low complexity algorithms are available
- Examples include Dijkstra's and Viterbi algorithms

Our problem



- Edge weights are independent Gaussian random variables $\mathbf{X}_i$ with $\boldsymbol{\mu}_i > \mathbf{0}$ and $\boldsymbol{\sigma}_i^2$
- A path weight from ***s*** to ***t*** is a sum **S** of Gaussian random variables

Find the *<u>Best</u>* path from *s* to *t*

What does the *best* path mean ?

Best path optimizes some feature of the path weight

- Tail probability

  The path minimizes the value of **Pr(S > *Σ*)**
  The path minimizes ***Σ*** such that **Pr(S > *Σ*) = ε**

- Intersection

  The path minimizes **ŝ = max { s | Pr(S = s) ≥ ε }**
  In some sense, the best path minimizes the worst (maximum) **S**

- For simplicity, we use the intersection definition of “best”

Review of Dijkstra's algorithm

procedure DIJKSTRA $(G, \boldsymbol{p}, s)$

  for all $v \in V$ **do**

    $l[v] \leftarrow \infty$

    $\pi[v] \leftarrow \mathrm{NIL}$

  $Q \leftarrow V$

  $l[s] \leftarrow 0$

  while $Q \neq \varnothing$ **do**

    $u \leftarrow \textsc{Min}(Q)$

    for all node $v \in N(u)$ **do**

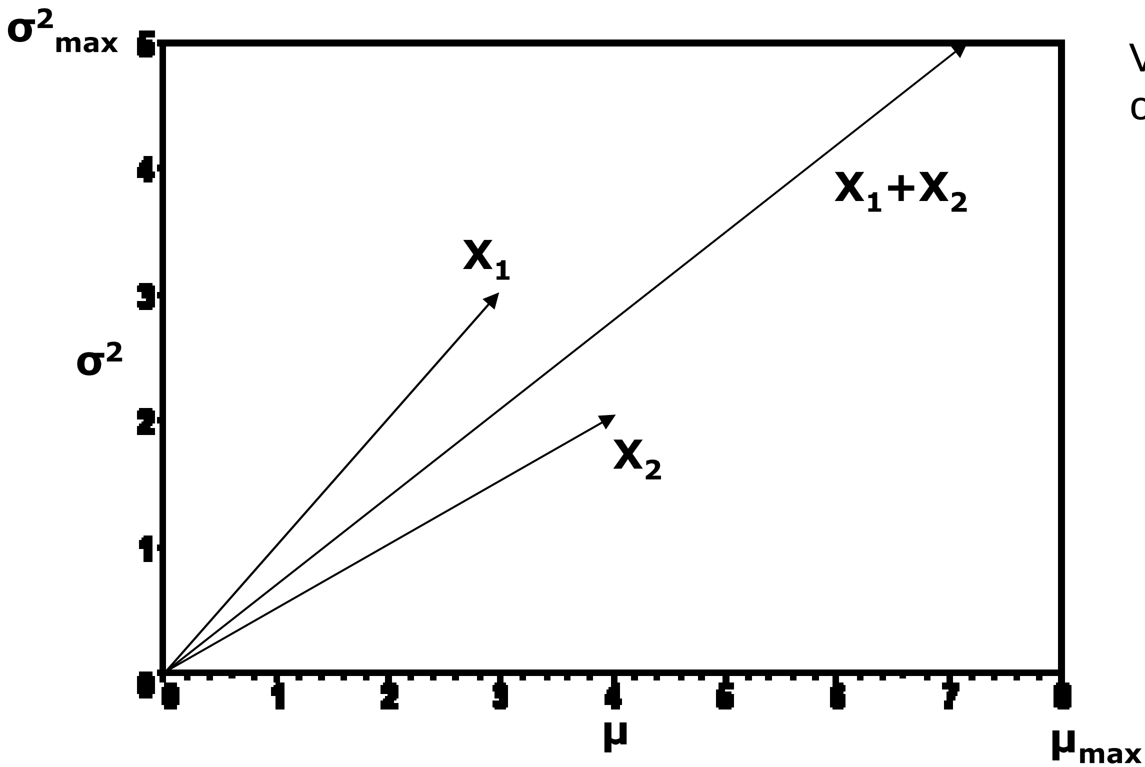
      if $l[v] \succ l[u] \oplus \boldsymbol{p}(u, v)$ **then**

        $l[v] \leftarrow l[u] \oplus \boldsymbol{p}(u, v)$

        $\pi[v] \leftarrow u$

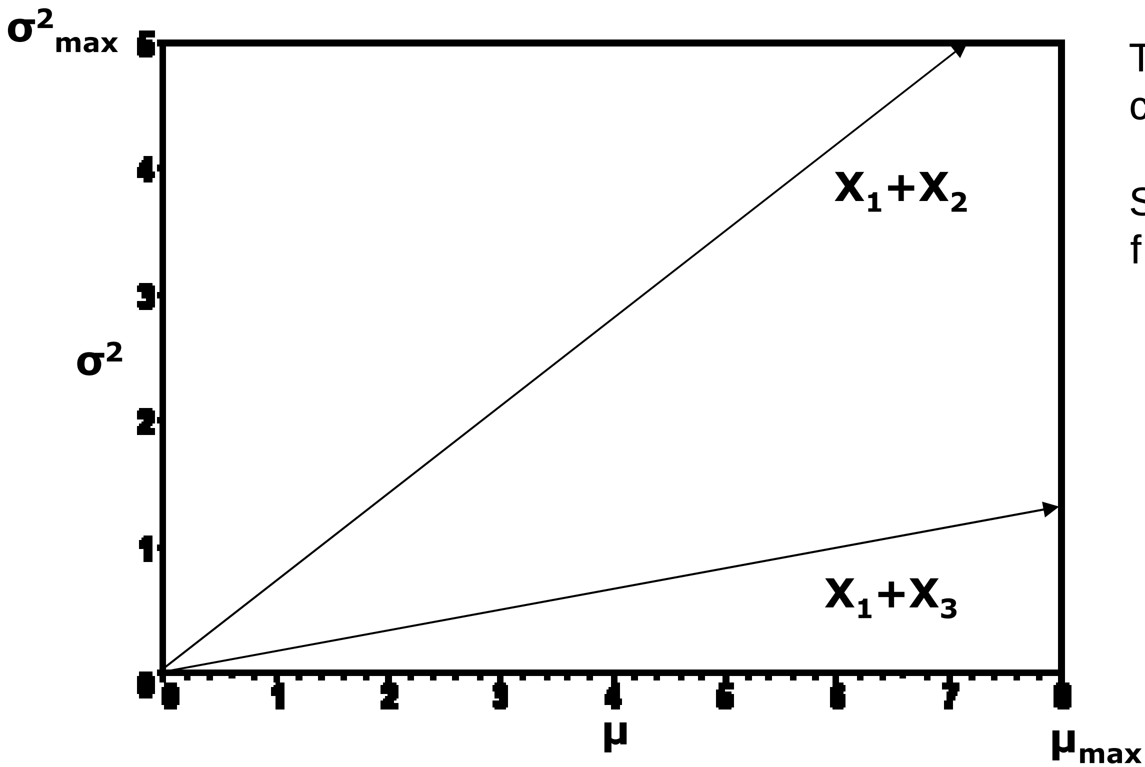
AJI AND McELIECE: THE GENERALIZED DISTRIBUTIVE LAW
IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 46, NO. 2, MARCH 2000

Addition



Vector addition within a box of μ_{max} and σ^2_{max}

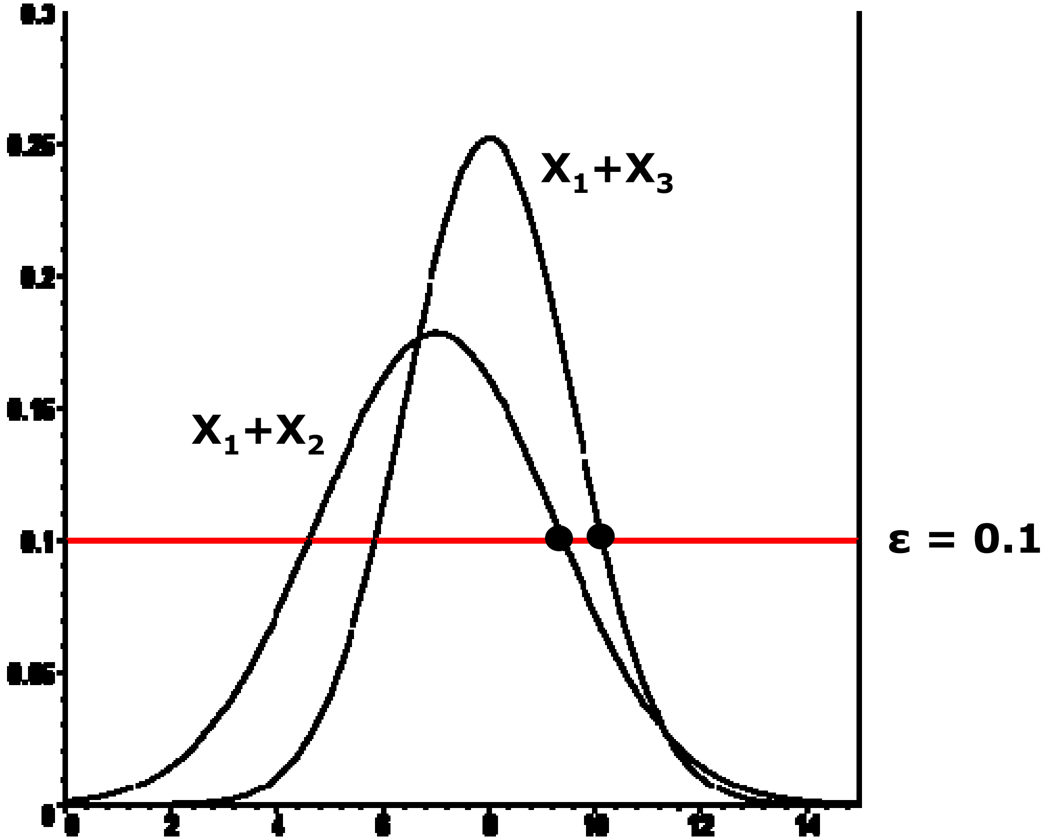
Comparison



The algorithm also needs to compare two paths

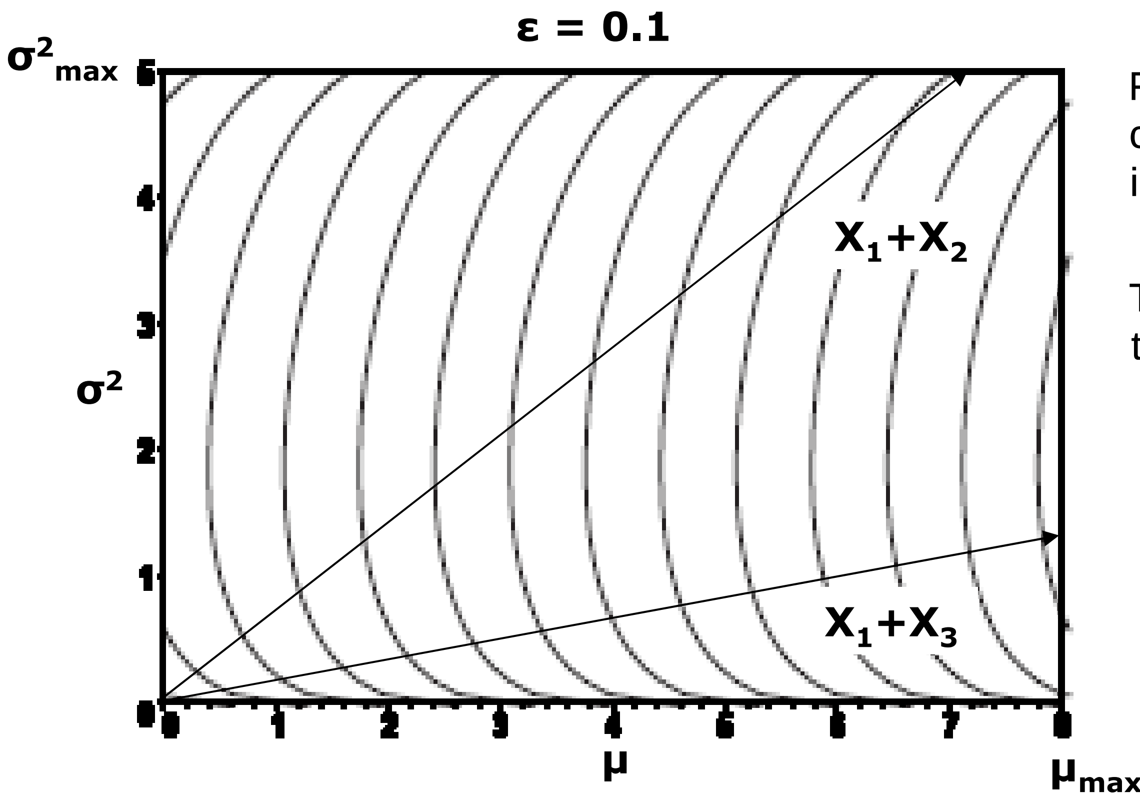
Solution: use the definition for the best path

Intersection



From the picture, at a threshold value of 0.1 the path **X1+X2** is best in the intersection, i.e., minimum worst-case interpretation

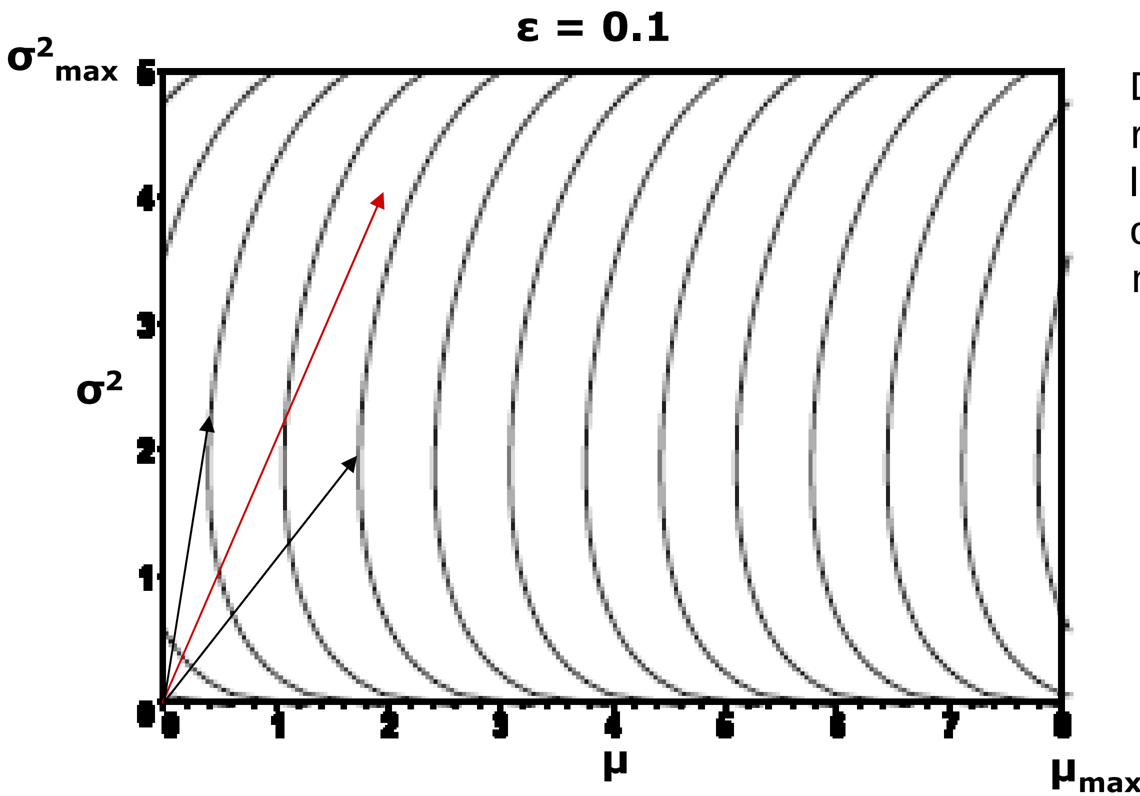
Isocontour



From left to right, we have contour of equal point of intersection **ŝ**

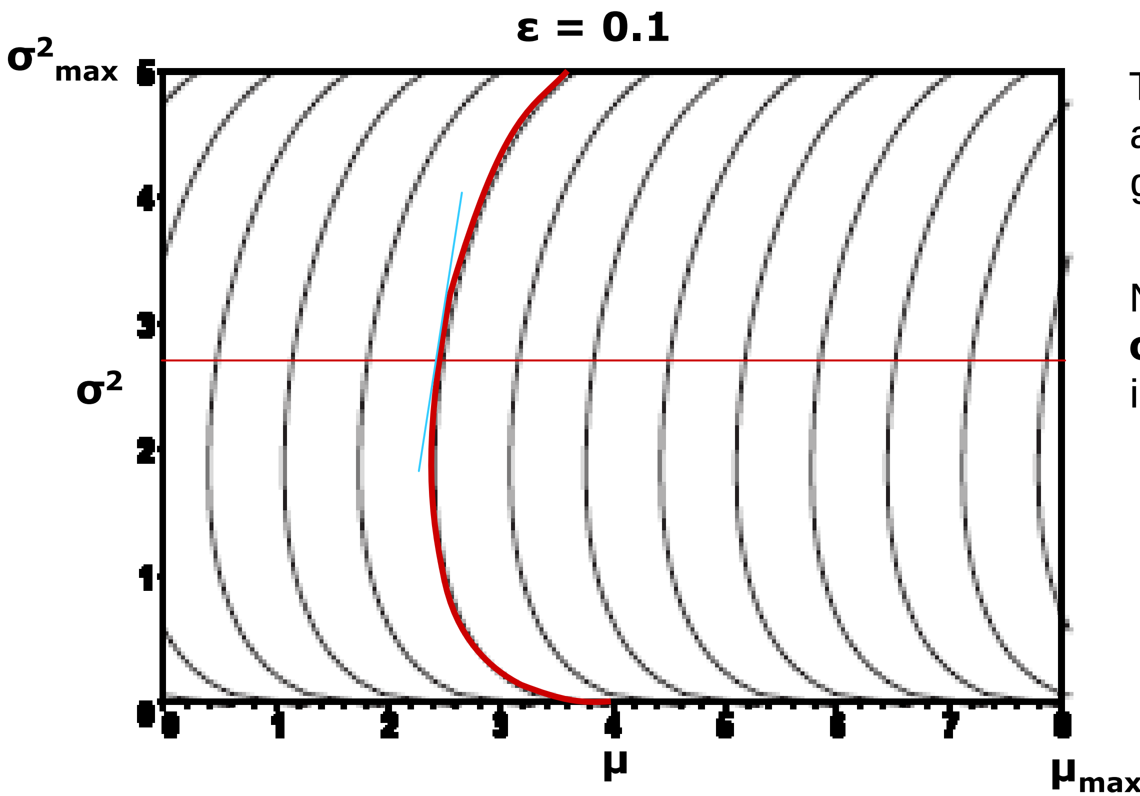
The path with minimum **ŝ** is the best path

Monotonicity



Dijkstra's Algorithm also requires that the sum be larger than the summands, otherwise the algorithm might produce loops

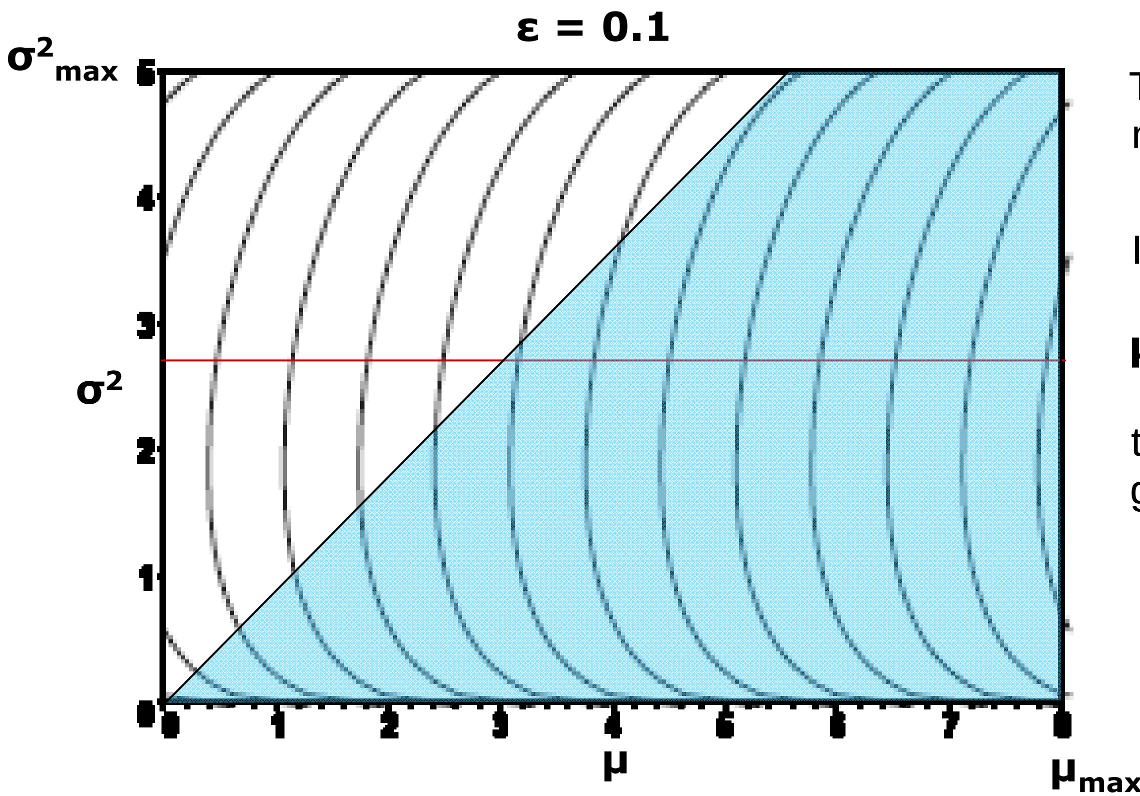
Domain



Think of each contour line as a function **$\mu(\sigma^2)$** for a given value of **ε** and **$\hat{s}$**

Note that at each value of **σ^2** , the slope **$\mu'(\sigma^2)$** is identical for all values of **$\hat{s}$**

Domain (2)



This slope $\mu'(\sigma^2)$ is maximized at σ^2_{max}

In the domain where

$\mu/\sigma^2 > \mu'(\sigma^2_{max})$

the sum is guaranteed to be greater than the summands

Conclusion

- We have outlined a framework to apply Dijkstra's algorithm for network optimization problems where edge weights are random variables
- Problem is relevant in many areas of application, including finance, risk and performance analysis, mission critical network design, etc.
- Model is extensible to other types of random variables and definition of the best path (including tail probabilities)
- For rigorous proofs, refer to:

Soedarmadji E, McEliece R.J., "Optimal Worst-Case QoS Routing in Constrained AWGN Channel Network", to be presented at IEEE ICC 2007, Glasgow, Scotland.