Optimal Routing in the Worst-Case-Error Metric

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Abstract—This paper considers the problem of finding the path with minimum (zero) worst possible number of errors in a network with \( V \) nodes where (1) some nodes are capable of correcting up to a maximum number of \( x_{\text{max}} \) errors, (2) the nodes are connected by \( q \)-ary Symmetric Channels \( e_i \) parametrized by their Bit Error Ratios (BER) \( p_i \). We introduce (1) the BER and Worst-Case Error (WCE) metrics and (2) an algebra that allows us to compute the path BER length from its edge lengths, and use them to measure network QoS. The WCE and BER metrics can be used with a generalized Dijkstra’s Algorithm to compute the path of minimum WCE length. Finally, we present an algorithm that solves the above problem in the worst-case time complexity of \( O(V^3) \).

I. INTRODUCTION

In mission critical systems, the penalty for any error could be extremely high. In such systems, improving the network Quality of Service (QoS) involves comparing network paths (and edges) and selecting the optimal path that minimizes the worst possible number of errors (Worst-Case Error, or WCE) received — not a trivial task, especially in wireless, sensor and ad-hoc networks where the Bit Error Ratio (BER) is really high.

This paper introduces a new QoS metric, which we call the WCE metric. With this metric, network path “lengths” (the WCE lengths) are measured and compared to select the optimal path. A path’s WCE length depends on its edges’ WCE lengths. If each edge represents a \( q \)-ary Symmetric Channel (q-SC), then its WCE length can be determined from its BER. Thus, the BER can be used to measure edge and path lengths (the BER lengths), and is a metric — the BER metric. We prove that a path’s WCE length is a non-decreasing function of its BER length, and hence, the minimum BER path is also the minimum WCE path. To calculate a path’s BER length from its edges’ BER lengths, we introduce a new BER algebra and prove that it can be used with the Generalized Dijkstra’s Algorithm [1] to compute the shortest BER path.

We show that if some network nodes are capable of correcting edge errors, then it is possible to find a path (or paths) of zero WCE length. In wireless and ad-hoc networks where compute power and energy are constrained, available resource for complex mathematical operations used by high-performance FEC (such as finite fields arithmetic and iterative algorithms) is limited. It is therefore desirable to have only as many such “overhead” nodes as needed for reliability. In addition, compared to repeater nodes, error-correcting nodes incur delay and consume bandwidth resources. In this paper, we present a \( O(V^3) \) algorithm that routes information with zero WCE through the appropriate error-correcting nodes.

How relevant are error-correcting nodes in modern networks? For long trailing the Automated Repeat reQuest (ARQ) as the method of choice for improving network reliability, Forward Error Correction (FEC) methods have recently been shown to improve the QoS of wireless multimedia applications [2][3][4][5] beyond what could be achieved using only ARQ.

Unlike TCP and ARQ, FEC does not use return requests and thus consumes less bandwidth [6], especially in large multicast networks [7][8]. Even in peer-to-peer networks, FEC methods that use strategically positioned erasure-correcting nodes proved to be superior compared to replication-based strategies [9]. Other multicast algorithms such as Digital Fountain [10] and Buller [11] employ FEC-based source erasure codes. Most recently, a network-coding FEC algorithm has also been proposed [12]. In response to criticisms that FEC consumes bandwidth and computation to process its redundant data, adaptive [4][5] and hybrid (ARQ-FEC) QoS-driven algorithms have been proposed [13][14][15] to dynamically adjust FEC activity to network conditions.

To the best of our knowledge, the problem we pose in the abstract has not been answered in the literature. In addition, the BER metric, WCE metric, and the minimum WCE routing algorithm presented in this paper are new and applicable to many contemporary network problems. Although in this paper the WCE metric is used to calculate the zero WCE path, it can be used in other QoS network optimizations that involve measuring (or minimizing) the occurrence probability of non-typical, but possibly highly catastrophic, events.

This paper is organized as follows: the notations are introduced in section II. The main results and theorems are given in section III. Finally, the algorithm and a theorem of correctness are presented in section IV.

II. FORMULATION AND NOTATION

We model a network as a digraph \( G = (V, E) \), where \( V \) and \( E \) are the node, edge and path sets of \( G \), respectively. \( Q \) is the \( q \)-ary alphabet \( \{0, \ldots, q-1\} \). The nodes \( s, d \in V \) are the source and destination nodes, and \( \Pi \subset \Pi \) is the set of all paths from \( s \) to \( d \).

A path \( \pi \) whose nodes \( V_\pi \) are connected by \( E_\pi \) is denoted by \( \langle v_0, \ldots, v_J \rangle \), \( \langle e_1, \ldots, e_J \rangle \), or \( \langle v_0, e_1, \ldots, e_J, v_J \rangle \). Denote by \( |\pi|_v \) (or \( |\pi|_e \)) the number of nodes (or edges) in \( \pi \) from \( v_i \) to \( v_i+1 \). For (non-) adjacent nodes, \( \langle v_i, v_{i+1} \rangle \) denotes the edge (path) connecting the two nodes. A partial path \( \pi_j \) is \( \langle v_0, \ldots, v_j \rangle \), with \( 0 < j \leq J \), and a truncated path \( \bar{\pi}_j \) is \( \langle v_0, e_1, \ldots, v_{j-1}, e_j \rangle \).

Denote a data block by \( B \in Q^n \), with \( B_t \in Q \), and \( l = 1 \ldots n \). \( B \) can also be defined as a block with \( n \) packets.
of \( m \) symbols each. Instead of an alphabet, \( Q \) is the set of possible packet states encoding QoS metrics such as delay, loss, jitter, etc. Using these generalizations, our symbol- and alphabet-based results can be applied to other packet- and state-based QoS problems.

Let \( B_t \) (and \( b_t = B_b \)) denote \( B \) (and \( B_0 \)) as it departs from \( v_t \); and let \( B_t \) (and \( b_t = B_b \)) denote \( B \) (and \( B_0 \)) as it leaves \( e_t \). Both \( v_t \) and \( e_t \) are part of a path \((v_0, e_1, \ldots, e_j, v_j)\), along which \( B \) evolves as follows:

\[
B_0 \xrightarrow{e_1} B_1 \xrightarrow{v_1} B_2 \xrightarrow{e_2} \cdots \xrightarrow{e_j} B_j \xrightarrow{v_j} B_j
\]

where \( v_t \) and \( e_t \) corresponds to the operators \( v_t, e_t \in E : Q^n \rightarrow Q^n \) given by \( B_t = v_t(B_t) \) and \( B_{t+1} = e_t(B_t) \). The operator for \( \pi \) is \( v_j \circ e_j \circ \cdots \circ e_1 \circ v_0 \). For \( \pi_j \), it is \( \pi_j = v_j \circ e_j \circ \cdots \circ e_1 \circ v_0 \), and for \( \pi_j \), it is \( \pi = e_j \circ v_{j-1} \cdots \circ e_1 \circ v_0 \). Thus, \( \pi_j(B_0) = B_j \) and \( \pi(B_0) = B_j \). The number of errors in \( B_i \) relative to \( B_0 \) is denoted by \( X_i = X_i(B_i) = \{ l | b_i \neq b_i \} \). Also, \( X_i \) is a shorthand for \( X(B_i, \pi) \).

The nodes \( u_i \in U \subseteq V \) implement (possibly different) \( q \)-ary, error-correcting codes of length \( n \) that can correct up to \( x_{max} \) errors in \( B \), relative to \( B_0 \). By default, \( d \in U \). For packet-based systems, \( U \) are network nodes that can restore up to \( x_{max} \) packets in unfavorable states back to favorable states.

and for a partial path \( \pi_j \):

\[
\begin{align*}
X_{0,j} &= 0, & v_j \in U \text{ and } X_{0,j} \leq \max x_j \\
X_{0,j} &= 0, & v_j \in U \text{ and } X_{0,j} > \max x_j \\
X_{0,j} &= x_j \in V \setminus U
\end{align*}
\]

If \( v_j \in U \) and \( B_j \) does not have more than \( x_{max} \) errors, then \( v_j \) transforms \( B_j \) back to \( B_0 \). Otherwise, if \( x_{max} \) is exceeded, \( v_j \) may increase the number of error in \( B \). If \( v_j \notin U \), it simply repeats the content of \( B_j \) into \( B_j \).

Each edge \( e = (v_i, v_{i+1}) \) is a \( q \)-ary Symmetric Channel with a BER of \( p \in P = [0, 1] \cup \infty \), or \( q\)-SC\( (p) \). The value \( p = \infty \) means \( v_i \) is not connected to \( v_{i+1} \).

For each symbol in \( B \), the transition probability is:

\[
P(b_{i+1} | b_i) = \left\{ \begin{array}{ll}
1-p, & b_i = b_{i+1} \\
p/(q-1), & b_i \neq b_{i+1} 
\end{array} \right.
\]

(2)

The probability of \( B_{i+1} = e_i(B_i) \) having \( x \) errors relative to \( B_i \) is \( P(X_{i+1} = x) = P(x,p) \) given by:

\[
P(x,p) = \left\{ \begin{array}{ll}
\binom{n}{x} p^x (1-p)^{n-x}, & p \in (0,1) \\
\delta(x), & p = 0 \\
\delta(x-n), & p = \infty \end{array} \right.
\]

(3)

The BER metric uses \( p \), which determines \( P(x,p) \), to measure the QoS of \( e \). The WCE metric measures the worst “possible” (defined as \( P(x,p) \) above a threshold \( \epsilon \) number of errors (the Worst-Case Error, or WCE) \( \bar{x} \) in \( B_{i+1} \) relative to \( B_i \), denoted by \( \bar{x} \in X = [0, n] \):

\[
\bar{x}(p, \epsilon) = \max \{x \mid P(x,p) \geq \epsilon, 0 \leq x \leq n\}
\]

(4)

From (3), \( \bar{x}(1, \epsilon) = \bar{x}(\infty, \epsilon) = n, \bar{x}(0, \epsilon) = 0, \) and \( \bar{x}(p, \epsilon) \) is not continuous. The functions \( \beta : \Pi \rightarrow P \) and \( \omega : \Pi \rightarrow X \) measure the BER and WCE path lengths.

Consider a path \( \pi = (e_1, \ldots, e_j) \in \Pi \) and its partial path \( \pi_j = (e_1, \ldots, e_j) \) with \( 1 \leq j \leq J \). Let \( p_j = \beta(e_j) \) and \( \bar{x}_j = \omega(e_j) \). The path QoS length \( p_\pi \) (or \( \bar{x}_\pi \)) depends on \( p_j \) (or \( \bar{x}_j \)), but in general, \( p_\pi \neq \sum p_j \) and \( \bar{x}_\pi \neq \sum \bar{x}_j \).

Denote the BER and WCE addition operators by \( + \). If \( p_1 = \beta(e_1), p_2 = \beta(e_2), \bar{x}_1 = \omega(e_1), \bar{x}_2 = \omega(e_2), \) and \( \pi = (e_1, e_2), \) then \( p_\pi = p_1 + p_2 \) or \( \bar{x}_\pi = \bar{x}_1 + \bar{x}_2 \).

For q-SC\( (p) \), the \( \oplus \) operators are defined below:

\[
p_1 \oplus p_2 = 1 - (1-p_1)(1-p_2) = (p_1p_2) / (q-1) \]

\[
\bar{x}_1 \oplus \bar{x}_2 = \max \{x \mid P(x, p_1 \oplus p_2) \geq \epsilon\}
\]

(5)

We can now define \( p_\pi \) and \( \bar{x}_\pi \) in terms of \( p_j \) and \( \bar{x}_j \) using a generalized summation: \( p_\pi = \bigoplus p_e \) and \( \bar{x}_\pi = \bigoplus \bar{x}_e \). The pairing of \( + \) and \( \oplus \) forms an algebraic structure (a magma) which we call the BER algebra \( B \). Likewise, the pairing of \( \oplus \) and \( \oplus \) forms the WCE algebra \( W \).

Between two points, the BER (or WCE) optimal path \( \pi^* \) is the path with the “least” BER (or WCE) path length. To calculate \( \pi^* \), we need to compare path lengths. Therefore, we need a total order \( \leq \) on \( P \) (or \( X \)) to evaluate expressions like \( p_\pi \leq p_{\pi'} \) (or \( \bar{x}_\pi \leq \bar{x}_{\pi'} \)).

\[
\bar{x}^* = \min \{ \bar{x} \mid \bar{x} = \bar{x}(p_e, \epsilon) \mid \pi \in \Pi \}
\]

\[
p^* = \min \{ p \mid p \in \Pi \}
\]

(6)

The lengths \( p_\pi \) and \( \bar{x}_\pi \) are non-decreasing functions of \( j \) — an assertion we prove later in this paper. This means that on any path, the worst possible \( X_{0,j} \) also increases as \( B \) traverses more edges. If \( G \) cannot correct any errors, reliable communication (as far as the worst-case scenario is concerned) becomes very difficult to achieve except for very small values of \( p_j < 1 \).

III. MAIN RESULTS

The problem of finding the optimal path of minimum WCE in \( G \) bears many similarities to the Shortest Path Problem (SPP) that can be solved using Dijkstra’s Algorithm (DA). The Generalized Dijkstra’s Algorithm (GDA) below [1] is the perfect tool for the problem:

**Algorithm 1 (GDA):** The GDA is practically identical to DA except for the relaxation step, where \( + \) and \( \oplus \) operators act on a general metric space \( M \) (instead of the step in DA, where \( + \) and \( \oplus \) operators act on \( R \)):

1: **procedure** DIJKSTRA \((G,p,s)\)
2: for all \( v \in V \) do
3: \( l[v] \leftarrow \infty \)
4: \( \pi[v] \leftarrow \text{NIL} \)
5: \( Q \leftarrow V \)
6: \( l[s] \leftarrow 0 \)
7: while \( Q \neq \emptyset \) do
8: \( u \leftarrow \text{Min}(Q) \)
9: for all node \( v \in N(u) \) do
10: if \( l[v] > l[u] \oplus p(u,v) \) then
11: \( l[v] \leftarrow l[u] \oplus p(u,v) \)
12: \( \pi[v] \leftarrow u \)


On line 9, $N(u)$ denotes the set of all nodes adjacent to $u$. The argument $p$ is the BER lengths of the edges in $G$, each of which is an element in $M$, and $p(u, v)$ is the BER of $(u, v)$. Lines 10–12 perform the relaxation step of the GDA. This step depends on the definitions of $M$, $\oplus$, and $\preceq$. If the definitions are such that the GDA (incorrectly) returns the path in $G$ with minimum length measured in $M$, then $(M, \oplus)$ and $\preceq$ are said to be (in)compatible with the GDA.

**Proposition 1:** An algebra $A = (M, \oplus)$ and a total order $\preceq$ is compatible with the GDA if and only if it satisfies all the properties in the set denoted by $P$ below:

- **P1** is a commutative monoid, that is, for $a, b, c \in M$:
  - $M$ is closed under $\oplus$: $a \oplus b \in M$;
  - $\oplus$ is associative: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;
  - $0$ is the identity: $a \oplus 0 = 0 \oplus a = a$;
  - $\oplus$ is commutative: $a \oplus b = b \oplus a$.

**P2** There exists $\infty \in M$ such that $a \oplus \infty = \infty \oplus a = \infty$.

**P3** $\preceq$ is a total order on $M$, i.e., $\preceq$ is:
- reflexive: $a \preceq a$;
- anti-symmetric: if $a \preceq b$ and $b \preceq a$ then $a = b$;
- transitive: if $a \preceq b$ and $b \preceq c$ then $a \preceq c$;
- total: for every $a, b \in M$ either $a \preceq b$ or $b \preceq a$.

**P4** There exists the least element $0$ that satisfies $0 \preceq a$.

**P5** $a \oplus c \preceq b \oplus c$ if $a \preceq b$ and $c \in M - \{\infty\}$.

**Proof:** Refer to [1] for a complete proof.

**Theorem 2:** The algebra $B = (P, \oplus)$ and the total order $\preceq$ defined in section II satisfy all the properties in $P$, and thus compatible with the GDA.

**Proof:** Recall the definition of $\oplus$ from equation (5), which is repeated here for convenience:

$$a \oplus b = 1 - (1 - a)(1 - b) - (ab) / (q - 1)$$  \hspace{1cm} (5)

**P1** Except for closure, the monoid properties can be proven with algebraic manipulation of equation (5). Recall that $P = [0, 1] \cup \infty$. Closure is obvious if $a, b \in [0, 1]$. If $a = \infty$ or $b = \infty$, then by the definition of $\infty$ as a special designation for the absence of connection between two nodes, then we must have $a \oplus b = \infty$.

**P2** The proof is derived from closure on $\infty$.

**P3** The proof follows the definition of $P$.

**P4** Same as above.

**P5** The proof is obvious if $b = \infty$. However, if $b \neq \infty$, then by substituting $a, b,$ and $c$ into (5) we obtain the two values $a \oplus c$ and $b \oplus c$ in $[0, 1]$ given by:

$$a \oplus c = 1 - (1 - a)(1 - c) - ac / (q - 1)$$

$$b \oplus c = 1 - (1 - b)(1 - c) - bc / (q - 1).$$

Since both are in $[0, 1]$, the order $\prec$ is just $<$, and the expression $a \oplus c \prec b \oplus c$ is equivalent to the inequality $(a \oplus c) - (b \oplus c) < 0$, which can be simplified into:

$$(1 - cq / (q - 1))(a - b) < 0$$

If $c \leq \frac{q - 1}{q}$, then $0 < (1 - cq / (q - 1)) \leq 1$, and $(a - b) < 0$ (because $a < b$ and $b \neq \infty$). Thus, the above inequality is true, and we have proven all the properties in $P$.

Given two nodes $v_0, v_J \in V$ and the set of all paths $\Pi_j$ connecting them, the path length of $\pi \in \Pi_j$ is given by $p_\pi = \beta(\pi)$. Using the GDA with $(B, \preceq)$, we can compute the shortest path $\pi^* \in \Pi_j$ where $p_{\pi^*} \leq p_{\pi_j}, \forall \pi \in \Pi_j$. Again, if $\pi_j$ denotes the path length of $\pi$, $\Pi_j$.

**Proposition 3:** If $V_{\pi_j} \cap U = \emptyset$, then $\beta(\pi_j)$ is a non-decreasing function of $j$. The minimum $\beta(\pi_j)$ is 0 possible if the edge lengths $p_j = 0$ for all $j = 0 \ldots J$.

**Proof:** $P5$ with $0 = a \preceq b = p_j$, and $c = p_{\pi_j-1}$ gives us $\beta(\pi_j-1) = p_{\pi_j-1} < p_{\pi_j-1} \oplus p_j = \beta(\pi_j)$, proving that $\beta(\pi_j)$ is a non-decreasing function of $j$. The second part of the proof can be derived directly from $P1$.

The preceding proposition shows the practical benefit of including error-correcting nodes in the network, and of routing information through them. If the path includes $u \in U$, then for some $j \in [0, J]$, we can have $\beta(\pi_j) = 0$ even with $p_j \neq 0$ for all $j$. Next, we prove that within an admissible range of $c \in [0, \epsilon]$, the WCE function $\bar{x}(p, c)$ is a non-decreasing (albeit discontinuous) function of $p$.

**Lemma 4:** For a given $n$ and a fixed $x$, the probability function $P(x, p)$ is maximized at $p = \frac{\bar{x}}{n}$. Furthermore, $P* = P((\bar{x}), \frac{\bar{x}}{n})$ minimizes $P(x, \bar{x})$ over all $x \in X$.

**Proof:** From the definition of $P(x, p)$ in equation (3), the lemma is true for $x = 0$ and $x = n$. Consider:

$$\frac{\partial P(x, p)}{\partial p} = P(x, p) \left( \frac{\bar{x}}{n} - \frac{n - x}{1 - p} \right) = 0$$

Solving $\left( \frac{x}{n} - \frac{n - x}{1 - p} \right) = 0$ for $p$ gives us $p = \frac{x}{n}$, which maximizes the function $P(x, p)$ for $x \in (0, 1)$.

The next question is, for $0 < x < n$, which $x$ minimizes $P(x, \bar{x})$? Unlike with $p$, we cannot differentiate $P(x, p)$ with respect to $x$ because it is a discrete variable. Instead of approximating $P(x, p)$ with a Gaussian distribution, which is valid only for certain $n$ and $p$, we resort to the upper and lower bounds for $(\binom{n}{x})$ in [16],

$$\xi(n) = \frac{e^{n+1}}{(2\pi)^{n}} \frac{1}{(n-x)^{n-x-1}} \frac{1}{x^{(n-x)}} \frac{1}{(n-x+x-1)} (7)$$

$$\lambda_{nx} = \lambda(x, n) \left[ \frac{1}{12n} \frac{1}{12n+1} \frac{1}{12n-1} \frac{1}{12n-x+1} \right]$$

$$\mu_{nx} = \mu(x, n) \left[ \frac{1}{12n} \frac{1}{12n+1} \frac{1}{12n-1} \frac{1}{12n-x+1} \right]$$

Using (7), $P(x, p)$ is now bounded by two continuous and differentiable functions in $x$.

$$\lambda_{nx} p^x (1-p)^{n-x} \leq P(x, p) \leq \mu_{nx} p^x (1-p)^{n-x}$$

Since $p = \frac{x}{n}$, we substitute $x = np$ into the equation below and solve the $p$ roots of the $p$ derivatives of both the lower and upper bounds of $P(x, p)$ to find the minima with respect to $p$. 
The lower and upper bounds are minimized at $p = \frac{1}{2}$. Since $p = \frac{x}{n}$, then $x = \frac{2p}{n}$:

$$
\sqrt{\frac{2}{n+1}} e^{-\frac{2p}{n+1}} < P\left(\frac{1}{2}, \frac{1}{2}\right) < \sqrt{\frac{2}{n+1}} e^{-\frac{2p}{n+1}}
$$

(8)

in which for large values of $n$ the lower and upper bounds converge. In fact, in (7), $\lambda_{nx}$ converge to $\mu_{nx}$ for all $x$, including at the discrete points $0 < x < n \in N$. Thus the minima for $\lambda_{nx}$ and $\mu_{nx}$ over the continuous $x$ must also be the minimum for $P(x, x/n)$ over the discrete $x$, where it is denoted by $P_* = P\left(\frac{1}{2}, \frac{1}{2}\right)$.

A function $f(x)$ is unimodal over $x \in [a, b]$ if there exists an $x_0$ such that $f(x)$ is monotonically increasing for $x < x_0$ and monotonically decreasing for $x > x_0$.

**Lemma 5:** $P(x, p)$ is unimodal over $x$ and $p$.

**Proof:** To prove unimodality over $x$, solve $P(x, p) < P(x + 1, p)$ for $x$, giving $x < np - (1 - p)$ or $x < x_0 = \lfloor np \rfloor$. Similarly, $P(x, p) > P(x + 1, p)$ or $x > x_0$. Unimodality over $p$ is proven with calculus:

$$
P'(x, p) = \frac{\partial}{\partial p} P(x, p) = P(x, p) \left(\frac{p}{n} - \frac{x}{n^2}\right)
$$

$P'(x, p_0) = 0$ at $p_0 = \frac{x}{n}$. For $p < p_0$, we have $P'(x, p) > 0$ and for $p > p_0$, we have $P'(x, p) < 0$.

**Corollary 6:** The set $\epsilon$ of admissible $\epsilon$ is $[0, \bar{\epsilon} = P_*]$.

**Proof:** Recall that $\bar{x}(p, \epsilon) = \max\{ x \mid P(x, p) \geq \epsilon \}$. To ensure $\bar{x}(p, \epsilon)$ is valid for all $p \in P$, for each value of $p$ we must have $P(x, p) \geq \epsilon$ for some $x$. This condition is trivially met if $\epsilon \leq 0$ because $P(x, p) \geq 0$. If $\epsilon > P_*$, then $\{ x \mid P(x, \epsilon) \geq \epsilon \} = \emptyset$, and $\bar{x}(p, \epsilon)$ is invalid. Thus, the set of admissible $\epsilon$ is given by $[0, P_*]$.

Denote by $p_i(x, \epsilon) = p_i(x) = \{ p \in P \mid P(x, p) \geq \epsilon \}$ the $i$-th roots of $P(x, p) = \epsilon$ for given $n$ and $\epsilon$. From Lemma 5, if $\epsilon \in \epsilon$, then there is at least one root. At $x = 0$, the single root is $p_0(0) = 0$ and at $x = n$, it is $p_1(n) = 1$. Except for another special case when $P_* = \epsilon$, where at the midpoint $x_m = \frac{1}{2}$ there is only one root to $p_0(x_m) = p_1(x_m) = \frac{1}{2}$, in general, there are two distinct roots $p_0(x), p_1(x) \in [0, 1]$, with $p_0(x) < \frac{x}{n} < p_1(x)$. If $\epsilon$ goes toward 0, then $p_0(x)$ and $p_1(x)$ decrease, except $p_0(0) = 0$ and $p_0(1) = 1$. If $\epsilon$ goes toward $P_*$, then $p_0(x)$ increases, while $p_1(x)$ decreases.

Define the sets $P_0$ and $P_1$, each define the $n + 1$ values of $p_0(x)$ and $p_1(x)$, for $0 \leq x \leq n$. These values are also referred to as $x_0(p)$ and $x_1(p)$, for $0 \leq p = \frac{x}{n} \leq n$.

**Lemma 7:** The roots $p_0(x)$ and $p_1(x)$ are non-decreasing functions of $x$ with $p_0(x) = p_1(x)$ only at $x = 0$ and $x = n$ (or $x = x_m$ for $P_*$).

**Proof:** First, observe that $P(x, p) = P(x + 1, p)$ only has one root at $p = p^* = \frac{x + 1}{n + 1}$ between the maxima of $P(x, p)$ and $P(x + 1, p)$, i.e., $\frac{x}{n} < p^* < \frac{x + 1}{n + 1}$. From Lemma 5, this implies that if $p < p^*$ then $P(x, p) > P(x + 1, p)$, and if $p > p^*$, $P(x, p) < P(x + 1, p)$. Hence, $p_0(x) \leq p_0(x + 1)$ and $p_1(x) \leq p_1(x + 1)$, that is, $p_0(x)$ and $p_1(x)$ are both non-decreasing functions of $x$.

**Theorem 8:** The values $\bar{x}(p, \epsilon) = \max_{x} x_0(p)$ and $\max_{x} x_1(p)$ are non-decreasing functions of $p$.

**Proof:** First, the $\max_{x}$ function is used because it is possible to have $p_0(x) = p_0(x') \in P_1$ and $x \neq x'$. For example, if $\epsilon = 0$, $p_0(0) = \cdots = p_0(n - 1) = 0$ (the same argument applies to $P_1$). Just as $\max_{x}$ in $\bar{x}(p, \epsilon)$ isolates the largest $x$ satisfying $P(x, p) \geq \epsilon$, the function $\max_{x} x_0(p)$ isolates the largest $x$ satisfying $P(x, p) = \epsilon$. After establishing a unique $x$ for each $p$, the proof follows from the monotonicity of $p(x)$.

Theorem 8 proves that for any path $\pi$, the WCE length $\omega(\pi)$ is a non-decreasing function of its BER $\beta(\pi)$. Therefore, the minimum BER path between two nodes (computed by the GDA) is also the minimum WCE path. Therefore, from this point on, the terms “minimum BER” and “minimum WCE” are interchangeable.

In the next section, we finally present the algorithm MIN-WCE-PATH that computes the minimum (zero) WCE path in the presence of error-correcting nodes $U$.

**IV. ALGORITHM**

A path $\phi = (v_1, v_2)$ is feasible iff $\omega((s, v_1)) + \omega(\phi) \leq x_{\max}$ — the WCE at $v_1$ plus the WCE length of $\phi$ must be less than $x_{\max}$. Denote by $\Phi \subseteq \Pi$ the feasible paths in $\Pi$, and by $\Phi(v_1, v_2)$ the feasible paths between $v_1, v_2$.

**Theorem 9:** A path $\pi^*$ is the minimum WCE path iff it solves the SPP given by $G' = (V', E')$, where $V' = \{ s \} \cup U$. An edge connecting two nodes $v_1, v_2 \in V'$ represents the shortest path in $\Phi(v_1, v_2)$. Therefore

$$
E' = \{ \arg\min_{\phi} \{ \omega(\phi) \mid \phi \in \Phi(v_1, v_2) \} \mid v_1, v_2 \in V' \}.
$$

**Proof:** Suppose $\pi^*$ contains $n + 1$ segments $\phi_i$ connecting the nodes in $V'' = \{ s, U_{\pi^*}, d \}$, where $U_{\pi^*} = U \cap V_{\pi^*}$. In segment notation, $\pi^*$ is denoted by $s \rightarrow u_1 \rightarrow \cdots \rightarrow u_j \rightarrow d$, with $\{ u_j \} = U_{\pi^*}$, and $0 \leq j \leq |U|$.

Then $\phi_i$ must be the shortest feasible paths between adjacent nodes in $V''$, and $U_{\pi^*}$ must be the set that minimizes $\sum \beta(\phi_i)$. Otherwise, a better path $\xi^*$ can be obtained by modifying $\phi_i$ or $U_{\pi^*}$, contradicting the claim that $\pi^*$ is optimal. For the forward proof, note that $V'' = \{ s, d, U_{\pi^*} \} \subseteq V'$. Further, since each $\phi_i$ is a shortest path between nodes in $V'$, then $\phi_i \in E'$. Therefore $\pi^*$ solves the SPP given by $(V', E')$.

For the reverse proof, suppose $\xi^*$ is the SPP solution but is not the minimum WCE path $\pi^*$. From the forward proof, if $\pi^*$ minimizes WCE, then it also solves the SPP, thus $\omega(\pi^*) \leq \omega(\xi^*)$. However, if $\omega(\pi^*) < \omega(\xi^*)$, then $\xi^*$ is not the SPP solution — a contradiction.

Thus, $\omega(\pi^*) = \omega(\xi^*)$, and if path lengths are unique, $\pi^* = \xi^*$. Hence, the SPP solution $\pi^*$ is the minimum WCE path. □

Hence, to find the minimum WCE path we first compute the minimum WCE path for each pair of nodes in $V'$. These minimum paths are then converted into edges in $E'$, connecting the nodes in $V'$. The overall minimum WCE path is then computed from these edges using the GDA.
Algorithm 2: Theorem 9 proves the correctness of the optimization algorithm listed below:

1: procedure MIN-WCE-PATH $(G, p, s)$
2: \[ E' \leftarrow \{ e \in E \mid \omega(e) > x_{\text{max}} \} \]
3: \[ V' \leftarrow \{ v \in V \mid \deg(v) = 0 \} \]
4: for \( v_1 \in V' \) do
5: \[ \text{SP}_1 = \text{GDA} (G, p, v_1) \]
6: for \( v_2 \in V' \neq v_1 \) do
7: \[ E' = E' \cup (v_1, v_2) \]
8: \[ E' \leftarrow \{ e' = (v_1, v_2) \in E' \mid e' \notin \Phi(v_1, v_2) \} \]
9: \[ V' \leftarrow \{ v' \in V' \mid \deg(v') = 0 \} \]
10: \[ \text{SP}_2 = \text{GDA} (G', p, s) \]

On lines 2 and 3, the algorithm prunes all the infeasible edges. Then, on line 5, it runs the GDA on all \( v_1 \in V' \), every time producing a shortest path tree \( \text{SP}_1 \) rooted at \( v_1 \). On line 7, the edges connecting \( v_1 \) and \( v_2 \in V' \) are added into \( E' \) based on \( \text{SP}_1 \). The first stage is finished and the second stage begins. On lines 8 and 9, infeasible edges in \( E' \) are pruned and any isolated nodes in \( V' \) are removed. Finally, \( \pi^* \) is obtained.

Denoting \( \lvert V' \rvert \) by \( \alpha \), the first stage produces a complete graph with \( \alpha \) nodes and \( \alpha(\alpha - 1) \) directed edges by executing the GDA \( \alpha \) times on line 5, and thus has a time complexity of \( O(\alpha V^2) \) if the GDA is implemented using a generalized Fibonacci heap, then its complexity could reach \( O(V \log V + E) \) [17]). Lines 8 and 9 search linearly over them with \( O(\alpha^2) \) time complexity. The GDA on line 10 has a time complexity \( O(\alpha^2) \). Hence, overall time complexity is \( O(\alpha V^2 + \alpha^2) < O(V^3) \).

V. CONCLUSION

In this paper, we considered the problem of finding the path with minimum (or zero) worst possible number of errors in mission-critical communication networks. We made the assumptions that some network nodes are capable of correcting up to a maximum number of \( x_{\text{max}} \) errors in a block of \( n \) symbols, and that the nodes are connected by \( q \)-ary Symmetric Channels parametrized by the Bit Error Ratios (BER).

We introduced (1) the BER and Worst-Case Error (WCE) metrics and (2) a BER algorithm that allows for computation of the path BER length from its edge lengths. The metric-algebra pair was then used to optimize the network QoS in the BER and WCE metrics using a generalized Dijkstra’s Algorithm in the worst-case time complexity of \( O(V^3) \), where \( V \) is the number of nodes in the network.

Future research can explore the issues of whether the approach outlined in this paper can be generalized to other types of distributions and scenarios, the incorporation of the algorithm presented here into existing networking protocols, as well as simulation and experimental verification of the algorithm on standard network models.

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