Equilibrium of Heterogeneous Congestion Control: Existence and Uniqueness

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***Abstract*— When heterogeneous congestion control protocols that react to different pricing signals share the same network, the resulting equilibrium may no longer be interpreted as a solution to the standard utility maximization problem. We prove the existence of equilibrium in general multi-protocol networks under mild assumptions. For almost all networks, the equilibria are locally unique, and finite and odd in number. They cannot all be locally stable unless there is a globally unique equilibrium. Finally, we show that if the price mapping functions, which map link prices to effective prices observed by the sources, are sufficiently similar, then global uniqueness is guaranteed.**

I. Introduction

A. Motivation

Congestion control protocols have been modelled as distributed algorithms for network utility maximization, e.g., [10], [11], [13], [14], [22], [34]. With the exception of a few limited analysis on very simple topologies [8], [12], [13], [21], existing literature generally assumes that all sources are homogeneous in that, even though they may control their rates using different algorithms, they all adapt to the same type of congestion signals, e.g., all react to loss probabilities, as in TCP Reno, or all to queueing delay, as in TCP Vegas or FAST [9]. When sources with *heterogeneous* protocols that react to different congestion signals share the same network, the current duality framework is no longer applicable. With more congestion control protocols being proposed and ideas of using congestion signals other than packet losses, including explicit feedbacks, being developed in the networking community, we need a mathematically rigorous framework to understand the behavior of large-scale networks with heterogeneous protocols. This paper proposes and develops such a framework.

Our emphasis is on general networks with multiple sources and links that use a large class of algorithms to adapt their rates and congestion prices. Often, interesting and counter-intuitive behaviors arise only in a network setting where sources interact through shared links in intricate and surprising ways, e.g., [28]. Such behaviors are absent in single-link models and are usually hard to discover or explain without a fundamental understanding of the underlying structure. Given the scale and heterogeneity of the Internet, it is conceivable that such behaviors are more common than we realize, but remain difficult to measure due to the complexity of the infrastructure and our inability to monitor it closely. A mathematical framework thus becomes indispensable in exploring structures, clarifying ideas, and suggesting directions. Some of the theoretical predictions in this paper have already been demonstrated experimentally in [31].

B. Summary

A congestion control protocol generally takes the form [1]

$$\dot{p}_l = g_l\left(\sum_{j:l\in L(j)} x_j(t), p_l(t)\right) \qquad (1)$$

$$\dot{x}_j = f_j\left(x_j(t), \sum_{l\in L(j)} m_l^j(p_l(t))\right) \qquad (2)$$

Here, $L(j)$ denotes the set of links used by source j, and $g_l(\cdot)$ models a queue management algorithm that updates the price $p_l(t)$ at link l, often implicitly, based on its current value and the sum of source rates $x_j(t)$ that traverse link l. The prices may represent loss probabilities, queueing delays, or quantities explicitly calculated by the links and fed back to the sources. The function f_j models a TCP algorithm that adjusts the transmission rate $x_j(t)$ of source j based on its current value and the sum of "effective prices" $m_l^j(p_l(t))$ in its path. The effective prices $m_l^j(p_l(t))$ are functions of the link prices $p_l(t)$, and the functions m_l^j in general can depend on the links and sources.

When all algorithms use the same pricing signal, i.e., $m_l^j = m_l$ are the same for all sources j, the equilibrium properties of (1)–(2) turn out to be very simple. Indeed, under mild conditions on g_l and f_j, the equilibrium of (1)–(2) exists and is unique [13]. This is proved by identifying the equilibrium of (1)–(2) with the unique solution of the utility maximization problem defined in [10] and its Lagrange dual problem [14]. Here, the equilibrium prices p_l play the role of Lagrange multipliers, one at each link. This utility maximization problem thus provides a simple and complete characterization of the equilibrium of a single-protocol network and also leads to a relatively simple dynamic behavior.

When heterogeneous algorithms that use different pricing signals share the same network, i.e., m_l^j are different for different sources j, the situation is much more complicated. For instance, when TCP Reno and TCP Vegas or FAST share the same network, neither loss probability nor queueing delay can serve as the Lagrange multiplier at the link, and (1)–(2) can no longer be interpreted as solving the standard network utility maximization problem. Basic questions, such as the existence and uniqueness of equilibrium, its local and global stability, need to be re-examined.

We focus in this paper on the existence and uniqueness of equilibrium. We prove that equilibrium still exists, under

[1] Delay is omitted to simplify the whole system. Note that this paper focus on equilibrium properties that don't depend on delay.

mild conditions, despite the lack of an underlying convex optimization problem (Section III). In contrast to the single-protocol case, even when the routing matrix has full row rank, there can be uncountably many equilibria (Example 1 in Section IV) and the set of bottleneck links can be non-unique (Example 2 in Section IV). However, we prove that almost all networks have finitely many equilibria and they are necessarily locally unique (Section IV). The number of equilibria is always odd, though can be more than one (Section IV). Moreover, these equilibria cannot all be locally stable unless the equilibrium is globally unique (Section IV). Finally, we provide two sufficient conditions for global uniqueness of network equilibrium (Sections V and VI). The first condition implies that if the price mapping functions that map link prices to effective prices observed by the sources do not differ too much ("degree of heterogeneity" is sufficiently small), then global uniqueness is guaranteed. The second condition generalizes the full-rank condition on routing matrix for global uniqueness from single-protocol networks to multi-protocol networks. Throughout the paper, we provide numerical examples to illustrate equilibrium properties or how a theorem can be applied. In [31], we demonstrate experimentally the phenomenon of multiple equilibria using TCP Reno and TCP Vegas/FAST in ns-2 simulator and Dummynet testbed. More properties of heterogeneous congestion control, including optimality, fairness and methods to achieve them, have recently been obtained in [32].

C. Related work

Our formulation is close to the general equilibrium theory in economics from which we borrow ideas and techniques [18]. See [3], [4], [5], [6], [7], [24], [25], [33] and [1], [17] for a fairly complete treatment of related works in economics literature. A typical model of the pure exchange economy consists of L commodities and N consumers. Each consumer i has an initial endowment vector $\omega_i = (\omega_{il} \geq 0, l = 1, \ldots, L)$ and its goal is to choose a consumption vector $x_i = (x_{il}, l = 1, \ldots, L)$ to maximize its utility subject to its wealth constraint, i.e.,

$$\max_{x_i \geq 0} \; U_i(x_i) \quad \text{subject to} \quad p^T x_i \leq p^T \omega_i$$

where $p = (p_l, l = 1, \ldots, L)$ are unit prices for the goods and ^{T} denotes matrix transpose. For each good $l = 1, \cdots, L$, demand and supply are balanced if

$$\sum_{i=1}^{N} x_{il} \;\; = \;\; \sum_{i=1}^{N} \omega_{il}$$

A consumption vector $x^* = (x_i^*, i = 1, \ldots, N)$ and a price vector p^* are called a *competitive equilibrium* (or *Walrasian equilibrium*) if x_i^* maximizes i's utility and demand equals supply for all goods.

In general equilibrium theory, consumers are assumed to be price takers. This aspect is similar to our model where sources do not take into account how their decisions affect the link prices or each other. Both problems are concerned with characterizing fixed points of a continuous mapping, and hence there are considerably similarities in terms of the characterizations and the mathematical tools to derive them. The main mathematical tools used in this paper are the Nash theorem in game theory [2], [23], which is an application of Kakutani's generalized fixed point theorem, and results from differential topology, especially the Poincare-Hopf Index Theorem [20]. They are used to prove existence and study uniqueness of network equilibrium, respectively.

However, there are several important differences. First, the effective prices to different sources (consumers) are generally different in our model, whereas the prices in the economic model are independent of consumers. Differential pricing is what makes networks with heterogeneous protocols much more difficult. Second, in the economic model, there is a concept of initial endowment that defines both the demand-supply relation and a consumer's consumption possibility through the wealth constraint. In our model, the wealth constraint is replaced by the link capacity constraint. Third, in the economic model, consumers maximize their utilities whereas in our model, sources maximize their utilities minus bandwidth costs. Finally, in our model, every source consumes exactly the same amount of bandwidth at each link in its path ($x_{il} = x_i$, for all $l \in L(i)$), whereas, in the economic model, consumers can consume different goods at different amounts. This guarantees that the demand for every good is exactly balanced by its supply in a pure exchange economy, yet in networks, the set of bottleneck links where demand for and supply of bandwidth is balanced can be non-unique and a strict subset of all links. The property $x_{il} = x_i$ is the key structure that allows us to obtain interesting results on global uniqueness in fairly general settings. In contrast, global uniqueness in general equilibrium analysis usually requires very strong conditions and most literature focuses on local uniqueness [1], [3], [5].

II. Model

A. Notation

A network consists of a set of L links, indexed by $l = 1, \ldots, L$, with finite capacities c_l. We often abuse notation and use L to denote both the number of links and the set $L = \{1, \ldots, L\}$ of links. Each link has a price p_l as its congestion measure. There are J different protocols indexed by superscript j, and N^j sources using protocol j, indexed by (j, i) where $j = 1, \ldots, J$ and $i = 1, \ldots, N^j$. The total number of sources is $N := \sum_j N^j$.

The $L \times N^j$ routing matrix R^j for type j sources is defined by $R^j_{li} = 1$ if source (j, i) uses link l, and 0 otherwise. The overall routing matrix is denoted by

$$R = \left[\begin{array}{cccc} R^1 & R^2 & \cdots & R^J \end{array} \right]$$

Even though different classes of sources react to different prices, e.g. Reno to packet loss probability and Vegas/FAST to queueing delay, the prices are related. We model this relationship through a price mapping function that maps a common price (e.g. queue length) at a link to different prices (e.g. loss probability and queueing delay) observed by different sources. Formally, every link l has a price p_l. A type j source reacts to the "effective price" $m^j_l(p_l)$ in its path, where m^j_l is a price mapping function, which can depend on both the link and the protocol type. The exact form of m_l^j depends on the AQM algorithm used at the link; see [31] for links with RED.[2] Let $m^j(p) = (m_l^j(p_l), l = 1, \ldots L)$ and $m(p) = (m^j(p_l), j = 1, \ldots J)$. The aggregate prices for source (j,i) is defined as

$$q_i^j \;=\; \sum_l R_{li}^j m_l^j(p_l) \tag{3}$$

Let $q^j = (q_i^j, i = 1, \ldots, N^j)$ and $q = (q^j, j = 1 \ldots, J)$ be vectors of aggregate prices. Then $q^j = \left(R^j\right)^T m^j(p)$ and $q = R^T m(p)$.

Let x^j be a vector with the rate x_i^j of source (j,i) as its ith entry, and x be the vector of x^j

$$x \;=\; \left[\; (x^1)^T, (x^2)^T, \ldots, (x^J)^T \;\right]^T$$

Source (j,i) has a utility function $U_i^j(x_i^j)$ that is strictly concave increasing in its rate x_i^j. Let $U = (U_i^j, i = 1, \ldots, N^j, j = 1, \ldots, J)$.

In general, if z_k are defined, then z denotes the (column) vector $z = (z_k, \ \forall k)$. Other notations will be introduced later when they are encountered. We call (c, m, R, U) a *network*.

B. Network equilibrium

A network is in equilibrium, or the link prices p and source rates x are in equilibrium, when each source (j,i) maximizes its net benefit (utility minus bandwidth cost), and the demand for and supply of bandwidth at each bottleneck link are balanced. Formally, a network equilibrium is defined as follows.

Given any prices p, we assume in this paper that the source rates x_i^j are uniquely determined by

$$x_i^j\left(q_i^j\right) \;=\; \left[\left(U_i^j\right)'^{-1}\left(q_i^j\right)\right]^+$$

where $\left(U_i^j\right)'$ is the derivative of U_i^j, and $\left(U_i^j\right)'^{-1}$ is its inverse which exists since U_i^j is strictly concave. Here $[z]^+ = \max\{z, 0\}$. This implies that the source rates x_i^j uniquely solve

$$\max_{z \geq 0} \;\; U_i^j(z) - z q_i^j$$

As we will see, under the assumptions in this paper, $\left(U_i^j\right)'^{-1}\left(q_i^j\right) > 0$ for all the prices p that we consider, and hence we can ignore the projection $[\cdot]^+$ and assume without loss of generality that

$$x_i^j\left(q_i^j\right) \;=\; \left(U_i^j\right)'^{-1}\left(q_i^j\right) \tag{4}$$

As usual, we use $x^j\left(q^j\right) = \left(x_i^j\left(q_i^j\right), i = 1, \ldots, N^j\right)$ and $x(q) = \left(x^j\left(q^j\right), j = 1, \ldots, J\right)$ to denote the vector-valued functions composed of x_i^j. Since $q = R^T m(p)$, we often abuse notation and write $x_i^j(p), x^j(p), x(p)$.[3]

Define the aggregate source rates $y(p) = (y_l(p), l = 1, \ldots, L)$ at links l as:

$$y^j(p) \;=\; R^j x^j(p), \qquad y(p) \;=\; Rx(p) \tag{5}$$

In equilibrium, the aggregate rate at each link is no more than the link capacity, and they are equal if the link price is strictly positive. Formally, we call p an *equilibrium price*, a *network equilibrium*, or just an *equilibrium* if it satisfies (from (3)–(5))

$$P(y(p) - c) = 0, \quad y(p) \leq c, \quad p \geq 0 \tag{6}$$

where $P := \text{diag}(p_l)$ is a diagonal matrix. The goal of this paper is to study the existence and uniqueness properties of network equilibrium specified by (3)–(6). Let E be the equilibrium set:

$$E \;=\; \{p \in \Re_+^L \,|\; P(y(p) - c) = 0, \; y(p) \leq c\} \tag{7}$$

For future use, we now define an active constraint set and the Jacobian for links that are actively constrained. Fix an equilibrium price $p^* \in E$. Let the *active constraint set* $\hat{L} = \hat{L}(p^*) \subseteq L$ (with respect to p^*) be the set of links l at which $p_l^* > 0$. Consider the reduced system that consists only of links in $\hat{L}$, and denote all variables in the reduced system by $\hat{c}$, $\hat{p}$, $\hat{y}$, etc. Then, since $y_l(p) = c_l$ for every $l \in \hat{L}$, we have $\hat{y}(\hat{p}) = \hat{c}$. Let the Jacobian for the reduced system be $\boldsymbol{\hat{J}}(\hat{p}) = \partial \hat{y}(\hat{p}) / \partial \hat{p}$. Then

$$\boldsymbol{\hat{J}}(\hat{p}) \;=\; \sum_j \hat{R}^j \frac{\partial x^j}{\partial \hat{q}^j}(\hat{p}) \left(\hat{R}^j\right)^T \frac{\partial \hat{m}^j}{\partial \hat{p}}(\hat{p}) \tag{8}$$

where

$$\frac{\partial x^j}{\partial \hat{q}^j} \;=\; \text{diag}\left(\left(\frac{\partial^2 U_i^j}{\partial (x_i^j)^2}\right)^{-1}\right) \tag{9}$$

$$\frac{\partial \hat{m}^j}{\partial \hat{p}} \;=\; \text{diag}\left(\frac{\partial \hat{m}_l^j}{\partial \hat{p}_l}\right) \tag{10}$$

and all the partial derivatives are evaluated at the generic point $\hat{p}$.

C. Current theory: $J = 1$

In this subsection, we briefly review the current theory for the case where there is only one protocol, i.e., $J = 1$, and explain why it cannot be directly applied to the case of heterogeneous protocols.

When all sources react to the same price, then the equilibrium described by (3)–(6) is the unique solution of the following utility maximization problem defined in [10] and its Lagrange dual [14]:

$$\max_{x \geq 0} \qquad \sum_i U_i(x_i) \tag{11}$$

$$\text{subject to} \qquad Rx \;\leq\; c \tag{12}$$

[2] One can also take the price p_l^j used by one of the protocols, e.g. queueing delay, as the common price p_l. In this case the corresponding price mapping function is the identity function, $m_l^j(p_l) = p_l$.

[3] Hence we can effectively modify user utility functions and influence rate allocations through the choice of price mapping functions m_l^j. In particular, linear link-independent m_l^j scale user utility functions linearly; see Theorem 13.

where we have omitted the superscript $j = 1$. The strict concavity of U_i guarantees the existence and uniqueness of the optimal solution of (11)–(12). The basic idea to relate the utility maximization problem (11)–(12) to the equilibrium equations (3)–(6) is to examine the dual of the utility maximization problem, and interpret the effective price $m_l(p_l)$ as a Lagrange multiplier associated with each link capacity constraint (see, e.g., [13], [14], [22]). As long as $m_l(p_l) \geq 0$ and $m_l(0) = 0$, one can replace p_l in (6) by $m_l(p_l)$. The resulting equation together with (3)–(5) provides the necessary and sufficient condition for $x_i(p)$ and $m_l(p_l)$ to be primal and dual optimal respectively.

This approach breaks down when there are $J > 1$ types of prices because there cannot be more than one Lagrange multiplier at each link. In general, an equilibrium no longer maximizes aggregate utility, nor is it unique. However, as shown in the next section, existence of equilibrium is still guaranteed under the following assumptions:

A1: Utility functions U_i^j are strictly concave increasing, and twice continuously differentiable in their domains. Price mapping functions m_l^j are continuously differentiable in their domains and strictly increasing with $m_l^j(0) = 0$.

A2: For any $\epsilon > 0$, there exists a number $p_{\max}$ such that if $p_l > p_{\max}$ for link l, then

$$x_i^j(p) < \epsilon \text{ for all } (j, i) \text{ with } R_{li}^j = 1$$

These are mild assumptions. Concavity and monotonicity of utility functions are often assumed in network pricing for elastic traffic. Moreover, most TCP algorithms proposed or deployed turn out to have strictly concave increasing utility functions; see e.g. [13]. The assumption on m_l^j preserves the relative order of prices and maps zero price to zero effective price. Assumption A2 says that when p_l is high enough, then every source going through link l has a rate less than ϵ.

III. Existence of equilibrium

In this section, we prove the existence of network equilibrium. We start with a lemma that bounds the equilibrium prices.

Lemma 1. *Suppose A1 and A2 hold. Given a network (c, m, R, U), there is a scalar $p_{\max}$ that upper bounds any equilibrium price p, i.e., $p_l \leq p_{\max}$ for all l.*

Proof. Choose $\epsilon = \min_l c_l/N$, and let $p_{\max}$ be the corresponding scalar in A2. Suppose that there exists an equilibrium price p and a link l, such that $p_l > p_{\max}$. A2 implies that the aggregate equilibrium rate at link l satisfies

$$\sum_j \sum_i R_{li}^j x_i^j(p) < N\epsilon = \min_l c_l$$

Therefore, we get a link with $p_l > 0$ but not fully utilized. It contradicts the equilibrium condition (6). □

The following theorem asserts the existence of equilibrium for a multi-protocol network.

Theorem 2. *Suppose A1 and A2 hold. There exists an equilibrium price p^* for any network (c, m, R, U).*

Proof. Let $p_{\max}$ be the scalar upper bound in Lemma 1. For any $p \in [0, p_{\max}]^L$, define a vector function

$$F(p) := Rx(p) - c \tag{13}$$

For any link l, let

$$p_{-l} \ := \ (p_1, ..., p_{l-1}, p_{l+1} ... p_L)^T$$

Then we may write $F(p)$ as $F(p_l, p_{-l})$. Define function h_l as

$$h_l(p_l, p_{-l}) \ := \ -F_l^2(p_l, p_{-l}) \tag{14}$$

We claim that $h_l(p_l, p_{-l})$ is a quasi-concave function in p_l for any fixed p_{-l}. By the definition of quasi-concavity in [23], we only need to check that the set

$$A_l \ := \ \{ \ p_l \mid h_l(p_l, p_{-l}) \geq a \ \}$$

is convex for all $a \in \Re$. If $a > 0$, clearly $A_l = \emptyset$ by (14). When $a \leq 0$, the set A_l can be rewritten as

$$A_l \ = \ \left\{ \ p_l \ \middle| \ -\sqrt{|a|} \leq F_l(p_l, p_{-l}) \leq \sqrt{|a|} \right\}$$

Since $F_l(p_l, p_{-l})$ is a non-increasing function in p_l for any fixed p_{-l}, the set A_l is convex. Therefore $h_l(p_l, p_{-l})$ is quasi-concave in p_l.

Since $[0, p_{\max}]$ is a nonempty compact convex set, by the theorem of Nash [23], the quasi-concavity of $h_l(p_l, p_{-l})$ guarantees that there exists a $p^* \in [0, p_{\max}]^L$ such that for all $l \in \{1, 2 ... L\}$

$$p_l^* \ = \ \arg \max_{p_l \in [0, p_{\max}]} h_l(p_l, p_{-l}^*)$$

We now argue that, for all l, either 1) $F_l(p^*) = 0$, or 2) $F_l(p^*) < 0$ and we can take $p_l^* = 0$. These conditions imply (6), and hence p^* is an equilibrium price.

Case 1: $F_l(0, p_{-l}^*) > 0$. Since U_i^j is strictly concave, $F_l(p_l, p_{-l}^*)$ is non-increasing[4] in $[0, p_{\max}]$. Moreover, the proof of Lemma 1 shows that $F_l(p_{\max}, p_{-l}^*) < 0$. Therefore, there exists a point p_l^* in $[0, p_{\max}]$ where $F_l(p_l, p_{-l}^*) = 0$. This p_l^* maximizes $h_l(p_l, p_{-l}^*)$.

Case 2: $F_l(0, p_{-l}^*) \leq 0$. Since $F_l(p_l, p_{-l}^*)$ is a non-increasing function in p_l, we have that

$$F_l(p_l, p_{-l}^*) \leq 0 \text{ for all } p_l \in [0, p_{\max}]$$

If $-c_l < F_l(0, p_{-l}^*) \leq 0$, then $F_l(p_l, p_{-l}^*)$ and $h_l(p_l, p_{-l}^*)$ are strictly decreasing in p_l and hence

$$p_l^* = \arg \max_{p_l \in [0, p_{\max}]} h_l(p_l, p_{-l}^*) = 0$$

Otherwise we have $F_l(0, p_{-l}^*) = -c_l$ from (13). In this situation, *all* x_i^j going through link l are zero, and hence we can set $p_l^* = 0$ without affecting any other prices. More precisely, a (possibly) new price vector $\tilde{p}$ with $\tilde{p}_l = 0$ and $\tilde{p}_k = p_k^*$ for $k \neq l$ is also a Nash equilibrium that maximizes $h_k(p_k, \tilde{p}_{-k})$ for $k = 1, \ldots, L$.

Thus we have proved that, for $l = 1, \ldots, L$,

$$p_l^* F_l(p_l^*, p_{-l}^*) \ = \ 0, \quad F_l(p_l^*, p_{-l}^*) \ \leq \ 0, \quad p^* \geq 0$$

which is (6). □

[4] $F_l(p_l, p_{-l}^*)$ is strictly decreasing unless some $x_i(p)$ becomes zero.

IV. Regular networks

Theorem 2 guarantees the existence of network equilibrium. We now study its uniqueness properties.

A. Multiple equilibria: examples

In a single-protocol network, if the routing matrix R has full row rank, then there is a unique active constraint set $\hat{L}$ and a unique equilibrium price p associated with it. If R does not have full row rank, then equilibrium prices p may be non-unique but the equilibrium rates $x(p)$ are still unique since the utility functions are strictly concave.

In contrast, there can be multiple equilibrium prices associated with the same active constraint set (Example 1). Moreover, the active constraint set in a multi-protocol network can be nonunique even if R has full row rank (Example 2). Clearly, the equilibrium prices associated with different active constraint sets are different.

Example 1: unique active constraint set but uncountably many equilibria

In this example, we assume all the sources use the same utility function

$$U_i^j(x_i^j) \quad = \quad -\frac{1}{2}\left(1 - x_i^j\right)^2 \qquad (15)$$

Then the equilibrium rates x^j of type j sources are determined by the equilibrium prices p as

$$x^j(p) \quad = \quad \mathbf{1} - (R^j)^T m^j(p)$$

where $\mathbf{1}$ is a vector of appropriate dimension whose entries are all 1s. We use linear price mapping functions:

$$m^j(p) \quad = \quad K^j p$$

where K^j are $L \times L$ diagonal matrices. Then the equilibrium rate vector of type j sources can be expressed as

$$x^j(p) \quad = \quad \mathbf{1} - (R^j)^T K^j p$$

When only links with strictly positive equilibrium prices are included in the model, we have

$$y(p) \ = \ \sum_{j=1}^{J} R^j x^j(p) \quad = \quad c$$

Substituting in $x^j(p)$ yields

$$\sum_{j=1}^{J} R^j (R^j)^T K^j p \quad = \quad \sum_{j=1}^{J} R^j \mathbf{1} - c$$

which is a linear equation in p for given R^j, K^j, and c. It has a unique solution if the determinant is nonzero, but has no or multiple solutions if

$$\det\left(\sum_{j=1}^{J} R^j (R^j)^T K^j\right) \quad = \quad 0$$

When $J = 1$, i.e., there is only one protocol, and R^1 has full row rank, $\det(R^1(R^1)^T K^1) > 0$ since both $R^1(R^1)^T$ and K^1 are positive definite. In this case, there is a unique equilibrium price vector. When $J = 2$, there are networks whose determinants are zero that have uncountably many equilibria. See [29] for an example where R does not have full row rank. We provide here an example with $J = 3$ where R still has full row rank.

The network is shown in Figure 1 with three unit-capacity links, $c_l = 1$. There are three different protocols with the

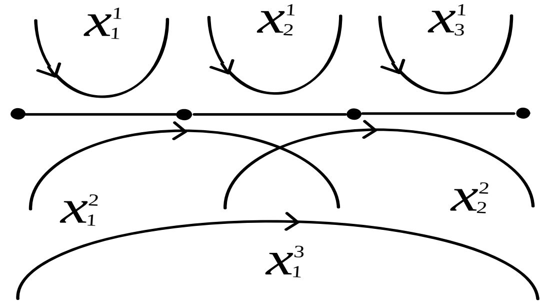

Fig. 1. Example 1: uncountably many equilibria.

corresponding routing matrices

$$R^1 = I, \qquad R^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^T, \qquad R^3 = (1,1,1)^T$$

The linear mapping functions are given by

$$K^1 = I, \quad K^2 = \text{diag}(5,1,5), \quad K^3 = \text{diag}(1,3,1)$$

It is easy to calculate that

$$\sum_{i=1}^{3} R^i (R^i)^T K^i \quad = \quad \begin{bmatrix} 7 & 4 & 1 \\ 6 & 6 & 6 \\ 1 & 4 & 7 \end{bmatrix}$$

which has determinant 0. Using the utility function defined in (15), we can check that the following are equilibrium prices for all $\epsilon \in [0, 1/24]$:

$$p_1^1 = p_3^1 = 1/8 + \epsilon \qquad p_2^1 = 1/4 - 2\epsilon$$

The corresponding rates are

$$x_1^1 = x_3^1 = 7/8 - \epsilon \qquad x_2^1 = 3/4 + 2\epsilon$$
$$x_1^2 = x_2^2 = 1/8 - 3\epsilon \qquad x_1^3 = 4\epsilon$$

All capacity constraints are tight with these rates. Since there is a one-link flow at every link, the active constraint set is unique and contains every link. Yet there are uncountably many equilibria.

Example 2: multiple active constraint sets each with a unique equilibrium

Consider the symmetric network in Figure 2 with 3 flows. There are two protocols in the network with the following routing matrices

$$R^1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad R^2 = (1,1,1)^T$$

Flows $(1,1)$ and $(1,2)$ have identical utility function U^1 and source rate x^1, and flow $(2,1)$ has a utility function U^2 and source rate x^2.

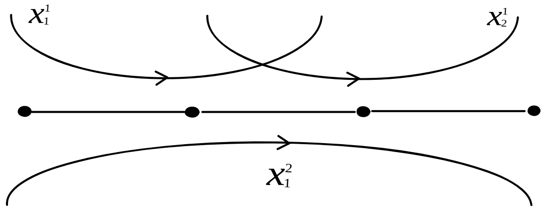

Fig. 2. Example 2: two active constraint sets.

Links 1 and 3 both have capacity c_1 and price mapping functions $m_1^1(p) = p$ and $m_1^2(p)$ for protocols 1 and 2 respectively. Link 2 has capacity c_2 and price mapping functions $m_2^1(p) = p$ and $m_2^2(p)$.

In [31], we prove that when assumption A1 holds, the network shown in Figure 2 has at least two equilibria provided:

1) $c_1 < c_2 < 2c_1$;
2) for $j = 1, 2$, $(U^j)'(x^j) \to \overline{p}^j$, possibly ∞, if and only if $x^j \to 0$.
3) for $l = 1, 2$, $m_l^2(p_l) \to \overline{p}^2$ as $p_l \to \overline{p}^1$, and satisfy

$$2m_1^2((U^1)'(c_2 - c_1)) \; < \; (U^2)'(2c_1 - c_2) \\ < \; m_2^2((U^1)'(c_2 - c_1))$$

By manipulating buffer sizes and RED parameters, i.e., carefully designing the price mapping functions m_l^j, we have demonstrated experimentally in [31] the phenomenon of multiple equilibria for this example using TCP Reno, which reacts to loss probability, and TCP Vegas/FAST, which react to delay. [5]

B. Regular networks

Examples 1 and 2 show that global uniqueness is generally not guaranteed in a multi-protocol network. We now show, however, that local uniqueness is basically a generic property of the equilibrium set. We present our main results on the structure of the equilibrium set here, providing conditions for the equilibrium points to be locally unique, finite and odd in number, and globally unique. We prove these results in the next subsection.

Consider an equilibrium price $p^* \in E$. Recall the active constraint set $\hat{L}$ defined by p^*. The equilibrium price $\hat{p}^*$ for the links in $\hat{L}$ is a solution of

$$\hat{y}(\hat{p}) \quad = \quad \hat{c} \qquad (16)$$

By the inverse function theorem, the solution of (16), and hence the equilibrium price $\hat{p}^*$, is *locally unique* if the Jacobian matrix $\hat{\boldsymbol{J}}(\hat{p}^*) = \partial \hat{y} / \partial \hat{p}$ is nonsingular at $\hat{p}^*$. We call a network (c, m, R, U) *regular* if all its equilibrium prices are locally unique.

The next result shows that almost all networks are regular, and that regular networks have finitely many equilibrium prices. This justifies restricting our attention to regular networks.

[5]It is pointed out in [26] that there is actually a third equilibrium for this network where all links are actively constrained. However, unlike the other two equilibria, the third is not locally stable and hence did not manifest itself in the experiments reported in [31].

Theorem 3. *Suppose assumptions A1 and A2 hold. Given any price mapping functions m, any routing matrix R and utility functions U,*

1) *the set of link capacities c for which not all equilibrium prices are locally unique has Lebesgue measure zero in $\Re_+^L$.*
2) *the number of equilibria for a regular network (c, m, R, U) is finite.*

For the rest of this subsection, we narrow our attention to networks that satisfy an additional assumption:

A3: Every link l has a single-link flow (j, i) with $\left(U_i^j\right)'(c_l) > 0$.

Assumption A3 says that when the price of link l is small enough, the aggregate rate through it will exceed its capacity. This ensures that the active constraint set contains all links and facilitates the application of Poincare-Hopf theorem by avoiding equilibrium on the boundary (some $p_l = 0$). [6]

Since all the equilibria of a regular network have nonsingular Jacobian matrices, we can define the *index* $I(p)$ of $p \in E$ as

$$I(p) \quad = \quad \begin{cases} 1 & \text{if } \det(\boldsymbol{J}(p)) > 0 \\ -1 & \text{if } \det(\boldsymbol{J}(p)) < 0 \end{cases}$$

Then, we have

Theorem 4. *Suppose assumptions A1–A3 hold. Given any regular network, we have*

$$\sum_{p \in E} I(p) = (-1)^L$$

where L is the number of links.

We give two important consequences of this theorem.

Corollary 5. *Suppose assumptions A1–A3 hold. A regular network has an odd number of equilibria.*

Notice that Corollary 5 implies the existence of equilibrium. Although we proved this in Section III in a more general setting, this simple corollary shows the power of Theorem 4.

The next result provides a condition for global uniqueness. We say that an equilibrium $p^* \in E$ is *locally stable* if the corresponding Jacobian matrix $\boldsymbol{J}(p^*)$ defined in (8) is stable, that is, every eigenvalue of $\boldsymbol{J}(p^*) = \partial y(p^*)/\partial p$ has negative real part. For justification of this definition, local stability of p^* implies that the gradient algorithm (19) below converges locally.

[6]It is recently shown in [26] that A3 is not necessary and one can generalize Theorem 4 to

$$\sum_{p \in E} (-1)^{\hat{L}(p)} I(p) = 1$$

where $\hat{L}(p)$ is the number of links of the active constraint set associated with equilibrium p. Clearly, if $\hat{L}(p) = L$, it reduces to Theorem 4. This generalized theorem also allows [26] to conclude the number of equilibria is odd (and therefore existence) without A3. In this paper, although A3 is imposed, all results can be viewed with respect to a fixed active constraint set with appropriate modifications. In particular, the global uniqueness results in Section V directly apply without A3 since $\hat{\boldsymbol{J}}$ has a similar structure as $\boldsymbol{J}$ except with a smaller dimension.

Corollary 6. *Suppose assumptions A1–A3 hold. The equilibrium of a regular network is globally unique if and only if every equilibrium point in E has an index $(-1)^L$. In particular, if all equilibria are locally stable, then E contains exactly one point.*

This result may seem surprising at first sight as it relates the local stability of an algorithm to the uniqueness property of a network. This is because both equilibrium and local stability are defined in terms of the function $y(p)$: an equilibrium p^* satisfies $y(p^*) = c$ and the local asymptotic stability of p^* is determined by $\partial y(p^*)/\partial p$. The connection between these two properties is made exact by the index theorem. An implication of this result is that if there are multiple equilibria, then no algorithm $\dot{p} = f(p(t))$, whose linearization around each equilibrium $p^* \in E$ satisfies $\partial f(p^*)/\partial p = \partial y(p^*)/\partial p$, can be found to locally stabilize all of the equilibria.

Corollary 6 will be used in Section V to derive a sufficient condition on price mapping functions m for global uniqueness. We close this subsection with an example that illustrates the application of Theorem 4 and Corollary 5.

Example 3: illustration of Theorem 4 and Corollary 5, 6

We revisit Example 1 with modified utility functions. Recall that in Example 1, as ϵ varies from 0 to $1/24$, we trace out all equilibrium points. The components x_1^1 and $q_1^1 = p_1^1$ of these equilibrium points are shown by the (red) solid line in Figure 3. Other sources x_i^j and their effective end-to-end prices q_i^j also lie on similar straight lines. Since the network has uncountably many equilibrium points, it is not regular. To make it regular, suppose we change the utility functions of sources (j, i) to

$$U_i^j(x_i^j, \alpha_i^j) = \begin{cases} \beta_i^j (x_i^j)^{1-\alpha_i^j}/(1-\alpha_i^j) & \text{if } \alpha_i^j \neq 1 \\ \beta_i^j \log x_i^j & \text{if } \alpha_i^j = 1 \end{cases}$$

with appropriately chosen positive constants α_i^j and β_i^j. These utility functions can be viewed as a weighted version of the α-fairness utility functions proposed in [22].

The basic idea of how to choose α_i^j and β_i^j to generate only finitely many equilibrium points is as follows. First, we pick two points in the equilibrium set of Example 1, say, the points associated with $\epsilon = 0.01$ and $\epsilon = 0.04$. These choices of ϵ provide two distinct equilibrium points (q, x) and $(\tilde{q}, \tilde{x})$. For instance, $(q_1^1, x_1^1) = (0.135, 0.865)$ corresponds to $\epsilon = 0.01$ and $(\tilde{q}_1^1, \tilde{x}_1^1) = (0.165, 0.835)$ corresponds to $\epsilon = 0.04$, as illustrated in Figure 3. Then, for each source (j, i), find α_i^j and β_i^j such that (4) is satisfied by the two equilibrium points (q_i^j, x_i^j) and $(\tilde{q}_i^j, \tilde{x}_i^j)$ with the new utility functions. This is illustrated in Figure 3 where relation (4) with the new utility function is represented by the (blue) curve, and α_i^j, β_i^j are chosen so that the curve passes through the original equilibrium points (x_1^1, q_1^1) and $(\tilde{q}, \tilde{x})$. More specifically, given two equilibrium points (q_i^j, x_i^j) and $(\tilde{q}_i^j, \tilde{x}_i^j)$, choose

$$\alpha_i^j = \frac{\log(q_i^j) - \log(\tilde{q}_i^j)}{\log(\tilde{x}_i^j) - \log(x_i^j)} \qquad \beta_i^j = q_i^j \left(x_i^j\right)^{\alpha_i^j}$$

The resulting α_i^j and β_i^j for all flows (j, i) are shown in Table I.

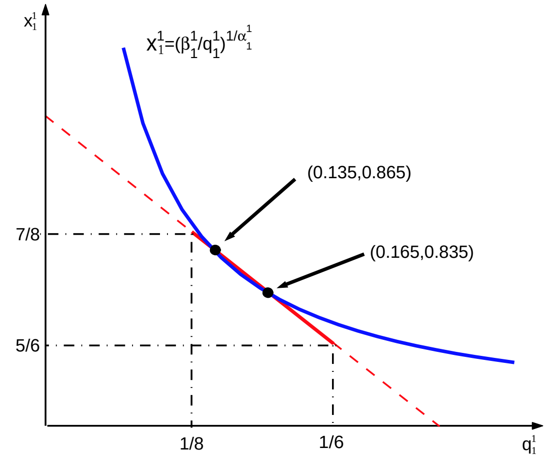

Fig. 3. Example 3: construction of multiple isolated equilibria.

TABLE I
EXAMPLE 3: α_i^j AND β_i^j.

Flows	α_i^j	β_i^j
x_1^1	5.6851	0.0592
x_2^1	4.0285	0.0803
x_3^1	5.6851	0.0592
x_1^2	0.0322	0.8389
x_2^2	0.0322	0.8389
x_1^3	0.0963	0.7041

By construction, both $(p_1^1 = 0.135, p_2^1 = 0.230)$ and $(p_1^1 = 0.165, p_2^1 = 0.170)$ are network equilibria. By Corollary 5, there is at least one additional equilibrium. Numerical search indeed located a third equilibrium with $(p_1^1 = 0.142,\ p_2^1 = 0.206)$.

We further check the local stability of these three equilibria under the gradient algorithm (19) to be introduced in Section IV-C. The eigenvalues and index for each equilibrium are shown in Table II. It turns out that the equilibrium $(p_1^1 = 0.142,\ p_2^1 = 0.206)$ is not stable and has index 1, while the other two are stable with index -1. The dynamics of this network under the gradient algorithm can be illustrated by a vector field. By symmetry, the equilibrium prices for the first and third link are always same. Therefore, we can draw the vector field restricted on the plane $p_1 = p_3$ to illustrate the system dynamics. The phase portrait is shown in Figure 4. The (red) dots represent the three equilibria. Note the equilibrium in the middle is a saddle point, and therefore unstable. The (red) arrows give the direction of this vector field. Individual trajectories are plotted with slim (blue) lines.

TABLE II
EXAMPLE 3: STABILITY AND INDICES OF EQUILIBRIA.

Equilibria (p_1, p_2, p_3)	Eigenvalues	Index
$(0.135, 0.23, 0.135)$	$-0.21, -17.43, -26.73$	-1
$(0.142, 0.206, 0.142)$	$0.21, -12.32, -22.40$	1
$(0.165, 0.17, 0.165)$	$-12.41, -1.67, -0.67$	-1

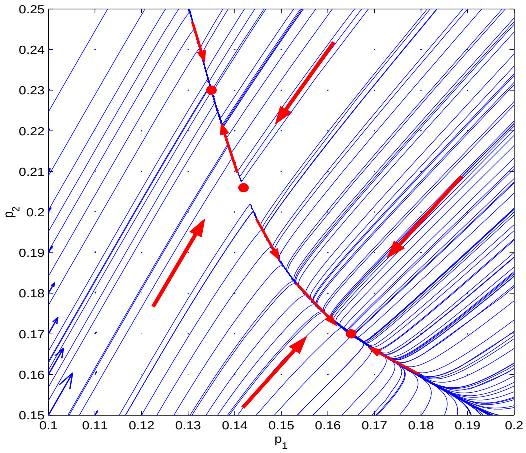

Fig. 4. Example 3: vector field of (p_1, p_2).

C. Proofs

In this subsection we provide proofs for the results in Section IV-B.

Proof of Theorem 3. The main mathematical tool used in our proof is Sard's Theorem [4], [27], of which we quote a version here that is tailored to our problem. Let G be an open subset of $\Re^L_+$ and let F be a continuously differentiable function from G to $\Re^L_+$. A point $y \in G$ is a *critical point of* F if the Jacobian matrix $\partial F/\partial y$ of F at y is singular. A point $z \in \Re^L_+$ is a *critical value of* F if there is a critical point $y \in G$ with $z = F(y)$. A point in $\Re^L_+$ is a *regular value of* F if it is not a critical value.

Sard's theorem. If $F : G \to \Re^L_+$ is continuously differentiable on the open subset $G \subseteq \Re^L_+$, then the set of critical values of F has Lebesgue measure zero in $\Re^L_+$.

Fix a routing matrix R and utility functions U. There are at most $2^L - 1$ different active constraint sets. Let $\hat{L} \subseteq L$ be such a combination with $\hat{L}$ links. Consider the set of all possible link capacities $c = (c_l, l \in L)$ under which the active constraint set is $\hat{L}$, i.e., with such a capacity vector c, an equilibrium price p has $p_l > 0$ if $l \in \hat{L}$ and $p_l = 0$ otherwise. Fix such an equilibrium point p^*. Again let $\hat{p}$ denote the price vector only for links in $\hat{L}$. Then $\hat{p}^*$ is not locally unique if the function $\hat{y} : \Re^{\hat{L}}_+ \to \Re^{\hat{L}}_+$ defined by $\hat{y}(\hat{p}) = \hat{R}x(\hat{p})$ has a singular Jacobian matrix $\partial\hat{y}/\partial\hat{p}$ at $\hat{p}^*$, i.e., if $\hat{p}^*$ is a critical point of $\hat{y}$. The set of such capacity vectors $\hat{c} \in \Re^{\hat{L}}_+$ under which all links in $\hat{L}$ have active constraints in equilibrium satisfy

$$\hat{y}(\hat{p}^*) = \hat{c}$$

and hence are critical values of $\hat{y}$. Since $\hat{y}$ is continuously differentiable by assumption A1, we can apply Sard's theorem and conclude that the set of such capacity vectors $\hat{c}$ has zero Lebesgue measure in $\Re^{\hat{L}}_+$. The extension to $\Re^L_+$ for all link capacities clearly also has zero Lebesgue measure in $\Re^L_+$.

Since we only have a finite number of different active constraint sets, the union of link capacity vectors that give rise to locally nonunique equilibria still has zero Lebesgue measure. This proves the first part of the theorem.

The equilibrium set E defined in (7) is closed because $y(p)$ is continuous, and is bounded by Lemma 1. Hence E is compact. Since (c, m, R, U) is a regular network, every $p \in E$ is locally unique, i.e., for each $p \in E$ we can find an open neighborhood such that it is the only equilibrium in that open set. The union of these open sets forms a cover for set E. Since E is compact, it admits a finite subcover [16], i.e., E can be covered by a finite number of open sets each containing a single equilibrium. Hence, the number of equilibria is finite. □

Proof of Theorem 4. By assumption A3, we can always find $p_{\min} > 0$ such that for any price p and link l with $p_l < p_{\min}$, we have

$$\sum_j \sum_i R^j_{li} x^j_i(p) > c_l$$

Let $G := [p_{\min}, p_{\max}]^L$ where $p_{\max}$ is defined in Lemma 1. Clearly, all equilibria are in the set G. To prove our result, we will invoke a version of the Poincare-Hopf Index Theorem tailored to our problem [20], [33].

Poincare-Hopf index Theorem. Let D be an open subset of $\Re$ and $v : D^L \to \Re^L$ be a smooth vector field, with nonsingular Jacobian matrix $\partial v/\partial p$ at every equilibrium. If there is a $G \subseteq D^L$ such that every trajectory moves inward of region G, then the sum of the indices of the equilibria in G is $(-1)^L$.

Gradient project algorithm. To construct the vector field v required by the index theorem, let $D^L = G$ and consider the following gradient algorithm from G to G proposed in [14]. The prices are updated at time t according to

$$\dot{p}(t) \;=\; \Lambda\left(Rx(t) - c\right) \tag{17}$$

where $\Lambda > 0$ is an $L \times L$ diagonal matrix whose elements represent stepsizes. A source updates its rate based on the end-to-end price

$$x(t) \;=\; x(p(t)) \tag{18}$$

A consequence of assumption A3 is that $p(t) \geq p_{\min} > 0$ for all t under the gradient algorithm (17)–(18). This guarantees a unique active constraint set that is L. Hence the equilibrium set E defined in (7) is equivalent to $E = \{p \in \Re^L_+ \mid y(p) - c = 0\}$.

Combining (17)–(18) with $y(p(t)) = Rx(t)$ yields the required vector field v:

$$\dot{p}(t) \;=\; \Lambda(y(p(t)) - c) \;=:\; v(p(t)) \tag{19}$$

whose Jacobian matrix is:

$$\frac{\partial v}{\partial p}(p) \;=\; \Lambda \boldsymbol{J}(p) \;=\; \Lambda \frac{\partial y}{\partial p}(p) \tag{20}$$

where $\boldsymbol{J}(p)$ is given by (8). Clearly, p^* is an equilibrium point of v, i.e., $v(p^*) = 0$, if and only if p^* is a network equilibrium, i.e., $p^* \in E$. Since the network (c, m, R, U) is regular, $\boldsymbol{J}(p)$ is nonsingular at every network equilibrium $p^* \in E \subset G$. Since Λ is a positive diagonal matrix, $\partial v(p)/\partial p$ is also nonsingular by (20) at all its equilibrium points p in G, as the index theorem requires.

Consider any point p on the boundary of G. For any l, we have one of two cases:

1) If $p_l(t) = p_{\max}$, link l will be underutilized, $y_l(p(t)) < c_l$, and $\dot{p}_l < 0$ according to (19).
2) If $p_l(t) = p_{\min}$, the aggregate rate at link l will exceed c_l, $y_l(p(t)) > c_l$, and $\dot{p}_l > 0$ according to (19).

Therefore, every point p on the boundary of G will move inward and our result directly follows from the Poincare-Hopf index theorem. □

Proof of Corollary 5. Since both $I(p)$ and $(-1)^L$ are odd, the number of terms in the summation in Theorem 4 must be odd. □

Proof of Corollary 6. The first claim of the theorem directly follows from Theorem 4. We now claim that an equilibrium $p^* \in E$ which is locally stable has an index $I(p^*)$ of $(-1)^L$. To prove the claim, consider a locally stable equilibrium price p^*. All the eigenvalues of $\boldsymbol{J}(p^*)$ have negative real parts. Moreover, since $\boldsymbol{J}(p^*)$ has real entries, complex eigenvalues come in conjugate pairs. The determinant of $\boldsymbol{J}(p^*)$ is the product of all its eigenvalues. If there are k conjugate pairs of complex eigenvalues and $L-2k$ real eigenvalues, the product of all eigenvalues has the same sign as $(-1)^{L-2k}$ which has the same sign as $(-1)^L$. Hence the index of a locally stable equilibrium is $(-1)^L$. □

V. Global uniqueness: mapping functions $m(p)$

In this and the next sections, we provide sufficient conditions on the structure of the network for global uniqueness. We also provide some important special cases in Appendix VIII-A where global uniqueness is set up. In this section, we reveal that, under assumptions A1–A3, if the price mapping functions m_l^j are similar, then the equilibrium of a regular network is globally unique.

A. General result

To state the result concisely, we need the notion of permutation. We call a vector $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_L)$ a *permutation* if each σ_l is distinct and takes value in $\{1, \ldots, L\}$. Treating $\boldsymbol{\sigma}$ as a mapping $\boldsymbol{\sigma} : \{1, \ldots, L\} \to \{1, \ldots, L\}$, we let $\boldsymbol{\sigma}^{-1}$ denote its unique inverse permutation. For any vector $a \in \Re^L$, $\boldsymbol{\sigma}(a)$ denotes the permutation of a under $\boldsymbol{\sigma}$, i.e., $[\boldsymbol{\sigma}(a)]_l = a_{\sigma_l}$. If $a \in \{1, \ldots, L\}^L$ is a permutation, then $\boldsymbol{\sigma}(a)$ is also a permutation and we often write $\boldsymbol{\sigma} a$ instead. Let $\boldsymbol{l} = (1, \ldots, L)$ denote the identity permutation. Then $\boldsymbol{\sigma l} = \boldsymbol{\sigma}$. See [19] for more details. Finally, denote dm_l^j/dp_l by $\dot{m}_l^j$.

Theorem 7. *Suppose assumptions A1–A3 hold. If, for any vector $\boldsymbol{j} \in \{1, \ldots, J\}^L$ and any permutations $\boldsymbol{\sigma}, \boldsymbol{k}, \boldsymbol{n}$ in $\{1, \ldots, L\}^L$,*

$$\prod_{l=1}^{L} \dot{m}_l^{[\boldsymbol{k}(\boldsymbol{j})]_l} \;+\; \prod_{l=1}^{L} \dot{m}_l^{[\boldsymbol{n}(\boldsymbol{j})]_l} \;\;\geq\;\; \prod_{l=1}^{L} \dot{m}_l^{[\boldsymbol{\sigma}(\boldsymbol{j})]_l} \qquad (21)$$

then the equilibrium of a regular network is globally unique.

Proof. See Appendix VIII-B.

Theorem 7 implies that if the (slopes of the) price mapping functions are "similar", then global uniqueness is guaranteed, as the following corollary shows: if m_l^j do not differ much across source types at each link, or they do not differ much along links in every source's path, the equilibrium is unique.

Corollary 8. *Suppose assumptions A1–A3 hold. The equilibrium of a regular network is globally unique if any one of the following conditions holds:*

1) *For each $l = 1, \ldots, L$, $j = 1, \ldots, J$*

$$\dot{m}_l^j \;\in\; \left[a_l, 2^{\frac{1}{L}} a_l\right] \qquad \text{for some } a_l > 0 \qquad (22)$$

2) *For each $j = 1, \ldots, J$, $l = 1, \ldots, L$*

$$\dot{m}_l^j \;\in\; \left[a^j, 2^{\frac{1}{L}} a^j\right] \qquad \text{for some } a^j > 0 \qquad (23)$$

Proof. If (22) holds, we have for any $j_l, \hat{j}_l, \tilde{j}_l$ in $\{1, \ldots, J\}$

$$\prod_{l=1}^{L} \dot{m}_l^{j_l} + \prod_{l=1}^{L} \dot{m}_l^{\hat{j}_l} \;\;\geq\;\; 2\prod_{l=1}^{L} a_l = \prod_{l=1}^{L} 2^{\frac{1}{L}} a_l \;\;\geq\;\; \prod_{l=1}^{L} \dot{m}_l^{\tilde{j}_l}$$

which implies the sufficient condition in Theorem 7.

For the second assertion, fix any $\boldsymbol{j}$ in $\{1, \ldots, L\}^L$ and any permutations $\boldsymbol{\sigma}, \boldsymbol{k}, \boldsymbol{n}$ in $\{1, \ldots, L\}^L$. If (23) holds, we have

$$\begin{aligned}\prod_{l=1}^{L} \dot{m}_l^{[\boldsymbol{k}(\boldsymbol{j})]_l} \;+\; \prod_{l=1}^{L} \dot{m}_l^{[\boldsymbol{n}(\boldsymbol{j})]_l} \;\;&\geq\;\; 2\prod_{l=1}^{L} a^{j_l} = \prod_{l=1}^{L} 2^{\frac{1}{L}} a^{j_l} \\ &\geq\;\; \prod_{l=1}^{L} \dot{m}^{[\boldsymbol{\sigma}(\boldsymbol{j})]_l}\end{aligned}$$

which implies the sufficient condition in Theorem 7. □

Remarks:

1) Asymptotically when $L \to \infty$, both conditions (22) and (23) converge to a single point. Condition (22) reduces to $\dot{m}_l^j = a_l$ which essentially says that all protocols are the same ($J = 1$). Condition (23) reduces to $\dot{m}_l^j = a^j$, which is the linear link independent case discussed in Theorem 13.
2) The sufficient condition in Theorem 7 can be conservative because many r_π^j may be zero (no source of type j takes path π).
3) These link-based uniqueness results hold for a network whenever no flow uses more than L links.

B. Special case: $L = 3$ and $J = 2$

We now specialize our uniqueness result to the case of $L = 3$, $J = 2$. This case is of particular interest as it represents the smallest network that can exhibit non-unique equilibrium points if A1–A3 are satisfied; see Theorem 15 in the Appendix.

Theorem 9. *Suppose assumptions A1–A3 hold for a 3-links regular network with 2 protocols. If the following 6 inequalities hold, the network has a unique equilibrium:*

$$\lambda_2 + \lambda_3 \geq \lambda_1, \quad \lambda_1 + \lambda_3 \geq \lambda_2, \quad \lambda_1 + \lambda_2 \geq \lambda_3$$
$$\frac{1}{\lambda_2} + \frac{1}{\lambda_3} \geq \frac{1}{\lambda_1}, \quad \frac{1}{\lambda_1} + \frac{1}{\lambda_3} \geq \frac{1}{\lambda_2}, \quad \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \geq \frac{1}{\lambda_3}$$

where $\lambda_l := \dot{m}_l^1(p)/\dot{m}_l^2(p)$.

Proof. It is straightforward to check that only the following six $\rho(\boldsymbol{j},\boldsymbol{\pi})$ in (38) can have negative coefficients $D(\boldsymbol{j},\boldsymbol{\pi})$:

$$(\lambda_2+\lambda_3-\lambda_1)\dot{m}_1^2\dot{m}_2^2\dot{m}_3^2 r_{111}^1 r_{101}^2 r_{110}^2$$
$$(\lambda_1+\lambda_3-\lambda_2)\dot{m}_1^2\dot{m}_2^2\dot{m}_3^2 r_{111}^1 r_{011}^2 r_{110}^2$$
$$(\lambda_1+\lambda_2-\lambda_3)\dot{m}_1^2\dot{m}_2^2\dot{m}_3^2 r_{111}^1 r_{011}^2 r_{101}^2$$
$$(\frac{1}{\lambda_2}+\frac{1}{\lambda_3}-\frac{1}{\lambda_1})\dot{m}_1^1\dot{m}_2^1\dot{m}_3^1 r_{111}^2 r_{101}^1 r_{110}^1$$
$$(\frac{1}{\lambda_1}+\frac{1}{\lambda_3}-\frac{1}{\lambda_2})\dot{m}_1^1\dot{m}_2^1\dot{m}_3^1 r_{111}^2 r_{011}^1 r_{110}^1$$
$$(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}-\frac{1}{\lambda_3})\dot{m}_1^1\dot{m}_2^1\dot{m}_3^1 r_{111}^2 r_{011}^1 r_{101}^1$$

The condition in the theorem guarantees that these terms are all nonnegative. By (38), $\det(-\boldsymbol{J}(p)) \geq 0$. Since the network is regular, we have $\det(-\boldsymbol{J}(p)) > 0$ for all equilibria p. Hence the equilibrium is globally unique. □

A straightforward corollary is the following

Corollary 10. *Suppose assumptions A1–A3 hold. For a 3-links regular network with 2 protocols, if, for all l, $\lambda_l \in [a, 2a]$ for some constant $a > 0$, the network admits a globally unique equilibrium.*

Remark: If $\dot{m}_l^j = k^j$ are link independent , then $\lambda_l = k^1/k^2 \in [a, 2a]$ for any $k^1/2k^2 \leq a \leq k^1/k^2$. Hence global uniqueness is guaranteed, which agrees with Theorem 13.

We illustrate in Figures 5 and 6 the regions of λ_l in Theorem 9 and Corollary 10. They are both cones. The first one is the projection to $\lambda_1 - \lambda_2$ plane and the second one is the cross-section cut by plane $\lambda_1+\lambda_2 = 1$. These two figures clearly show the bound on the "degree of heterogeneity" within which global uniqueness hold.

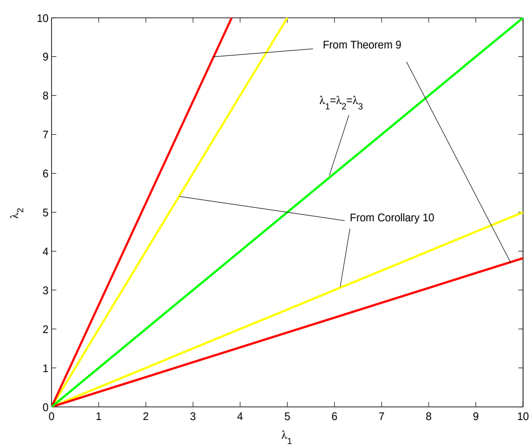

Fig. 5. Region of λ_l for global uniqueness: projection to $\lambda_1 - \lambda_2$ plane.

VI. Global uniqueness: Jacobian $\mathbf{J}(p)$

In a single-protocol network, for the equilibrium price to be unique, it is sufficient that the routing matrix R has full row rank. Otherwise, only the source rates are unique, not necessarily the link prices. In a multi-protocol network, this is no longer sufficient. We now provide another sufficient condition that plays the same role in a multi-protocol network as the rank condition on R does in a single-protocol network (see also the remark after Theorem 12).

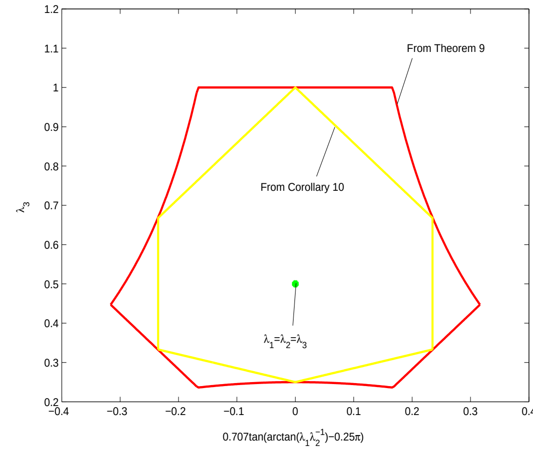

Fig. 6. Region of λ_l for global uniqueness: cross-section cut by plane $\lambda_1 + \lambda_2 = 1$.

Let $f = (f_1, ..., f_n)$ be a vector of real-valued functions defined on $\Re^n$. Let $G := \{z \in \Re^n | f(z) = 0\}$ and co$G$ be its convex hull. Define a set $V(G)$ of vectors as

$$V(G) := \{v | v = \phi - \psi \text{ for } \psi, \phi \in \text{co}G\} \tag{24}$$

as a function of the set G.

Lemma 11. *If for every $z \in coG$, the Jacobian matrix $\boldsymbol{J}(z) = \partial f(z)/\partial z$ exists and $v^T \boldsymbol{J}(z) v < 0$ for all $v \in V(G)$, then G contains at most one point.*

Proof. For the sake of contradiction, assume there are two distinct points ϕ and ψ in G such that $f(\phi) = f(\psi) = 0$. Let

$$g(\theta) := \phi + \theta(\psi - \phi) \text{ where } \theta \in [0, 1]$$

Then

$$\frac{df(g(\theta))}{d\theta} = \boldsymbol{J}(g(\theta))\frac{dg(\theta)}{d\theta} = \boldsymbol{J}(g(\theta))(\psi - \phi)$$

Hence,

$$f(\psi) - f(\phi) = \int_0^1 \boldsymbol{J}(g(\theta))(\psi - \phi) d\theta$$

Multiplying both sides by $(\psi - \phi)^T$ yields

$$(\psi-\phi)^T(f(\psi) - f(\phi)) = \int_0^1 (\psi - \phi)^T \boldsymbol{J}(g(\theta))(\psi - \phi) d\theta$$

The left hand-side of the above equation is 0, and the right-hand side is negative under the assumption of the theorem. This contradiction proves the theorem. □

Let $f = y$, and let $G = E$ be the set of network equilibria. Then Lemma 11, together with Theorem 2, provides a sufficient condition for global uniqueness of network equilibrium.

Theorem 12. *Suppose assumptions A1–A3 hold. If for every price vector $p \in coE$, the Jacobian matrix $\boldsymbol{J}(p)$ defined in (8)* *exists and $v^T \boldsymbol{J}(p) v < 0$ for all $v \in V(E)$, then there exists a globally unique network equilibrium.*

In the single-protocol case, a similar result has been obtained in [22]. However, for that case, the Jacobian matrix is negative definite when R has full row rank. Then the condition in Theorem 12 always holds and the equilibrium is unique. In the multi-protocol case, the Jacobian matrix is in general not symmetric and hence not negative definite. Therefore R having full row rank is no longer sufficient for the condition in the theorem to hold.

Since we do not know the equilibrium set E, the condition in the theorem cannot be directly applied to prove global uniqueness. To use the theorem, however, it is sufficient to find a convex superset $\tilde{E}$ of E and a superset $\tilde{V}$ of $V(E)$ such that $v^T \boldsymbol{J}(p) v < 0$ for all $p \in \tilde{E}$ and $v \in \tilde{V}$. This implies the condition in Theorem 12 and hence global uniqueness. We illustrate this procedure in the next example.

Example 4: application of Theorem 12 to verify global uniqueness

We visit Example 1 for the third time but using $\log$ utility functions for all sources, i.e.,

$$U_i^j(x_i^j) = \log(x_i^j) \quad \text{for all} \quad (j, i) \tag{25}$$

Let the Jacobian matrix be

$$\boldsymbol{J}(p) = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

where $J_{kl} = J_{kl}(p)$ are functions of prices p given by (8). For example

$$J_{11} = -\frac{1}{p_1^2} - \frac{5}{(5p_1 + p_2)^2} - \frac{1}{(p_1 + p_3 + 3p_2)^2}$$

It can be seen that $\boldsymbol{J}(p)$ is not negative definite for general p unlike in the single-protocol case. Even though E can be hard to find, we demonstrate how to find a simple convex superset $\tilde{E}$ of E and a simple superset $\tilde{V}$ of $V(E)$.

Consider the convex set

$$\tilde{E} := \{p \in \Re_+^3 \mid 1 \le p_1 = p_3 \le 2, \ 1 \le p_2 \le 2\}$$

We claim that $E \subseteq \tilde{E}$. To see this, let p be an equilibrium price. If $p_1 < 1$, then $x_1^1 = 1/p_1$ will exceed the link capacity 1, and hence $p_1 \ge 1$. A similar argument gives $p_2 \ge 1$. To see $p_1 \le 2$, assume it is not true. Then

$$\begin{aligned} x_1^1 &= 1/p_1 < 1/2 \\ x_1^2 &= 1/(5p_1 + p_2) < 1/11 \\ x_1^3 &= 1/(2p_1 + 3p_2) < 1/7 \end{aligned}$$

Summing them yields $x_1^1 + x_1^2 + x_1^3 < 1$. Hence the network is not in equilibrium, contradicting that p is an equilibrium price. Hence $p_1 \le 2$. The argument for $p_2 \le 2$ is similar.

Using the definition of $\tilde{E}$, we can bound all $J_{kl}(p)$ for $p \in \tilde{E}$. The results are collected in Table III.

Let

$$\tilde{V} := \{v \in \Re_+^3 \mid v_1 = v_3\}$$

TABLE III

EXAMPLE 4: BOUNDS ON ELEMENTS OF $\boldsymbol{J}(p)$

Elements	Upperbound	Lowerbound
J_{11}	-0.2947	-1.1789
J_{22}	-0.2939	-1.1756
J_{33}	-0.2947	-1.1789
J_{23}	-0.0447	-0.1789
J_{32}	-0.0369	-0.1478
J_{12}	-0.0369	-0.1478
J_{21}	-0.0447	-0.1789
J_{13}	-0.0100	-0.0400
J_{31}	-0.0100	-0.0400

We claim that $V(E) \subseteq \tilde{V}$. To show this, note that co$E \subseteq \tilde{E}$ since co$E$ is the smallest convex set that contains E. Hence $V(E) \subseteq V(\tilde{E})$. Since $p_1 = p_3$ at equilibrium, $v_1 = v_3$ holds for any $v \in V(\tilde{E})$ from the definition of $\tilde{E}$. Hence, $V(\tilde{E}) \subseteq \tilde{V}$ and therefore $V(E) \subseteq \tilde{V}$.

We now check that $v^T \boldsymbol{J}(p) v < 0$ for all $p \in \tilde{E}$ and $v \in \tilde{V}$. For any $v \in \tilde{V}$, $v^T \boldsymbol{J}(p) v$ is the following quadratic form in v_1 and v_2:

$$\begin{aligned} v^T \boldsymbol{J}(p) v &= v_1^2 (J_{11} + J_{33} + J_{13} + J_{31}) + \\ &\quad v_1 v_2 (J_{12} + J_{21} + J_{23} + J_{32}) + v_2^2 J_{22} \end{aligned}$$

If v_1 and v_2 have the same signs, then since J_{kl} are all negative from Table III, $v^T \boldsymbol{J}(p) v < 0$. If v_1 and v_2 have opposite sign, then a sufficient condition for $v^T \boldsymbol{J}(p) v < 0$ is

$$(J_{12} + J_{21} + J_{23} + J_{32})^2 < 4J_{22}(J_{11} + J_{33} + J_{13} + J_{31})$$

Using Table III, it is easy to check that the maximum value of $(J_{12} + J_{21} + J_{23} + J_{32})^2 - 4J_{22}(J_{11} + J_{33} + J_{13} + J_{31})$ is -0.2895. Therefore we have found a superset $\tilde{E}$ of co$E$ and a superset $\tilde{V}$ of $V(E)$ such that $v^T \boldsymbol{J}(p) v < 0$ for all $p \in \tilde{E}$ and all $v \in \tilde{V}$. This implies the condition of Theorem 12 and hence the global uniqueness of network equilibrium. □

VII. CONCLUSION

When sources sharing the same network react to different pricing signals, the widely used duality model of congestion control no longer explains the equilibrium of bandwidth allocation. We have introduced a new, mathematical framework of network equilibrium for multi-protocol networks and studied several fundamental properties, such as existence, local uniqueness, number of equilibria, and global uniqueness. We prove that equilibria exist, and are almost always locally unique. The number of equilibria is almost always finite and must be odd. Finally the equilibrium is globally unique if the price mapping functions are similar (small "degree of heterogeneity"), or the $\boldsymbol{J}(p)$ is negative definite for vectors in a certain set.

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REFERENCES

[1] K. Arrow and F. Hahn. *General Competitive Analysis*. Holden-Day, 1971.
[2] T. Basar and G. Olsder. *Dynamic Noncooperative Game Theory*. 2nd Edtion, SIAM, 1999.
[3] G. Debreu. *Theory of Value*. New York:Wiley, 1959.
[4] G. Debreu. Economies with a finite set of equilibria. *Econometrica*, 38(3): 387-392, May,1970.
[5] G. Debreu. Economic theory in the mathematical mode. Noble Memorial lecture, 8 December, 1983.
[6] E. Dierker. Two remarks on the number of equilibria of an economy. *Econometrica*, 40(5): 751-753, September,1972.
[7] E. Dierker and H. Dierker. The local uniqueness of equilibria. *Econometrica*, 40(5): 867-881, September,1972.
[8] L. Grieco and S. Mascolo. Performance Evaluation and Comparison of Westwood+, New Reno, and Vegas TCP Congestion Control ACM SIGCOMM Computer Communications Review,34(2):25-38, April 2004
[9] D. Wei, C. Jin, S. H. Low and S. Hegde FAST TCP: motivation, architecture, algorithms, performance. *to appear in IEEE/ACM Transactions on Networking*, 2007. http://netlab.caltech.edu/FAST/
[10] F. Kelly, A. Maoulloo, and D. Tan. Rate control for communication networks: shadow prices, proportional fairness and stability. *Journal of the Operational Research Society*, 49:237–252,1998.
[11] S. Kunniyur and R. Srikant. End-to-end congestion control: utility functions, random losses and ECN marks. *IEEE/ACM Transactions on Networking*, 11(5):689-702, Oct 2003.
[12] K. Kurata, G. Hasegawa, and M. Murata. Fairness comparisons between TCP Reno and TCP Vegas for future deployment of TCP Vegas. *Proceedings of INET*, 2000.
[13] S. H. Low. A duality model of TCP and queue management algorithms. *IEEE/ACM Trans. on Networking*, 11(4):525-536, August 2003.
[14] S. H. Low and D. Lapsley. Optimization flow control, I: basic algorithm and convergence. *IEEE/ACM Transactions on Networking*, 7(6):861-874, December 1999.
[15] S. H. Low, F. Paganini, and J. Doyle. Internet congestion control. *IEEE Control Systems Magzine*, February 2002.
[16] J. Marsden and M. Hoffman. *Elementary Classical Analysis*. Second edition, W.H.Freeman and Company, New York 1993.
[17] A. Mas-Colell. *The Theory of General Economic Equilibrium. A Differentiable Approach*. Cambridge University Press, 1985.
[18] A. Mas-Colell, M. Whinston and J. Green. *Microeconomic Theory*. Oxford University Press, 1995.
[19] C. Meyer. *Matrix Analysis and Applied Linear Algebra*. SIAM, 2000.
[20] J. Milnor. *Topology from the Differentiable Viewpoint*. The University Press of Virginia, Charlottesville, 1972.
[21] J. Mo, R. La, V. Anantharam, and J. Walrand. Analysis and comparison of TCP Reno and Vegas. *Proceedings of IEEE Infocom*, March 1999.
[22] J. Mo and J. Walrand. Fair end-to-end window-based congestion control. *IEEE/ACM Transactions on Networking*, 8(5):556-567, October 2000.
[23] M. Osborne and A. Rubinstein. *A Course in Game Theory*. The MIT Press, 1994.
[24] I. Pearce and J. Wise. On the uniqueness of competitive equilibrium: part I, unbounded demand. *Econometrica*,41(5): 817-828, September, 1973.
[25] I. Pearce and J. Wise. On the uniqueness of competitive qquilibrium: part II, bounded demand. *Econometrica*,42(5): 921-932, September, 1974.
[26] A. Simsek, A. Ozdaglar and D. Acemoglu. Generalized Poincare-Hopf Theorem for compact nonsmooth regions. submitted for publication, 2005.
[27] S. Sternberg. *Lectures on Differential Geometry*. Prentice-Hall, 1964.
[28] A. Tang, J. Wang and S. H. Low. Counter Intuitive Throughput Behaviors in Networks under End-to-end Control. *IEEE/ACM Transactions on Networking*, 14(2):355-368, June 2006.
[29] A. Tang, J. Wang, S. H. Low and M. Chiang. Network Equilibrium of Heterogeneous Congestion Control Protocols. *Proceedings of IEEE Infocom*, 2005.
[30] A. Tang, J. Wang, S. H. Low and M. Chiang. Equilibrium of Heterogeneous Congestion Control Protocols. *Caltech CS Technical Report CaltechCSTR:2005.005*, April 2005. http://resolver.caltech.edu/CaltechCSTR:2005.005
[31] A. Tang, J. Wang, S. Hedge and S. H. Low. Equilibrium and fairness of networks shared by TCP Reno and Vegas/FAST. Telecommunication Systems, 30(4):417-439, December 2005.
[32] A. Tang, D. Wei, S. H. Low and M. Chiang. Bandwidth Allocation for Networks with Heterogeneous Congestion Control Protocols. *Proceedings of IEEE ICNP*, 2006.
[33] H. Varian. A third remark on the number of equilibria of an economy *Econometrica*, 43(5/6):985-986, September-November,1975.
[34] H. Yaiche, R. Mazumdar, and C. Rosenberg. A game theoretic framework for bandwidth allocation and pricing in broadband networks. *IEEE/ACM Transactions on Networking*, 8(5), October 2000.

VIII. APPENDIX

A. Global uniqueness of special networks

In this section, we present special networks that have globally unique equilibrium. The proofs of these results can be found in [29].

1) Case 1: linear link-independent m^j: When the price mapping functions are linear and link-independent, i.e., $m_l^j(p_l) = k^j p_l$ for some scalars $k^j > 0$, it is easy to show that we have an unusual situation in the theory of heterogeneous protocols where the equilibrium rate vector x solves the following concave maximization problem

$$\max_x \sum_{i,j} k^j U_i^j(x_i^j) \qquad \text{s. t. } Rx \leq c$$

Therefore, such a network always has a globally unique equilibrium when U_i^j are strictly concave. In [29] we provide another proof using Theorem 12.

Theorem 13. *Suppose assumptions A1–A3 hold and R has full row rank. If for all j and l, $m_l^j(p_l) = k^j p_l$ for some scalars $k^j > 0$, then there is a unique network equilibrium.*

2) Case 2: linear network: Consider the classic linear network shown in Figure 7.

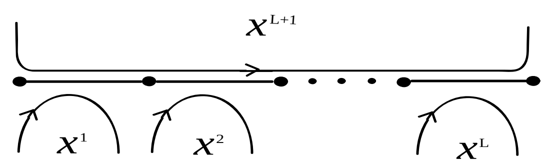

Fig. 7. Corollary 14: linear network.

Theorem 14. *Suppose assumptions A1–A2 hold. The linear network in Figure 7 has a unique equilibrium.*

The theorem can be generalized to include more than one multi-hop flows, provided they all belong to the same type $L+1$ and the sets of links they traverse are nested, i.e., $L(x_1^{L+1}) \supseteq L(x_2^{L+1}) \supseteq \cdots \supseteq L(x_n^{L+1})$ for n multi-hop flows. This result implies that the two 2-link flows in Example 3 are necessary to demonstrate non-uniqueness.

3) Case 3: networks with no flow using more than 2 links: Theorem 6 implies the global uniqueness of equilibrium for any network with no more than 2 links. In this case, the Jacobian matrix $\boldsymbol{J}(p)$ is strictly diagonally dominant with negative diagonal entries, and hence its determinant is $(-1)^L$.

Theorem 15. *Suppose assumptions A1–A2 hold and R has full row rank. A network that has multiple equilibria must have at least three links.*

If R does not have full row rank, then there are two-link networks that have multiple equilibria; see [29].

B. Proof of Theorem 7

By Corollary 6, we only need to prove that $I(p) = (-1)^L$ for any equilibrium $p \in E$. Since $\det(\boldsymbol{J}(p)) = (-1)^L \det(-\boldsymbol{J}(p))$, the condition reduces to $\det(-\boldsymbol{J}(p)) > 0$. Now

$$\begin{aligned} -\boldsymbol{J}(p) &= -\sum_j R^j D^j(p) \left(R^j\right)^T \frac{\partial m^j}{\partial p}(p) \\ &= \sum_j B^j M^j \end{aligned}$$

where $M^j = M^j(p) = \frac{\partial m^j}{\partial p}(p)$ is a diagonal matrix, and $B^j = B^j(p)$ is defined by its elements

$$B^j_{kl} = \sum_i R_{ki} R_{li} \left(-\frac{\partial^2 U_i^j}{\partial (x_i^j)^2}\right)^{-1} \tag{26}$$

Hence

$$\begin{aligned} \det(-\boldsymbol{J}(p)) &= \det \left[\sum_j B^j M^j\right] \\ &= \sum_{\boldsymbol{k}} \mathrm{sgn}\boldsymbol{k} \prod_{l=1}^{L} \sum_{j=1}^{J} \left[B^j M^j\right]_{k_l l} \end{aligned} \tag{27}$$

Here, the summation over $\boldsymbol{k} = (k_1, \ldots, k_L) \in \{1, \ldots, L\}^L$ is over all $L!$ permutations of the L items $\{1, \ldots, L\}$. The function $\mathrm{sgn}\boldsymbol{k}$ is 1 if the minimum number of pairwise interchanges necessary to achieve the permutation $\boldsymbol{k}$ starting from $(1, 2, \ldots, L)$ is even and -1 if it is odd.

Let π denote an L-bit binary sequence that represents the path consisting of exactly those links k for which the kth entries of π are 1, i.e., $\pi_k = 1$. Let $\Pi(k, l) := \{\pi | \pi_k = \pi_l = 1\}$ be the set of paths that contain both links k and l. Let $I^j_\pi = \{i | R^j_{li} = 1 \text{ if and only if } \pi_l = 1\}$ be the set of type j sources on path π, possibly empty. Let

$$r^j_\pi = r^j_\pi(p) = \sum_{i \in I^j_\pi} \left(-\frac{\partial^2 U_i^j}{\partial (x_i^j)^2}\right)^{-1} \tag{28}$$

where r^j_π is zero if I^j_π is empty. Since all utility functions are assumed concave, $r^j_\pi \geq 0$. Then we have from (26) and (28)

$$B^j_{kl} = \sum_{\pi \in \Pi(k,l)} r^j_\pi \tag{29}$$

This together with (27) implies

$$\det(-\boldsymbol{J}(p)) = \sum_{\boldsymbol{k}} \mathrm{sgn}\boldsymbol{k} \prod_{l=1}^{L} \sum_{j=1}^{J} \left(\dot{m}^j_l \sum_{\pi \in \Pi(k_l, l)} r^j_\pi\right) \tag{30}$$

Consider any sequence $a_{ij}, j \in J_i, i = 1, \ldots, I$, where J_i is a finite index set that depends on i. We have

$$\prod_{i=1}^{I} \sum_{j \in J_i} a_{ij} = \sum_{\boldsymbol{j}} \prod_{i=1}^{I} a_{i j_i} \tag{31}$$

where $\boldsymbol{j}$ denotes the vector index $\boldsymbol{j} = (j_1, \ldots, j_I)$ and the summation is over all values in $J_1 \times \ldots \times J_I$.

Using (31) to change the order of product over l and summation over j in (30), we have

$$\det(-\boldsymbol{J}(p)) = \sum_{\boldsymbol{k}} \mathrm{sgn}\boldsymbol{k} \sum_{\boldsymbol{j}} \prod_{l=1}^{L} \left(\dot{m}^{j_l}_l \sum_{\pi \in \Pi(k_l, l)} r^{j_l}_\pi\right)$$

where the vector index $\boldsymbol{j} = (j_1, \ldots, j_L)$ ranges over $\{1, \ldots, J\}^L$. Applying (31) again to change the order of product over l and summation over the index π, we have

$$\det(-\boldsymbol{J}(p)) = \sum_{\boldsymbol{k}} \mathrm{sgn}\boldsymbol{k} \sum_{\boldsymbol{j}} \mu(\boldsymbol{j}) \sum_{\boldsymbol{\pi} \in \Pi(\boldsymbol{k}, \boldsymbol{l})} \rho(\boldsymbol{j}, \boldsymbol{\pi}) \tag{32}$$

where

$$\mu(\boldsymbol{j}) := \prod_{l=1}^{L} \dot{m}^{j_l}_l \tag{33}$$

$$\rho(\boldsymbol{j}, \boldsymbol{\pi}) := \prod_{l=1}^{L} r^{j_l}_{\pi^l} \tag{34}$$

The last summation in (32) is over the vector index $\boldsymbol{\pi} = (\pi^1, \ldots, \pi^L)$ that takes value in the set $\{$ all L-bit binary sequences $\}^L$. As mentioned above, $\boldsymbol{l} = (1, \ldots, L)$ denotes the identity permutation, and "$\boldsymbol{\pi} \in \Pi(\boldsymbol{k}, \boldsymbol{l})$" is a shorthand for "$\pi^l \in \Pi(k_l, l), l = 1, \ldots, L$". Denote by $\mathbf{1}(a)$ the indicator function that is 1 if the assertion a is true and 0 otherwise. Then (32) becomes

$$\det(-\boldsymbol{J}(p)) = \sum_{\boldsymbol{j}} \sum_{\boldsymbol{\pi}} C(\boldsymbol{j}, \boldsymbol{\pi})\ \rho(\boldsymbol{j}, \boldsymbol{\pi}) \tag{35}$$

where

$$C(\boldsymbol{j}, \boldsymbol{\pi}) := \sum_{\boldsymbol{k}} \mathbf{1}(\boldsymbol{\pi} \in \Pi(\boldsymbol{k}, \boldsymbol{l}))\, \mathrm{sgn}\boldsymbol{k}\, \mu(\boldsymbol{j}) \tag{36}$$

Hence $\det(-\boldsymbol{J}(p))$ is a summation, over the index $(\boldsymbol{j}, \boldsymbol{\pi})$, of terms $\rho(\boldsymbol{j}, \boldsymbol{\pi})$ with coefficients $C(\boldsymbol{j}, \boldsymbol{\pi})$. We now show that only those terms for which the constituent r^j_π in the product $\rho(\boldsymbol{j}, \boldsymbol{\pi})$ are all distinct have nonzero coefficients.

Lemma 16. *Consider a term in the summation in (35) indexed by $(\boldsymbol{j}, \boldsymbol{\pi})$. If there are integers $a, b \in \{1, \ldots, L\}$ such that $j_a = j_b$ and $\pi^a = \pi^b$, then $C(\boldsymbol{j}, \boldsymbol{\pi}) = 0$.*

Proof. Fix any $(\boldsymbol{j}, \boldsymbol{\pi})$. Suppose without loss of generality that $j_1 = j_1$ and $\pi^1 = \pi^2$ and $\rho(\boldsymbol{j}, \boldsymbol{\pi}) \neq 0$. We now show that its coefficient $C(\boldsymbol{j}, \boldsymbol{\pi}) = 0$.

Consider any permutation $\boldsymbol{k}$ in (36) that gives a nonzero coefficient in $C(\boldsymbol{j}, \boldsymbol{\pi})$:

$$\mathbf{1}(\boldsymbol{\pi} \in \Pi(\boldsymbol{k}, \boldsymbol{l}))\, \mathrm{sgn}\boldsymbol{k}\, \mu(\boldsymbol{j}) = \mathrm{sgn}\boldsymbol{k} \mu(\boldsymbol{j}) \tag{37}$$

This means that

$$\pi^1 \in \Pi(k_1, 1) \quad \text{and} \quad \pi^2 \in \Pi(k_2, 2)$$

Hence, since $\pi^1 = \pi^2$, the path π^1 goes through all links $1, 2, k_1, k_2$. In particular

$$\pi^1 \in \Pi(k_2, 1) \quad \text{and} \quad \pi^2 \in \Pi(k_1, 2)$$

Therefore there is a permutation $\hat{\boldsymbol{k}}$ in (36) with $\hat{k}_1 = k_2$, $\hat{k}_2 = k_1$, and $\hat{k}_l = k_l$ for $l \geq 3$ for which $\mathbf{1}(\boldsymbol{\pi} \in \Pi(\hat{\boldsymbol{k}}, \boldsymbol{l})) = 1$ but $\mathrm{sgn}\hat{\boldsymbol{k}} = -\mathrm{sgn}\boldsymbol{k}$. This yields a term $-\mathrm{sgn}\boldsymbol{k}\, \mu(\boldsymbol{j})$ in $C(\boldsymbol{j}, \boldsymbol{\pi})$

which exactly cancels the term in (37). Since the argument applies to any $\boldsymbol{k}$ in (36), $C(\boldsymbol{j},\boldsymbol{\pi})=0$. $\square$

In view of Lemma 16, we will restrict the summation over the index $(\boldsymbol{j},\boldsymbol{\pi})$ in (35) to the largest subset of $\{1,\ldots,J\}^L$ where the constituent r_π^j in $\rho(\boldsymbol{j},\boldsymbol{\pi})$ are all distinct. Let Θ denote this subset. We abuse notation and define permutation $\boldsymbol{\sigma}\in\{1,\ldots,L\}^L$ on Θ by

$$\boldsymbol{\sigma}(\boldsymbol{j},\boldsymbol{\pi}) \;=\; (\boldsymbol{\sigma}(\boldsymbol{j}),\boldsymbol{\sigma}(\boldsymbol{\pi}))$$

Then let Θ_0 be the largest subset of Θ that is *permutationally distinct*, i.e., no vector in Θ_0 is a permutation of another vector in Θ_0. The set of permutations $\boldsymbol{\sigma}\in\{1,\ldots,L\}^L$ is in one-one correspondence with the set of $(\boldsymbol{j}',\boldsymbol{\pi}')$ that are permutations of a given $(\boldsymbol{j},\boldsymbol{\pi})$ in Θ_0.[7] This allows us to carry out the summation over $(\boldsymbol{j},\boldsymbol{\pi})$ in (35) first over $(\boldsymbol{j},\boldsymbol{\pi})$ that are permutationally distinct and then over all their permutations. Notice that, given any $(\boldsymbol{j},\boldsymbol{\pi})$ and any permutation $\boldsymbol{\sigma}$, we have from (34)

$$\rho(\boldsymbol{\sigma}(\boldsymbol{j}),\boldsymbol{\sigma}(\boldsymbol{\pi})) \;=\; \rho(\boldsymbol{j},\boldsymbol{\pi})$$

i.e., ρ is invariant to permutations. Hence, we can rewrite (35)–(36) as

$$\det(-\boldsymbol{J}(p)) \;=\; \sum_{(\boldsymbol{j},\boldsymbol{\pi})\in\Theta_0} D(\boldsymbol{j},\boldsymbol{\pi})\;\rho(\boldsymbol{j},\boldsymbol{\pi}) \qquad (38)$$

where

$$D(\boldsymbol{j},\boldsymbol{\pi}) \;=\; \sum_{\boldsymbol{\sigma}}\sum_{\boldsymbol{k}} \mathbf{1}(\boldsymbol{\sigma}(\boldsymbol{\pi})\in\Pi(\boldsymbol{k},\boldsymbol{l}))\,\mathrm{sgn}\boldsymbol{k}\,\mu(\boldsymbol{\sigma}(\boldsymbol{j})) \qquad (39)$$

In the above, L-vectors $\boldsymbol{\sigma}$ and $\boldsymbol{k}$ are permutations.

The next lemma converts a condition on $\boldsymbol{\sigma}(\boldsymbol{\pi})$ into one on $\boldsymbol{\pi}$. It follows directly from the definition of permutation.

Lemma 17. *For any $\boldsymbol{\pi}$ and any permutations $\boldsymbol{\sigma},\boldsymbol{k}$, we have*

$$\boldsymbol{\sigma}(\boldsymbol{\pi})\in\Pi(\boldsymbol{k},\boldsymbol{l}) \quad\Leftrightarrow\quad \boldsymbol{\pi}\in\Pi(\boldsymbol{\sigma}^{-1}\boldsymbol{k},\boldsymbol{\sigma}^{-1})$$

i.e., $[\boldsymbol{\sigma}(\boldsymbol{\pi})]_l \in \Pi(k_l,l)$ for all l if and only if $\pi^l \in \Pi(k_{\sigma_l^{-1}},\sigma_l^{-1})$ for all l.

Applying Lemma 17 to (39), we have

$$D(\boldsymbol{j},\boldsymbol{\pi}) \;=\; \sum_{\boldsymbol{\sigma}}\sum_{\boldsymbol{k}} \mathbf{1}(\boldsymbol{\pi}\in\Pi(\boldsymbol{\sigma}^{-1}\boldsymbol{k},\boldsymbol{\sigma}^{-1}))\,\mathrm{sgn}\boldsymbol{k}\,\mu(\boldsymbol{\sigma}(\boldsymbol{j}))$$

Since $\boldsymbol{k}$, and hence $\boldsymbol{\sigma}^{-1}\boldsymbol{k}$, range over all possible permutations, we can replace the index variable $\boldsymbol{\sigma}^{-1}\boldsymbol{k}$ by $\boldsymbol{k}$ to get

$$D(\boldsymbol{j},\boldsymbol{\pi}) \;=\; \sum_{\boldsymbol{\sigma}}\sum_{\boldsymbol{k}} \mathbf{1}(\boldsymbol{\pi}\in\Pi(\boldsymbol{k},\boldsymbol{\sigma}^{-1}))\,\mathrm{sgn}(\boldsymbol{k}\boldsymbol{\sigma})\,\mu(\boldsymbol{\sigma}(\boldsymbol{j})) \qquad (40)$$

We now use (40) to derive a sufficient condition under which $D(\boldsymbol{j},\boldsymbol{\pi})$ are nonnegative for all permutationally distinct $(\boldsymbol{j},\boldsymbol{\pi})$. The main idea is to show that for every negative term in the summation in (40), either it can be exactly canceled by a positive term, or we can find two positive terms whose sum has a larger or equal magnitude under the given condition. This lemma directly implies Theorem 7.

[7] The one-one correspondence fails to hold for permutations not in Θ.

Lemma 18. *Suppose assumptions A1–A3 hold. Suppose for any $\boldsymbol{j}\in\{1,\ldots,J\}^L$ and any permutations $\boldsymbol{\sigma},\boldsymbol{k},\boldsymbol{n}$ in $\{1,\ldots,L\}^L$, we have for a regular network*

$$\mu(\boldsymbol{k}(\boldsymbol{j}))+\mu(\boldsymbol{n}(\boldsymbol{j})) \quad\geq\quad \mu(\boldsymbol{\sigma}(\boldsymbol{j}))$$

Then, for all $(\boldsymbol{j},\boldsymbol{\pi})\in\Theta_0$, $D(\boldsymbol{j},\boldsymbol{\pi})\geq 0$.

Proof. Due to space limitation, one is referred to [30] for the proof. $\square$

Since the network is regular, $\det(-\boldsymbol{J}(p))\neq 0$. Lemma 18, together with (38), implies that $\det(-\boldsymbol{J}(p))>0$, or equivalently, $I(p)=(-1)^L$ for any $p\in E$, under the condition of the lemma. Theorem 7 then follows from Corollary 6. An illustration for the proof of Lemma 18 via a concrete example can be found in [30].

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