# MODELLING AND STABILITY OF FAST TCP*

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**Abstract.** We discuss the modelling of FAST TCP and prove four stability results. Using the traditional continuous-time flow model, we prove, for general networks, that FAST TCP is globally asymptotically stable when there is no feedback delay and that it is locally asymptotically stable in the presence of feedback delay provided a local stability condition is satisfied. We present an experiment on an emulated network in which the local stability condition is violated. While the theory predicts instability, the experiment shows otherwise. We believe this is because the continuous-time model ignores the stabilizing effect of self-clocking. Using a discrete-time model that captures this effect, we show that FAST TCP is locally asymptotically stable for general networks if all flows have the same feedback delay, no matter how large the delay is. We also prove global asymptotic stability for a single bottleneck link in the absence of feedback delay. The techniques developed here are new and applicable to other protocols.

**Key words.** FAST TCP, Modelling, Stability.

**AMS(MOS) subject classifications.** 68M10.

**1. Introduction.** Congestion control is a distributed feedback algorithm to allocate network resources among competing users. The algorithms in the current Internet, TCP Reno, have prevented severe congestion while the Internet underwent explosive growth during the last decade. It is well known however that TCP Reno's performance degrades steadily as networks continue to scale up in capacity and size [5, 12]. This has motivated several recent proposals for congestion control of high-speed networks, including HSTCP [4], Scalable TCP [10], FAST TCP [7, 8], and BIC TCP [19] (see [7, 8] for extensive references). The details of the architecture, algorithms, experimental evaluations of FAST TCP can be found in [7, 8]. A new discrete-time model of congestion control is also introduced in [7, 8] and a sufficient condition for the local asymptotic stability of FAST TCP is proved using the new model for the case of a single link in the absence of feedback delay. In this paper, we extend the analysis and prove four stability results.

Most of the stability analysis in the literature is based on the fluid model introduced in [5] (see surveys in [11, 9, 15] for extensions and related models). Two key features of these models are that a source controls its sending rate directly[1] and that the queueing delay at a link is proportional to the integral of the excess demand for its bandwidth.

*Partial and preliminary results have appeared in [17].

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[1] Even when the congestion window size is used as the control variable, sending rate is often taken to be the window normalized by a *constant* round-trip time, and hence a source still controls its rate directly.

In reality, a source dynamically sets its congestion window rather than its sending rate. These models do not adequately capture the self-clocking effect where a packet is sent only when an old one is acknowledged, except briefly and immediately after the congestion window is changed. This automatically constrains the input *rate* at a link to its link capacity, after a brief transient, no matter how large the congestion windows are set. The new discrete-time link model proposed in [7, 8] captures this effect. While the traditional continuous-time link model does not consider self-clocking, the new discrete-time link model ignores the fast dynamics at the links. We present both models of FAST TCP in Section 2. Experimental results are provided to show that, despite errors in these models, both of them seem to track the queue process reasonably well. Then we prove two stability results in each of these models.

In Section 3, we prove that FAST TCP is globally asymptotically stable in general networks when there is no feedback delay using the continuous-time model. We also derive a sufficient condition for local asymptotic stability in general networks with feedback delay, using the techniques developed in [13, 16].

This local stability condition becomes necessary when the network consists of a single link and the sources are homogeneous. We then present an experiment on an emulated network (Dummynet) in which the local stability condition is violated. While the theory, and the numerical simulation of the continuous-time model, predict instability, the experiment suggest that FAST TCP is stable. We believe that this discrepancy is due to the self-clocking effect that helps stability but is ignored in the continuous-time model.

In Sections 4, we analyze the stability of FAST TCP using the discrete-time model. First, we prove that a general network of FAST TCP is locally asymptotically stable if all sources have the same delay, no matter how large the delay is. Then we restrict ourselves to a single link without feedback delay and prove the global asymptotic stability of FAST TCP. The analysis technique developed for the discrete-time model is new and applicable to analyzing other protocols.

Finally, we conclude in Section 5 with limitations of this work.

## 2. Model.

**2.1. Notation.** A network consists of a set of $L$ links indexed by $l$ with finite capacity $c_l$. It is shared by a set of $N$ flows identified by their sources indexed by $i$. Let $R$ be the routing matrix where $R_{li} = 1$ if source $i$ uses link $l$, and 0 otherwise.

We use $t$ for time in the continuous-time model, and for time step in the discrete-time model. The meaning of $t$ should be clear from the context. FAST TCP updates its congestion window every fixed time period, which is used as the time unit.

Let $d_i$ denote the round-trip propagation delay of source $i$, and $q_i(t)$ denote the round-trip queueing delay. The round-trip time is given by $T_i(t) := d_i + q_i(t)$. We denote the forward feedback delay from source $i$ to link $l$ by $\tau_{li}^f$ and the backward feedback delay from link $l$ to source $i$ as $\tau_{li}^b$. The sum of forward delay from source $i$ to any link $l$ and the backward delay from link $l$ to source $i$ is fixed, i.e., $\tau_i := \tau_{li}^f + \tau_{li}^b$ for any link $l$ on the path of source $i$. We make a subtle assumption here. In reality, the feedback delays $\tau_{li}^f$, $\tau_{li}^b$ include queueing delay and are time-varying. We assume for simplicity that they are constant, and mathematically unrelated to $T_i(t)$. Later, when we analyze linear stability around the network equilibrium in the presence of feedback delay, we will interpret $\tau_i$ as the equilibrium value of $T_i(t)$.

Let $w_i(t)$ be source $i$'s congestion window at time $t$ (discrete or continuous-time). The sending rate of source $i$ at time $t$ is defined as

$$x_i(t) := \frac{w_i(t)}{T_i(t)} \tag{2.1}$$

where $T_i(t) = d_i + q_i(t)$. The aggregate rate at link $l$ is

$$y_l(t) := \sum_i R_{li} x_i(t - \tau_{li}^f). \tag{2.2}$$

Let $p_l(t)$ be the queueing delay at link $l$. The end-to-end queueing delay $q_i(t)$ observed by source $i$ is

$$q_i(t) = \sum_l R_{li} p_l(t - \tau_{li}^b). \tag{2.3}$$

A model of FAST TCP amounts to specifying how $w_i(t)$ and $p_l(t)$ evolve.

**2.2. Discrete and continuous-time models.** A FAST TCP source periodically updates its congestion window based on the average RTT and estimated queueing delay. The pseudo-code is

$$\texttt{w} \leftarrow (1-\gamma)\texttt{w} \; + \; \gamma\left(\frac{\texttt{baseRTT}}{\texttt{RTT}}\,\texttt{w} + \alpha\right)$$

where $\gamma \in (0, 1]$, `baseRTT` is the minimum RTT observed, and $\alpha$ is a constant. We model this by the following discrete-time equation

$$w_i(t+1) = \gamma\left(\frac{d_i w_i(t)}{d_i + q_i(t)} + \alpha_i\right) + (1-\gamma) w_i(t) \tag{2.4}$$

where $w_i(t)$ is the congestion window of source $i$, $\gamma \in (0, 1]$, and $\alpha_i$ is a constant that depends on source $i$. The corresponding continuous-time model is

$$\dot{w}_i(t) = \gamma\left(\alpha_i - \frac{q_i(t) w_i(t)}{d_i + q_i(t)}\right) \tag{2.5}$$

where the time is measured in the unit of update period in FAST TCP.

For the continuous-time model, queueing delay has been traditionally modelled by (e.g., [11])

$$\dot{p}_l(t) = \frac{1}{c_l}(y_l(t) - c_l). \tag{2.6}$$

In reality, TCP uses self-clocking to match the number of packets-in-flight to the congestion window size $w_i(t)$. When the congestion window is fixed, the source sends a new packet exactly after it receives an ACK packet. When the congestion window is increased, the source may send out more than one packet on the receipt of an ACK packet for the packet-in-flight to catch up with the new window size. When the congestion window is decreased, the source sends no packet for a short while for the packet-in-flight to drop. Therefore, one round-trip time after a congestion window is changed, packet transmission will be clocked at the same rate as the throughput the flow receives. We assume that the disturbance in the queues due to congestion window changes settles down quickly compared with the update period of the discrete-time model. A consequence of this assumption is that the link queueing delay vector, $p(t) = (p_l(t)$, for all $l)$, is determined implicitly by sources' congestion windows in a static manner:

$$\sum_i R_{li} \frac{w_i(t - \tau_{li}^f)}{d_i + q_i(t - \tau_{li}^f)} \begin{cases} = c_l & \text{if } p_l(t) > 0 \\ \le c_l & \text{if } p_l(t) = 0 \end{cases} \tag{2.7}$$

where $q_i$ is the end-to-end queueing delay given by (2.3).

In summary, the continuous-time model is specified by (2.5) and (2.6), and the discrete-time model is specified by (2.4) and (2.7), where the source rates and aggregate rates at links are given by (2.1) and (2.2), and the end-to-end delays are given by (2.3). While the continuous-time model does not take self-clocking into full account, the discrete-time model ignores the fast dynamics at the links. Before comparing these models, we clarify their common equilibrium structure by the following theorem cited from [7, 8].

THEOREM 2.1. *Suppose that the routing matrix $R$ has full row rank. A unique equilibrium $(x^*, p^*)$ of the network exists. Moreover, $x^*$ is the unique maximizer of*

$$\max_{x \ge 0} \sum_i \alpha_i \log x_i \quad \textit{subject to} \quad Rx \le c \tag{2.8}$$

*and $p^*$ is the unique minimizer of the Lagrangian dual problem. This implies in particular that the equilibrium rate $x^*$ is $\alpha_i$-weighted proportionally fair.*

**2.3. Validation.** The continuous-time link model implies that the queue takes an infinite amount of time to converge after a window change. On the other extreme, the discrete-time link model assumes that the queue

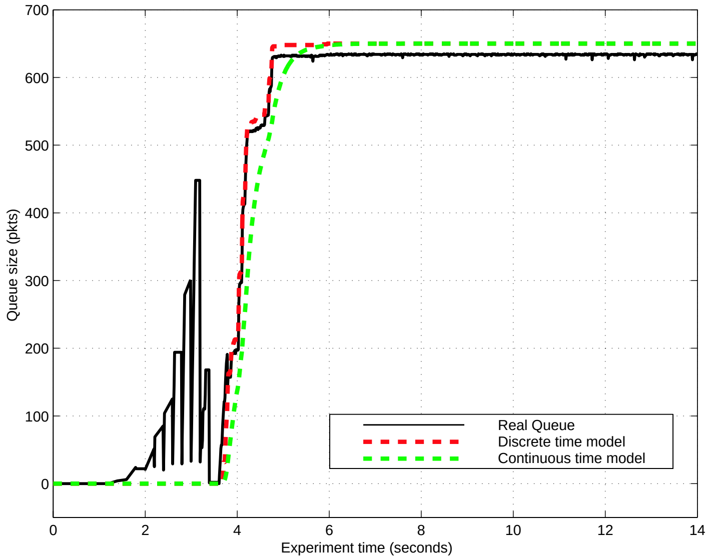



FIG. 1. *Model validation–closed loop.*

settles down in one sampling time. Neither is perfect, but we now present experimental results that suggest both track the queue dynamics well.

All the experiments reported in this paper are carried out on a Dummynet Testbed [14]. A FreeBSD machine is configured as a Dummynet router that provides different propagation delays for different sources. It can be configured with different capacities and buffer sizes. In our experiments, the bottleneck link capacity is 800Mbps, and the buffer size is 4000 packets with a fixed packet length of 1500 bytes. A Dummynet monitor records the queue size every 0.4 second. The congestion window size and RTT are recorded at the host every 50ms. TCP traffic is generated using *iperf*. The publicly released code of FAST is used in all experiments involving FAST. We present two experiments to validate the model, one closed-loop and one open-loop.

In the first (closed-loop) experiment, there are 3 FAST TCP sources sharing a Dummynet router with a common propagation delay of 100ms. The measured and predicted queue sizes are given in Figure 1. At the beginning of the experiment (before time $< 4$ seconds), the FAST sources are in the slow-start phase, and neither model gives accurate prediction. After the source enters the congestion avoidance phase, both models track the queue size well.

To eliminate the modelling error in the congestion window adjustment algorithm itself while validating the link models, we decouple the TCP and queue dynamics by using open-loop window control. The second experiment involves three sources with propagation delays 50ms, 100ms, and 150ms sharing the same Dummynet router.

We changed the Linux 2.4.19 kernel so that the sources vary their window sizes according to the schedules shown in Figure 2(a). The sequences of congestion window sizes are then used in (2.1)–(2.2) and (2.6) to compute the queueing delay predicted by the continuous-time model. We also use them in (2.1)–(2.2) and (2.7) to compute the predictions of the discrete-time model. The queueing delay measured from the Dummynet and those predicted by these two models are shown in Figure 2(b), which indicates that both models track the queue size well. We next analyze the stability properties of these two models.

**3. Stability analysis with the continuous-time model.** We present the stability analysis of the continuous model in general networks with and without feedback delays.

**3.1. Global stability without feedback delay.** In this subsection, we show that FAST is globally asymptotically stable for general networks by designing a Lyapunov function. When there is no feedback delay, the equations (2.2) and (2.3) can be simplified as

$$y_l(t) = \sum_i R_{li} x_i(t) \quad \text{and} \quad q_i(t) = \sum_l R_{li} p_l(t). \tag{3.1}$$

Suppose that $R$ is full row rank, and the system has unique equilibrium source rates and link prices. Let $w_i, p_l, q_i, \ldots$ be the equilibrium quantities, and denote $\delta w_i(t) := w_i(t) - w_i, \delta p_l(t) = p_l(t) - p_l, \delta q_i(t) = q_i(t) - q_i, \ldots$. From (2.5) the equilibrium window is given by

$$w_i = \frac{\alpha_i T_i}{q_i} \tag{3.2}$$

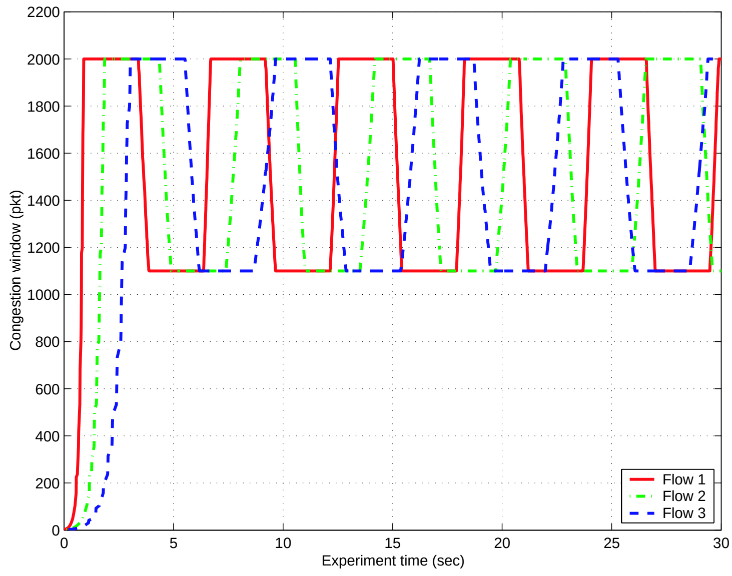
where $T_i = d_i + q_i$ is the equilibrium round-trip delay.

We can then rewrite (2.5) as

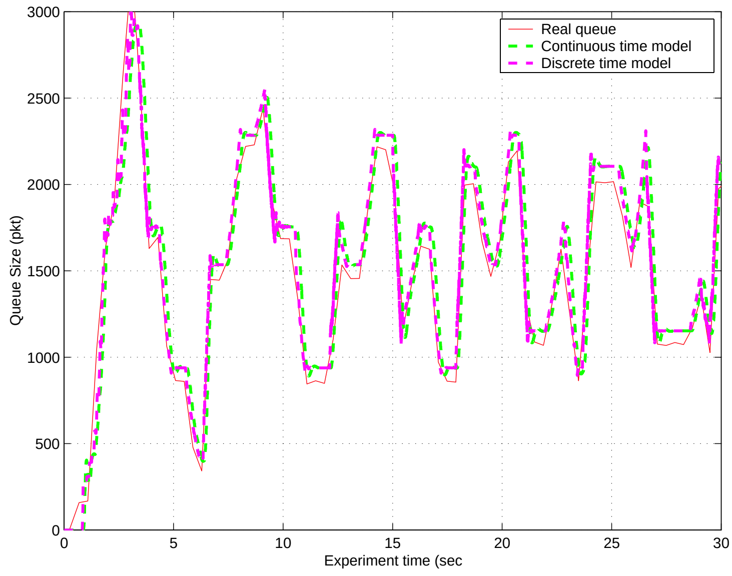
$$\begin{aligned}
\frac{1}{\gamma}\dot{w}_i(t) &= \alpha_i - \frac{q_i(t) w_i(t)}{T_i(t)} \\
&= \alpha_i - \frac{q_i(t)}{T_i(t)}(w_i + \delta w_i(t)) \\
&= -\frac{q_i(t)}{T_i(t)}\delta w_i(t) + \alpha_i \frac{T_i(t) q_i - q_i(t) T_i}{T_i(t) q_i} \\
&= -\frac{q_i(t)}{T_i(t)}\delta w_i(t) - \frac{\alpha_i d_i}{T_i(t) q_i}\delta q_i(t).
\end{aligned}$$

Therefore, we have

$$\frac{1}{\gamma}\delta \dot{w}_i(t) = -\frac{q_i(t)}{T_i(t)}\delta w_i(t) - \frac{\alpha_i d_i}{T_i(t) q_i}\delta q_i(t). \tag{3.3}$$




(a) Scheduled congestion window.




(b) Resulting queue size.

FIG. 2. *Model validation–open loop.*

Based on (2.1) and (3.2) we have

$$\begin{aligned}\delta x_i(t) &= \frac{w_i + \delta w_i(t)}{T_i(t)} - \frac{w_i}{T_i} \\ &= \frac{\delta w_i(t)}{T_i(t)} - (\frac{1}{T_i} - \frac{1}{T_i(t)})w_i \\ &= \frac{\delta w_i(t)}{T_i(t)} - \frac{\delta q_i(t)}{T_i(t)T_i}\frac{\alpha_i T_i}{q_i}.\end{aligned}$$

Therefore, we have

$$\delta x_i(t) = \frac{1}{T_i(t)}\delta w_i(t) - \frac{\alpha_i}{T_i(t)q_i}\delta q_i(t). \tag{3.4}$$

Based on (3.4) and (3.1), the derivative of link price is (from (2.6))

$$\dot{p}_l(t) = \frac{1}{c_l}\left(\sum_i R_{li}x_i(t) - c_l\right) = \frac{1}{c_l}\sum_i R_{li}\delta x_i(t). \tag{3.5}$$

From (3.4) and (3.5), we have

$$\delta\dot{p}_l(t) = \frac{1}{c_l}\sum_i R_{li}\left(\frac{1}{T_i(t)}\delta w_i(t) - \frac{\alpha_i}{T_i(t)q_i}\delta q_i(t)\right). \tag{3.6}$$

With these preliminary results, we prove the following theorem.

THEOREM 3.1. *The continues-time model of FAST TCP is globally asymptotically stable when there is no feedback delay and $R$ has full row rank.*

**Proof:** Considering the function $V(\hat{w}, \hat{p})$ defined as

$$V(\hat{w}, \hat{p}) = \frac{1}{2\gamma}\sum_i \frac{q_i}{\alpha_i d_i}(\hat{w}_i - w_i)^2 + \frac{1}{2}\sum_l c_l(\hat{p} - p_l)^2 \tag{3.7}$$

where $(w, p)$ is unique equilibrium point, which exists according to Theorem 2.1. Clearly, the function $V(\hat{w}, \hat{p})$ is non-negative for all $(\hat{w}, \hat{p})$ and zero if and only if $\hat{w} = w$ and $\hat{p} = p$. Taking time derivative of $V(w(t), p(t))$ along the solution trajectory of (3.6) and (3.3) yields

$$\begin{aligned}
\dot{V}(w(t),p(t)) &= \sum_i \frac{q_i}{\gamma\alpha_i d_i}\delta w_i(t)\delta\dot{w}_i(t) + \sum_l c_l \delta p_l(t)\delta\dot{p}_l(t) \\
&= \sum_i \frac{q_i}{\alpha_i d_i}\delta w_i(t)\left(-\frac{q_i(t)}{T_i(t)}\delta w_i(t) - \frac{\alpha_i d_i}{T_i(t) q_i}\delta q_i(t)\right) \\
&\quad + \sum_l \sum_i R_{li}\left(\frac{1}{T_i(t)}\delta w_i(t) - \frac{\alpha_i}{T_i(t)q_i}\delta q_i(t)\right)\delta p_l(t) \\
&= -\sum_i \frac{q_i q_i(t)}{T_i(t)\alpha_i d_i}\delta w_i(t)^2 - \sum_i \frac{1}{T_i(t)}\delta w_i(t)\delta q_i(t) \\
&\quad + \sum_i \frac{1}{T_i(t)}\delta w_i(t)\sum_l R_{li}\delta p_l(t) \\
&\quad - \sum_i \frac{\alpha_i}{T_i(t)q_i}\delta q_i(t)\sum_l R_{li}\delta p_l(t) \\
&= -\sum_i \frac{q_i q_i(t)}{T_i(t)\alpha_i d_i}\delta w_i(t)^2 - \sum_i \frac{\alpha_i}{T_i(t)q_i}\delta q_i(t)^2
\end{aligned}$$

where we have used $\delta q_i(t) = \sum_l R_{li}\delta p_l(t)$. Hence $V > 0$ and $\dot{V} < 0$ at all $(\hat{w},\hat{p})$ that is not the equilibrium $(w,p)$, and $V = \dot{V} = 0$ at the equilibrium $(w,p)$. Moreover, $V(\hat{w},\hat{p}) \to \infty$ as $\|(\hat{w},\hat{p})\| \to \infty$. This implies that the system specified by (3.6) and (3.3) is globally asymptotically stable. □

Note that the windows $w(t)$ and the end-to-end queueing delays $q(t)$ converge globally to their equilibrium values regardless of whether $R$ has full row rank. The link queueing delays $p(t)$ may not, unless $R$ has full row rank, in which case $p(t) = (RR^T)^{-1}Rq(t)$ is uniquely defined and must also converge globally.

**3.2. Local stability with feedback delay.** When there are feedback delays, the global stability analysis for FAST TCP in general networks is still open. In this subsection, we provide a sufficient condition for local asymptotic stability.

We make two assumptions in this subsection. First, $R$ has full row rank and hence there is a unique equilibrium point $(w,p)$. Second, the round-trip feedback delays $\tau_i = \tau_{li}^f + \tau_{li}^b$ in (2.2) and (2.3) equal the equilibrium values of $T_i := d_i + \sum_l R_{li}p_l$.

To linearize the model (2.5) and (2.6) around the unique equilibrium, define routing matrices with feedback delay in frequency domain as

$$[R_f(s)]_{li} := \begin{cases} e^{-\tau_{li}^f s} & \text{if } R_{li} = 1 \\ 0 & \text{if } R_{li} = 0 \end{cases}$$

$$[R_b(s)]_{li} := \begin{cases} e^{-\tau_{li}^b s} & \text{if } R_{li} = 1 \\ 0 & \text{if } R_{li} = 0. \end{cases}$$

Let $w_i, p_l$, $x_i$, $q_i$, and $T_i$ be the corresponding equilibrium values. The following Lemma provides the open-loop transfer function.

Lemma 3.1. *The open-loop transfer function of the linearized FAST TCP system is*

$$L(s) = DR_f(s)\Lambda(s)XR_f^T(-s) \tag{3.8}$$

*where*

$$D := diag\left(\frac{1}{c_l}\right), \quad X := diag(x_i), \quad \Lambda(s) := diag\left(\frac{e^{-T_i s}}{T_i s}\frac{T_i s + \gamma T_i}{T_i s + \gamma q_i}\right).$$

**Proof.** See Appendix A. □

The following theorem provides a sufficient condition for local stability.

Theorem 3.2. *The FAST TCP system described by (2.5) and (2.6) is locally asymptotically stable if*

$$\frac{M}{\phi}\sqrt{\frac{\phi^2 + \gamma^2 T_{\max}^2}{\phi^2 + \gamma^2 q_{\min}^2}} < 1 \tag{3.9}$$

*where* $M := \max_i \sum_l R_{li}$ *is the maximal number of links in the path of any source,* $q_{\min} = \min_i q_i$*,* $T_{\max} = \max_i T_i$ *and*

$$\phi := \min_i \left(\frac{\pi}{2} - \tan^{-1}\frac{1 - q_i/T_i}{2\sqrt{q_i/T_i}}\right). \tag{3.10}$$

**Proof.** It is sufficient to show that the eigenvalues of the open-loop transfer function do not encircle $-1$ in the complex plane for $s = j\omega$, $\omega \geq 0$ when the condition in the theorem is satisfied [3]. The proof is similar to that in [2]. Note that both $X$ and $\Lambda(s)$ are diagonal matrices and that $AB$ and $BA$ have the same nonzero eigenvalues for two matrices $A$ and $B$ of approriate dimensions. Hence the set of nonzero eigenvalues of $L(s)$ is the same as those of $\Lambda(s)\hat{R}^T(-j\omega)\hat{R}(j\omega)$, when $s = j\omega$, where $\hat{R}(j\omega)$ is defined as

$$\hat{R}(j\omega) := \text{diag}\left(\frac{1}{\sqrt{c_l}}\right) R_f(j\omega)\text{diag}(\sqrt{x_i}).$$

Following the argument of [13, 16], we study the convex hull of Nyquist trajectories and ensure it does not encircle the critical point $-1$. More specifically, the set $\sigma(L(j\omega))$ of eigenvalues of $L(j\omega)$ satisfies [16] (possibly ignoring the zero eigenvalue):

$$\begin{aligned}\sigma(L(j\omega)) &= \sigma\left(\Lambda(s)\hat{R}^T(-j\omega)\hat{R}(j\omega)\right)\\ &\subseteq \rho\left(\hat{R}^T(-j\omega)\hat{R}(j\omega)\right)\cdot \text{co}\left(0 \cup \{\Lambda_i(j\omega), i = 1, \ldots, N\}\right)\end{aligned}$$

where $\rho(A)$ denotes the spectral radius of matrix $A$, $\text{co}(\cdot)$ denotes the convex hull, and

$$\Lambda_i(j\omega) := \frac{e^{-j\omega T_i}}{j\omega T_i}\frac{j\omega T_i + \gamma T_i}{j\omega T_i + \gamma q_i}$$

Similar to [2], the spectral radius of $\hat{R}^T(-j\omega)\hat{R}(j\omega)$ is less than $M$, which is the maximal number of links in the path of any source, $M = \max_i \sum_l R_{li}$. This implies

$$\sigma(L(j\omega)) \subseteq M \cdot \mathrm{co}\left(0 \cup \{\Lambda_i(j\omega), \quad i = 1 \ldots N\}\right).$$

Therefore a sufficient condition for local stability is that $M\Lambda_i(j\omega)$ does not encircle $-1$ for any $i$. We now prove that when the phase of $M\Lambda_i(j\omega)$ reaches $-\pi$, its magnitude is strictly less than 1 and hence the trajectory of $M\Lambda_i(j\omega)$ will not encircle $-1$ as $\omega$ goes from 0 to $\infty$.

It is not hard to show that the largest phase lag (i.e., the minimum phase) of $(j\omega T_i + \gamma T_i)/(j\omega T_i + \gamma q_i)$ is produced when $\omega T_i = \sqrt{\gamma T_i \cdot \gamma q_i}$ , which is

$$\angle \frac{j\sqrt{\gamma T_i \cdot \gamma q_i} + \gamma T_i}{j\sqrt{\gamma T_i \cdot \gamma q_i} + \gamma q_i} = -\tan^{-1}\frac{1 - q_i/T_i}{2\sqrt{q_i/T_i}}.$$

The above equation yields

$$\angle\Lambda_i(j\omega) \geq -\omega T_i - \frac{\pi}{2} - \tan^{-1}\frac{1 - q_i/T_i}{2\sqrt{q_i/T_i}}.$$

Suppose that the phase of $\Lambda_i(j\omega)$ is $-\pi$ at frequency $\omega_i$. Then

$$-\pi = \angle\Lambda_i(j\omega_i) \geq -\omega_i T_i - \frac{\pi}{2} - \tan^{-1}\frac{1 - q_i/T_i}{2\sqrt{q_i/T_i}}.$$

The condition (3.10) in the theorem implies

$$\omega_i T_i \geq \phi \quad \text{for} \quad i = 1 \ldots N.$$

It is easy to check that the magnitude of $\Lambda_i(j\omega)$ is a decreasing function of $\omega$. Therefore under the condition in the theorem, we have

$$M|\Lambda_i(j\omega_i)| \leq M\left|\Lambda_i(j\frac{\phi}{T_i})\right| = \frac{M}{\phi}\sqrt{\frac{\phi^2 + \gamma^2 T_i^2}{\phi^2 + \gamma^2 q_i^2}} \leq \frac{M}{\phi}\sqrt{\frac{\phi^2 + \gamma^2 T_{max}^2}{\phi^2 + \gamma^2 q_{min}^2}} < 1$$

and $M\Lambda(j\omega_i)$ can not encircle $-1$. Hence the system is locally asymptotically stable if (3.9 ) is satisfied. □

The condition (3.10) can be hard to satisfy when $M$ is large. Nonetheless, it provides information on the effect of various parameters on stability. For example, it suggests that the equilibrium queueing delay should be large to guarantee stability.

**3.3. Numerical simulation and experiment.** In general, the condition in Theorem 3.2 is only sufficient. When there is only one link and all sources have the same feedback delay, it is necessary as well. The theorem implies that FAST TCP may become unstable in a single bottleneck network with homogeneous sources. We now present an experiment with a single bottleneck link where the local stability condition is violated. Numerical simulation of the continuous-time model exhibits instability confirming the theorem. Yet, the same network on Dummynet with real FAST TCP implementation is stable. This suggests that the discrepancy is not in the stability theorem but rather in the continuous-time model.

In our experiment, the sources have identical propagation delay of 100ms with a constant $\alpha$ value of 70 packets. They share a bottleneck with capacity of 800Mbps. The simulations and experiments consist of three intervals. The interval length is 10 seconds for the continuous-time model simulation and 100 seconds for the experiment[2]. Three sources are active from the beginning of the experiment, seven additional sources activate in the second interval, and in the last interval, all sources become inactive except five of them. The simulation and experimental results are shown in Figure 3 and Figure 4, respectively. Figure 3 confirms the theorem that the continuous-time model is unstable under the chosen condition that violates the stability condition of Theorem 3.2. However, as Figure 4 shows, the real FAST TCP implementation is actually stable.[3]

We believe that the discrepancy is largely due to the fact that the continuous-time model does not capture the self-clocking effect accurately. Self-clocking ensures that packets are sent at the same rate as the throughput the source receives, except briefly when the window size changes, and helps stabilize the system. Indeed, for the case of one source over one link, a discrete-event model is used in [18] to prove that FAST TCP and Vegas are always stable regardless of the feedback delay. It also provides justification for the discrete-time models in (2.4).

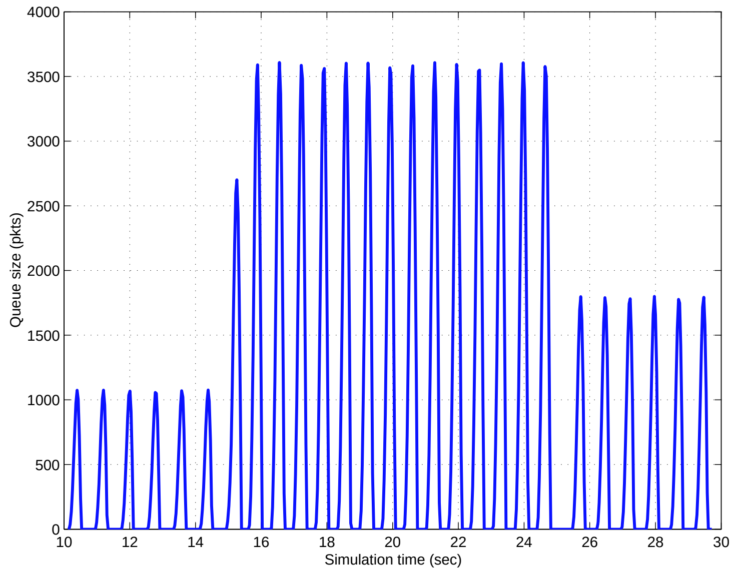
**4. Stability analysis with the discrete-time model.** We now analyze the stability of the discrete-time model. We first show that a network of homogeneous sources with the same feedback delay is locally stable no matter how large the delay is, agreeing with our experimental experience. We then show that at a single link, FAST TCP converges globally and exponentially in the absence of feedback delay.

**4.1. Local stability with feedback delay.** A network of FAST TCP sources is modelled by equations (2.3), (2.4), and (2.7). We assume $R$ has full row rank so that the equilibrium is unique. Since we are studying local stability around the equilibrium, we ignore all un-congested links (links where prices are zero in equilibrium) and assume that equality always holds in (2.7).

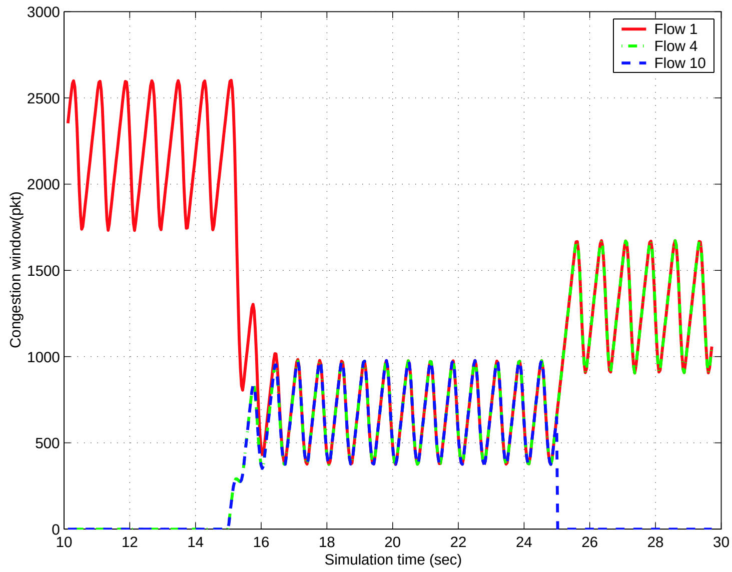
The main result of this section provides a sufficient condition for local stability in general networks with common feedback delay. This proof

[2] We use a longer duration in the Dummynet experiment because a FAST TCP source takes longer to converge due to slow-start, which is not included in our model.

[3] The regular spikes every 10 seconds in the queue size are probably due to a certain background task in the sending host.

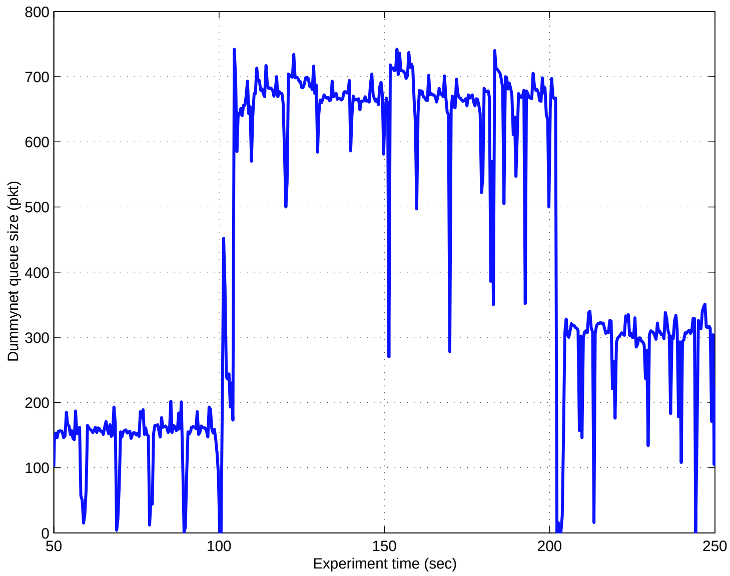



(a) Queue size.

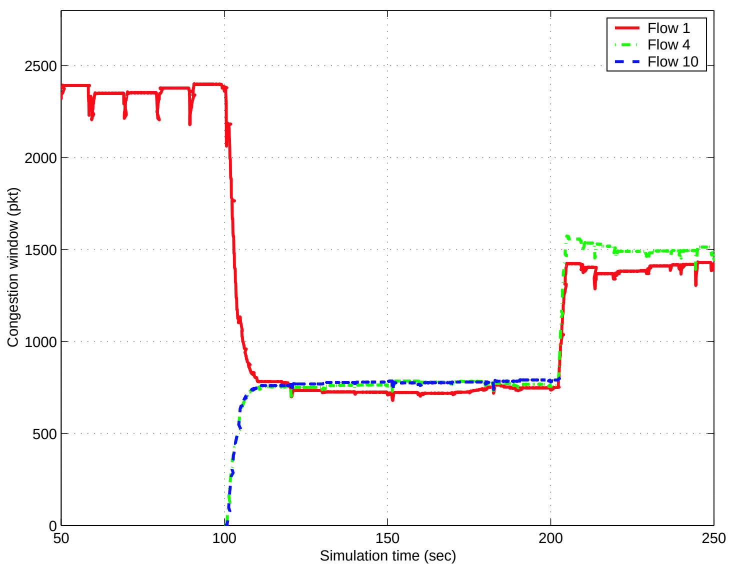



(b) Window size.

FIG. 3. *Numerical simulation of continuous-time model fo FAST TCP.*




(a) Queue size.




(b) Window size.

FIG. 4. *Dummynet experiments of FAST TCP.*

generalizes the technique in [7, 8] from a single link to a network and by including feedback delay.

THEOREM 4.1. *FAST TCP is locally asymptotically stable for arbitrary networks for any* $\gamma \in (0, 1]$ *and if all sources have the same round-trip feedback delay* $\tau_i = T_i = \tau$ *for all* $i$.

In particular, when all feedback delays are ignored, $\tau_i = 0$ for all $i$, then FAST TCP is locally asymptotically stable. This generalizes the stability result in [7, 8] from a single link to a network.

COROLLARY 4.1. *FAST TCP is locally asymptotically stable in the absence of feedback delay for general networks with any* $\gamma \in [0, 1)$.

The rest of this subsection is devoted to the proof of Theorem 4.1. We apply $Z$-transform to the linearized system, and use the generalized Nyquist criterion to derive a sufficient stability condition.

Define the forward and backward $Z$-transformed routing matrices $R_f(z)$ and $R_b(z)$ as

$$[R_f(z)]_{li} := \begin{cases} z^{-\tau_{li}^f} & \text{if } R_{li} = 1 \\ 0 & \text{if } R_{li} = 0 \end{cases}$$

$$[R_b(z)]_{li} := \begin{cases} z^{-\tau_{li}^b} & \text{if } R_{li} = 1 \\ 0 & \text{if } R_{li} = 0. \end{cases}$$

The relation $\tau_{li}^f + \tau_{li}^b = \tau_i = T_i$ gives

$$R_b(z) = R_f(z^{-1}) \cdot \text{diag}(z^{-\tau_i}). \tag{4.1}$$

Denote by $\delta w(z)$, $\delta q(z)$, and $\delta p(z)$ the corresponding $Z$-transforms of $\delta w(t)$, $\delta q(t)$, and $\delta p(t)$ for the linearized system, respectively. Let $q$ and $w$ be the end-to-end queueing delay and congestion window at equilibrium. Linearizing (2.7) yields

$$\sum_i R_{li} \left( \frac{\delta w_i(t - \tau_{li}^f)}{d_i + q_i} - w_i \frac{\delta q_i(t - \tau_{li}^f)}{(d_i + q_i)^2} \right) = 0$$

where equality is assumed in (2.7). The corresponding $Z$-transform in matrix form is

$$R_f(z) D^{-1} M \delta w(z) - R_f(z) B \delta q(z) = 0 \tag{4.2}$$

where the diagonal matrices are

$$B := \text{diag}\left( \frac{w_i}{(d_i + q_i)^2} \right), \; M := \text{diag}\left( \frac{d_i}{d_i + q_i} \right), \; D := \text{diag}(d_i).$$

Since $R_f(z)$ is generally not a square matrix, we cannot cancel it in (4.2).

Equation (2.3) is already linear, and the corresponding $Z$-transform in matrix form is

$$\delta q(z) = R_b(z)^T \delta p(z). \tag{4.3}$$

By combining (4.2) and (4.3), we obtain

$$\begin{pmatrix} I & -R_b^T(z) \\ R_f(z)B & 0 \end{pmatrix} \begin{pmatrix} \delta q(z) \\ \delta p(z) \end{pmatrix} = \begin{pmatrix} 0 \\ R_f(z)D^{-1}M \end{pmatrix} \delta w(z).$$

Solving this equation with block matrix inverse gives the transfer function from $\delta w(z)$ to $\delta q(z)$:

$$\frac{\delta q(z)}{\delta w(z)} = R_b^T(z)(R_f(z)BR_b^T(z))^{-1}R_f(z)D^{-1}M.$$

The $Z$-transform of the linearized congestion window update algorithm is

$$z\delta w(z) = \gamma\left(M\delta w(z) - DB\delta q(z)\right) + (1-\gamma)\delta w(z).$$

By combining the above equations, the open-loop transfer function $L(z)$ from $\delta w(z)$ to $\delta w(z)$ is:

$$\begin{aligned} L(z) = &-\gamma\left(M - DBR_b^T(z)(R_f(z)BR_b^T(z))^{-1}R_f(z)D^{-1}M\right)z^{-1} \\ &+(1-\gamma)z^{-1}I. \end{aligned}$$

A sufficient condition for local asymptotic stability can be derived based on the generalized Nyquist criterion [1, 3]. Since the open-loop system is stable, if we can show that the eigenvalue loci of $L(e^{jw})$ does not enclose $-1$ for $\omega \in [0, 2\pi)$, the closed-loop system is locally asymptotically stable. A sufficient condition for this is that the spectral radius of $L(e^{jw})$ is strictly less than 1 for $\omega \in [0, 2\pi)$.

When $z = e^{jw}$, the spectral radii of $L(z)$ and $-zL(z)$ are the same. Hence, we only need to study the spectral radius of

$$J(z) := \gamma(M - DBR_b^T(z)\left(R_f(z)BR_b^T(z)\right)^{-1}R_f(z)D^{-1}M + (1-\gamma)I.$$

Clearly, the eigenvalues of $J(z)$ are dependent on $\gamma$. For any given $z = e^{j\omega}$, let the eigenvalues of $J(z)$ be denoted by $\lambda_i(\gamma)$, $i = 1 \dots N$, as functions of $\gamma \in (0, 1]$. It is clear that

$$|\lambda_i(\gamma)| = |\gamma\lambda_i(1) + (1-\gamma)| \le \gamma|\lambda_i(1)| + (1-\gamma).$$

Hence if $\rho(J(z)) < 1$ for any $z = e^{j\omega}$ for $\gamma = 1$, it will also hold for all $\gamma \in (0, 1]$. Therefore, it suffices to study the stability condition for $\gamma = 1$.

Let $\mu_i = d_i/(d_i + q_i)$ be the $i$th diagonal entry of matrix $M$. Let $\mu_{\max} := \max_i \mu_i$. Since the end-to-end queueing delay $q_i$ cannot be zero at equilibrium (otherwise the rate will be infinitely large), we have $q_i > 0$ and $\mu_{\max} < 1$. The following key lemma characterizes the eigenvalues of $J(z)$ with $\gamma = 1$.

LEMMA 4.1. *When $z = e^{j\omega}$ with $\omega \in [0, 2\pi)$ and $\gamma = 1$, the eigenvalues of $J(z)$ have the following properties:*

1. *There are $L$ zero eigenvalues with the corresponding eigenvectors being the columns of the matrix $M^{-1}DBR_b^T(z)$.*
2. *The nonzero eigenvalues have moduli less than 1 if $\tau_{\max} - \tau_{\min} < 1/4$, where $\tau_{\max} = \max_i \tau_i$ and $\tau_{\min} = \min_i \tau_i$ .*

**Proof:** At $\gamma = 1$, the matrix $J(z)$ is

$$M - DBR_b^T(z)(R_f(z)BR_b^T(z))^{-1}R_f(z)D^{-1}M.$$

It is easy to check that

$$J(z)M^{-1}DBR_b^T(z) = DBR_b^T(z) - DBR_b^T(z) = 0.$$

Since $M^{-1}DBR_b^T(z)$ has full column rank, it consists of $L$ linearly independent eigenvectors of $J(z)$ with corresponding eigenvalue 0. This proves the first assertion.

For the second assertion, suppose that $\lambda$ is an eigenvalue of $J(z)$ for a given $z$. Define matrix $A$ as

$$A := J(z) - \lambda I = (M - \lambda I) - DBR_b^T(z)(R_f(z)BR_b^T(z))^{-1}R_f(z)D^{-1}M$$

which is singular by definition. Recall the matrix inversion formula (see, e.g., [6])

$$(J + EHS)^{-1} = J^{-1} - J^{-1}E(H^{-1} + SJ^{-1}E)^{-1}SJ^{-1}.$$

If $J + EHS$ is singular, then either $J$ or $H^{-1} + SJ^{-1}E$ must be singular. We can let

$$J := M - \lambda I, \qquad E := -DBR_b^T(z)$$
$$H := (R_f(z)BR_b^T(z))^{-1}, \qquad S := R_f(z)D^{-1}M.$$

Since $A = J+EHS$ is singular, either $J = M-\lambda I$ or $H^{-1}+SJ^{-1}E$ is singular. The second term can be rewritten as $R_f(z)(B-M(M-\lambda I)^{-1}B)R_b^T(z)$.

**Case 1:** $M - \lambda I$ is singular. Since $M$ is diagonal, then

$$0 < \lambda = \frac{d_i}{d_i + q_i} = \mu_i \le \mu_{\max} < 1.$$

**Case 2:** $R_f(z)(B - M(M - \lambda I)^{-1}B)R_b^T(z)$ is singular.

It is clear that

$$B - M(M - \lambda I)^{-1}B = \operatorname{diag}\left((1 - \mu_i(\mu_i - \lambda)^{-1})\beta_i\right) = -\lambda \operatorname{diag}\left(\frac{\beta_i}{\mu_i - \lambda}\right)$$

where $\beta_i$ is the $ith$ diagonal entry of matrix $B$. Hence, $\lambda = 0$ is always an eigenvalue, as shown above. If $\lambda$ is nonzero, it has to be true that

$$\det\left(R_f(z)\operatorname{diag}\left(\frac{\beta_i}{\mu_i - \lambda}\right)R_b^T(z)\right) = 0. \tag{4.4}$$

When $z = e^{j\omega}$, we have $z^{-1} = \overline{z}$. Hence, equation (4.1) can be rewritten as

$$R_b^T(z) = \text{diag}(z^{-\tau_i})R_f^T(\overline{z}) = \text{diag}(z^{-\tau_i})R_f^*(z).$$

Substituting the above equation into (4.4) with $z = e^{j\omega}$ yields

$$\det\left(R_f(z)\text{diag}\left(\frac{e^{-j\omega\tau_i}\beta_i}{\mu_i - \lambda}\right)R_f^*(z)\right) = 0. \tag{4.5}$$

Therefore, the following formula is also zero:

$$e^{-j(\omega\tau_{\max}+\psi)}\det\left(R_f(z)\text{diag}\left(\frac{e^{j(\theta_i+\psi)}\beta_i}{\mu_i - \lambda}\right)R_f^*(z)\right) = 0$$

where $\theta_i = (\tau_{\max} - \tau_i)\omega$, and $\psi$ can be any value. When $\tau_{\max} - \tau_{\min} < 1/4$, we have for $\omega \in [0, 2\pi)$

$$0 \leq \theta_i = (\tau_{\max} - \tau_i)\omega < \pi/2.$$

Suppose that there is a solution such that $|\lambda| \geq 1$. Based on Lemma 4.2, which will be presented later, there exists a $\psi$ s.t. $\text{Im}(\text{diag}\left(e^{j(\theta_i+\psi)}\beta_i/(\mu_i - \lambda)\right))$ is a positive diagonal matrix. Therefore the imaginary part of matrix

$$R_f(z)\text{diag}\left(\frac{e^{j(\theta_i+\psi)}\beta_i}{\mu_i - \lambda}\right)R_f^*(z)$$

is positive definite, and the real part is symmetric. From Lemma 4.3 below, it has to be nonsingular. This contradicts the equation

$$\det\left(R_f(z)\text{diag}\left(\frac{e^{j(\theta_i+\psi)}\beta_i}{\mu_i - \lambda}\right)R_f^*(z)\right) = 0.$$

Hence, we have $|\lambda| < 1$. □

The proof of Theorem 4.1 will be complete after the next two lemmas.

LEMMA 4.2. *Suppose that* $0 < \mu_i < 1$ *and* $0 \leq \theta_i < \pi/2$*. If* $|\lambda| \geq 1$ *, there exists a* $\psi$ *such that*

$$Im\left(\frac{e^{j(\theta_i+\psi)}\beta_i}{\mu_i - \lambda}\right) > 0 \quad for \quad i = 1 \ldots N.$$

**Proof:** See Appendix B. □

LEMMA 4.3. *If the real part of a complex matrix is symmetric, and the imaginary part is positive definite, then the matrix is nonsingular.*

**Proof:** See Appendix C. □

**4.2. Global stability for one link without feedback delay.** In the absence of feedback delay, when there is only one link, the FAST TCP model can be simplified into

$$w_i(t+1) = \gamma\left(\frac{d_i w_i(t)}{d_i + q(t)} + \alpha_i\right) + (1-\gamma)w_i(t) \tag{4.6}$$

$$\sum_i \frac{w_i(t)}{d_i + q(t)} \le c \quad \text{with equality if } q(t) > 0 \tag{4.7}$$

where $q(t)$ is the queueing delay at the link (subscript is omitted). The main result of this section proves that the system (4.6)–(4.7) is globally asymptotically stable and converges to the equilibrium exponentially fast starting from any initial point.

THEOREM 4.2. *On a single link, FAST TCP converges exponentially to the equilibrium, in the absence of feedback delay.*

In the rest of this subsection, we prove the theorem in several steps.

The first result is that equality always holds in (4.7) after some finite number $K_1$ of steps, i.e., and $q(t) > 0$ for any $t > K_1$. Define the normalized congestion window sum as $Y(t) := \sum_i w_i(t)/d_i$. From (4.7), it is clear that $q(t) > 0$ if and only if $Y(t) > c$.

LEMMA 4.4. *There exists $K_1 > 0$ such that the following are true for all $t > K_1$:*

1. $q(t) > 0$.
2. $\nu(t+1) = (1-\gamma)\nu(t)$ *where* $\nu(t) := Y(t) - c - \sum_i \alpha_i/d_i$ .

**Proof:** If initially $q(t) = 0$, which also means $Y(t) \le c$, from (4.6) we have $Y(t+1) = Y(t) + \gamma \sum_i \alpha_i/d_i$, which linearly increases with $t$. Then $Y(t) > c$ after some finite steps. Therefore, there exists a $K_1$ such that $Y(t) > c$ and $q(t) > 0$ at $t = K_1$.

We will show that $Y(t) > c$ implies $Y(t+1) > c$. Hence $q(t) > 0$ for all $t > K_1$. Moreover, $\nu(t)$ converges exponentially to 0.

Suppose $Y(t) > c$. From $\sum_i w_i(t)/(d_i + q_i(t)) = c$, we have

$$\begin{aligned}
\nu(t+1) &= \sum_i \frac{w_i(t+1)}{d_i} - \sum_i \frac{\alpha_i}{d_i} - c \\
&= (1-\gamma)\sum_i \frac{w_i(t) - \alpha_i}{d_i} + \gamma \sum_i \frac{w_i(t)}{d_i + q(t)} - c \\
&= (1-\gamma)\left(\sum_i \frac{w_i(t)}{d_i} - c - \sum_i \frac{\alpha_i}{d_i}\right) \; = \; (1-\gamma)\;\nu(t).
\end{aligned}$$

This proves the second assertion. Moreover it implies

$$Y(t+1) = (1-\gamma)Y(t) + \gamma\left(\sum_i \frac{\alpha_i}{d_i} + c\right)$$

Hence, $Y(t) > c$ implies $Y(t+1) > c$ and $q(t+1) > 0$. This completes the proof. ∎

For the rest of this subsection, we pick a fixed $\epsilon$ with $0 < \epsilon < \sum_i \alpha_i/d_i$. Define

$$q_{\min} := \frac{d_{\min}}{c}\left(\sum_i \frac{\alpha_i}{d_i} - \epsilon\right) \quad \text{and} \quad q_{\max} := \frac{d_{\max}}{c}\left(\sum_i \frac{\alpha_i}{d_i} + \epsilon\right)$$

where $d_{\min} := \min_i d_i$ and $d_{\max} := \max_i d_i$.

Then $q(t)$ is bounded by these two values after finite steps.

LEMMA 4.5. *There exists a positive $K_2$ such that $q_{\min} \leq q(t) \leq q_{\max}$ for any $t \geq K_2$.*

**Proof:** From Lemma 4.4, after finite steps $K_1$, $\nu(t+1) = (1-\gamma)\nu(t)$. Therefore, there exists a $K_2$ such that $|\nu(t)| < \epsilon$ for all $t \geq K_2$. It implies

$$\begin{aligned}\sum_i \frac{\alpha_i}{d_i} &< \sum_i \frac{w_i(t)}{d_i} - c + \epsilon = \sum_i \left(\frac{w_i(t)}{d_i} - \frac{w_i(t)}{d_i + q(t)}\right) + \epsilon \\ &\leq \sum_i \frac{q(t)w_i(t)}{d_{\min}(d_i + q(t))} + \epsilon = \frac{q(t)c}{d_{\min}} + \epsilon.\end{aligned}$$

Therefore

$$q(t) \geq \frac{d_{\min}}{c}\left(\sum_i \frac{\alpha_i}{d_i} - \epsilon\right) = q_{\min}.$$

The proof for $q_{\max}$ is the same. ∎

Define $\mu_i(t) := d_i/(d_i+q(t))$, and $\mu_{\max} := \max_i d_i/(d_i+q_{\min})$, $\mu_{\min} := \min_i d_i/(d_i+q_{\max})$. Based on Lemma 4.5, we have $0 < \mu_{\min} \leq \mu_i(t) \leq \mu_{\max} < 1$ for any $t \geq K_2$. Define

$$\eta_i(t) := \frac{w_i(t) - \alpha_i}{\alpha_i d_i} - \frac{1}{q(t)} \tag{4.8}$$

and denote $\eta_{\max}(t) := \max_i \eta_i(t)$, $\eta_{\min}(t) := \min_i \eta_i(t)$. We will show that the window update for source $i$ is proportional to $\eta_i(t)$, and the system is at equilibrium if and only if all $\eta_i(t)$ are zero. The next lemma gives bounds on $\eta_i(t)$.

LEMMA 4.6. *There exist two positive numbers $\delta_1$ and $\delta_2$ such that for all $t \geq K_2$*

$$\eta_{\max}(t) > -\delta_1(1-\gamma)^t \text{ and } \eta_{\min}(t) < \delta_2(1-\gamma)^t.$$

**Proof:** From (4.8), it is easy to check that $Y(t+1) - Y(t) = -\gamma\nu(t)$. By Lemma 4.4, when $t \geq K_2$ we have

$$Y(t+1) - Y(t) = -\gamma\nu(t) \leq \gamma(1-\gamma)^{t-K_2}|\nu(K_2)| = \kappa(1-\gamma)^t \tag{4.9}$$

where $\kappa := \gamma(1-\gamma)^{-K_2}|\nu(K_2)|$.

The update of source $i$'s congestion window is

$$\begin{aligned} w_i(t+1) - w_i(t) &= \gamma\left(\frac{d_i w_i(t)}{d_i + q(t)} + \alpha_i - w_i(t)\right) \\ &= \frac{\gamma q(t)}{d_i + q(t)}\left(\frac{\alpha_i d_i}{q(t)} - (w_i(t) - \alpha_i)\right) \\ &= -\frac{\gamma \alpha_i d_i q(t)}{d_i + q(t)}\left(\frac{w_i(t) - \alpha_i}{\alpha_i d_i} - \frac{1}{q(t)}\right) \\ &= -\gamma \alpha_i q(t) \mu_i(t) \eta_i(t). \end{aligned}$$

Choose $\delta_1$ large enough such that $\delta_1 N \gamma \alpha_{\min} q_{\min} \mu_{\min}/d_{\max} > \kappa$ where $\alpha_{\min} := \min_i \alpha_i$.

We now prove $\eta_{\max}(t) > -\delta_1(1-\gamma)^t$ for all $t \geq K_2$ by contradiction. Suppose that there is a time $t \geq K_2$ such that $\eta_{\max}(t) \leq -\delta_1(1-\gamma)^t$. Then all the $\eta_i(t)$ are negative, which implies

$$\begin{aligned} Y(t+1) - Y(t) &= \sum_i (w_i(t+1) - w_i(t))/d_i \\ &= \sum_i -\gamma \alpha_i q(t) \mu_i(t) \eta_i(t)/d_i \\ &\geq N(-\eta_{\max}(t)) \gamma \alpha_{\min} q_{\min} \mu_{\min}/d_{\max} \\ &\geq \delta_1 N (1-\gamma)^t \gamma \alpha_{\min} q_{\min} \mu_{\min}/d_{\max} > \kappa(1-\gamma)^t. \end{aligned}$$

This contradicts equation (4.9) and proves the claim. The proof for $\eta_{\min}(t)$ is similar. ☐

Define $L(t)$ as:

$$L(t) := \eta_{\max}(t) - \eta_{\min}(t). \tag{4.10}$$

The following lemma implies that the difference between different $\eta_i(t)$ goes to zero exponentially fast.

LEMMA 4.7. *There are two positive numbers $\delta_3$ and $\delta_4$, such that for $t \geq K_2$ we have*

1. $L(t) \geq 0$.
2. $L(t+1) \leq (1-\gamma+\gamma\mu_{\max})L(t) + \delta_3(1-\gamma)^t$.
3. $L(t) \leq \delta_4(1-\gamma+\gamma\mu_{\max})^t$.

**Proof:** See Appendix D. ☐

LEMMA 4.8. *Both $\eta_{\max}(t)$ and $\eta_{\min}(t)$ exponentially converge to zero.*

**Proof:** When $t \geq K_2$, combining Lemma 4.6 and Lemma 4.7 yields bounds for $\eta_{\max}(t)$:

$$-\delta_1(1-\gamma)^t < \eta_{\max}(t) = L(t) + \eta_{min}(t) \leq \delta_4(1-\gamma+\gamma\mu_{\max})^t + \delta_2(1-\gamma)^t.$$

Since both the upper and lower bounds of $\eta_{\max}(t)$ converge to zero exponentially fast, $\eta_{\max}(t)$ exponentially goes to zero. The proof for $\eta_{\min}(t)$ is similar. ☐

**Proof of Theorem 4.2:** The system is at equilibrium if and only if $w_i(t) = w_i(t+1)$ for all $i$. This is equivalent to $\eta_i(t) = 0$ for all $i$ because of the equation proved in Lemma 4.6:

$$w_i(t+1) - w_i(t) = -\gamma\alpha_i q(t)\mu_i(t)\eta_i(t).$$

Since both $\eta_{\max}(t)$ and $\eta_{\min}(t)$ converge to zero exponentially from any initial value, the system converges to the equilibrium defined by $\eta_i(t) = 0$ globally. □

**5. Conclusion.** we have proved that FAST TCP is globally asymptotically stable in a general network when there is no feedback delay using the traditional continuous-time model. When feedback delays are present, a sufficient condition is provided for local stability for general networks. Using a discrete-time model that captures the stabilizing effect of self-clocking, we have proved that FAST TCP is locally asymptotically stable in a general network as long as all flows have the same feedback delay, no matter how large it is. We have also proved that FAST TCP is globally asymptotically stable at a single link in the absence of feedback delay.

This work can be extended in several ways. First, the condition for local asymptotic stability derived appears more restrictive than our experiments suggest. Moreover, we have also found scenarios where predictions of the discrete-time model disagree with experiment. These discrepancies should be clarified. Second, it will be interesting to extend the global stability analysis to general networks with feedback delays. Finally, the new model and the analysis techniques here can be applied to analyze other congestion control algorithms.

## APPENDIX

**A. Proof of Lemma 3.1.** The FAST TCP model (2.1, 2.3, 2.5, 2.2) and (2.6) can be linearized into

$$\delta q_i(t) = \sum_l R_{li}\delta p_l(t - \tau_{li}^b), \quad \delta y_l(t) = \sum_i R_{li}\delta x_i(t - \tau_{li}^f)$$

$$\delta \dot{w}_i(t) = -\gamma\left(\frac{q_i \delta w_i(t)}{d_i + q_i} + \frac{d_i w_i \delta q_i(t)}{(d_i + q_i)^2}\right), \quad \delta \dot{p}_l(t) = \delta y_l(t)/c_l$$

$$\delta x_i(t) = \frac{\delta w_i(t)}{d_i + q_i} - \frac{w_i \delta q_i(t)}{(d_i + q_i)^2}$$

where $w_i$ and $q_i$ are equilibrium values. Since$\tau_i = \tau_{li}^f + \tau_{li}^b = T_i = d_i + q_i$ for all links $l$ on the path of source $i$, the following equation holds

$$R_b^T(s) = \text{diag}(e^{-T_i s}) R_f^T(-s). \tag{A.1}$$

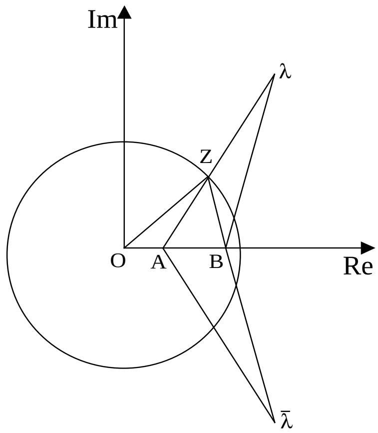



FIG. 5. *Illustration of Lemma 4.2.*

The Laplace transform of the linearized system in matrix form is

$$\begin{cases} \delta q(s) &= R_b(s)^T \delta p(s) \\ \delta x(s) &= D_3 \delta w(s) - D_4 \delta q(s) \\ s\delta w(s) &= -\gamma \left( D_2 D_3 \delta w(s) + D_1 D_4 \delta q(s) \right) \\ \delta y(s) &= R_f(s) \delta x(s) \\ s\delta p(s) &= D \delta y(s) \end{cases}$$

where the diagonal matrices are

$$D := \operatorname{diag}\left(\frac{1}{c_l}\right)$$
$$D_1 := \operatorname{diag}(d_i), \quad D_2 := \operatorname{diag}(q_i)$$
$$D_3 := \operatorname{diag}\left(\frac{1}{d_i + q_i}\right) \quad D_4 := \operatorname{diag}\left(\frac{w_i}{(d_i + q_i)^2}\right).$$

The open-loop transfer function from $\delta p(s)$ to $\delta p(s)$ can be derived based on the above equations as

$$\frac{1}{s} D R_f(s) \left( \gamma D_1 D_3 (sI + \gamma D_2 D_3)^{-1} + I \right) D_4 R_b^T(s).$$

By using the fact that $T_i = d_i + q_i$, $x_i = w_i / T_i$ and (A.1), we can simplify the open loop transfer function $L(s)$ into (3.8). □

**B. Proof of Lemma 4.2. Proof:** Consider the complex plane in Figure 5. Let the points $A$, $B$, and $\lambda$ represent the value of $\mu_{\min}$, $\mu_{\max}$, and $\lambda$, respectively. $Z$ is the intersection of segment $A\lambda$ and the unit circle, and $\overline{\lambda}$ stands for the complex conjugate of $\lambda$.

Let $\phi_i \in [0, 2\pi)$ be the phase of $1/(\mu_i - \lambda)$. Let $\phi_{\max} := \max_i \phi_i$ and $\phi_{\min} := \min_i \phi_i$. Clearly, $\phi_i \in [0, \pi)$ if $\mathrm{Im}(\lambda) > 0$, and $\phi_i \in (\pi, 2\pi)$ otherwise. Hence $0 \le \phi_{\max} - \phi_{\min} \le \pi$. Since every $\mu_i$ is in the range $[\mu_{\min}, \mu_{\max}]$, it is easy to check that every $\phi_i$ is in the range formed by the phases of $1/(\mu_{\min} - \lambda)$ and $1/(\mu_{\max} - \lambda)$. This implies

$$\begin{aligned} \phi_{\max} - \phi_{\min} &\le \left| \angle \frac{1}{\mu_{\min} - \lambda} - \angle \frac{1}{\mu_{\max} - \lambda} \right| \\ &= \angle A\bar{\lambda}B = \angle A\lambda B < \angle OZB \ < \ \pi/2. \end{aligned}$$

Let $\epsilon > 0$ be small enough such that $\phi_{\max} - \phi_{\min} < \pi/2 - \epsilon$. Choosing $\psi = -\phi_{\min} + \epsilon$ gives

$$\begin{aligned} \angle \frac{e^{j(\psi+\theta_i)}\beta_i}{\mu_i - \lambda} &= \phi_i + \psi + \theta_i \\ &= \phi_i - \phi_{\min} + \epsilon + \theta_i \quad \text{(greater than 0)} \\ &< \phi_{\max} - \phi_{\min} + \epsilon + \pi/2 < \pi. \end{aligned}$$

The fact that its phase is in $(0, \pi)$ implies that

$$Im\left(\frac{e^{j(\psi+\theta_i)}\beta_i}{\mu_i - \lambda}\right) > 0.$$

□

**C. Proof of Lemma 4.3.** Suppose that $A := A_r + jA_i$ where $A_r = A_r^T$ and $A_i$ is positive definite. If $A$ is singular, there exists a nonzero vector $v := v_r + jv_i$ such that $Av = 0$. Then $A_r v_r = A_i v_i$ and $A_i v_r = -A_r v_i$. Since $A_i > 0$ and $A_r = A_r^T$, we have

$$\begin{aligned} 0 \ < \ v_r^T A_i v_r &= -v_r^T A_r v_i = -v_r^T A_r^T v_i \\ &= -v_i^T A_r v_r = -v_i^T A_i v_i \ < \ 0 \end{aligned}$$

a contradiction. Hence $A$ is nonsingular. □

**D. Proof of Lemma 4.7.** It is obvious that $L(t) \ge 0$ because of its definition in (4.10). We start with the update of $\eta_i(t)$

$$\begin{aligned} \eta_i(t+1) - \eta_i(t) &= \frac{w_i(t+1) - w_i(t)}{\alpha_i d_i} - \frac{1}{q(t+1)} + \frac{1}{q(t)} \\ &= -\frac{\gamma \alpha_i q(t) \mu_i(t) \eta_i(t)}{\alpha_i d_i} - \frac{1}{q(t+1)} + \frac{1}{q(t)} \\ &= -\frac{\gamma q(t) \eta_i(t)}{d_i + q(t)} - \frac{1}{q(t+1)} + \frac{1}{q(t)} \\ &= -\gamma(1 - \mu_i(t))\eta_i(t) - \frac{1}{q(t+1)} + \frac{1}{q(t)}. \end{aligned}$$

For simplicity, we let $a_i(t) := 1-\gamma+\gamma\mu_i(t)$ and denote $a_{\max} := 1-\gamma+\gamma\mu_{\max}$, then $a_i(t) \le a_{\max}$. This definition simplifies the above equation into

$$\eta_i(t+1) = a_i(t)\eta_i(t) - \frac{1}{q(t+1)} + \frac{1}{q(t)}. \tag{D.1}$$

By comparing equation (D.1) for source $i$ and $j$, we obtain

$$\eta_i(t+1) - \eta_j(t+1) = a_i(t)\eta_i(t) - a_j(t)\eta_j(t). \tag{D.2}$$

Without loss of generality, suppose that at time $t+1$, the largest and smallest values of $\eta$ are achieved at sources $i$ and $j$, respectively. This assumption implies

$$L(t+1) = \eta_i(t+1) - \eta_j(t+1).$$

The upper bound of $L(t+1)$ is derived by considering the following three cases separately.
**Case 1:** $\eta_i(t)$ and $\eta_j(t)$ have different signs. It is easy to see that

$$\begin{aligned} L(t+1) &= a_i(t)\eta_i(t) - a_j(t)\eta_j(t) \le a_{\max}(\eta_i(t) - \eta_j(t)) \\ &= a_{\max}(\eta_{\max}(t) - \eta_{\min}(t)) = a_{\max}L(t). \end{aligned}$$

**Case 2:** Both $\eta_i(t)$ and $\eta_j(t)$ are positive. It yields

$$\begin{aligned} L(t+1) &= a_i(t)\eta_i(t) - a_j(t)\eta_j(t) \le a_{\max}\eta_{\max}(t) \\ &= a_{\max}L(t) + a_{\max}\eta_{\min}(t) \le a_{\max}L(t) + a_{\max}\delta_2(1-\gamma)^t \\ &\le a_{\max}L(t) + \delta_3(1-\gamma)^t \end{aligned}$$

as long as $\delta_3 \ge a_{\max}\delta_2$.
**Case 3:** Both $\eta_i(t)$ and $\eta_j(t)$ are negative. The proof is similar to that for Case 2.

Summarizing all the above cases, we have proved $L(t+1) \le a_{\max}L(t) + \delta_3(1-\gamma)^t$ for all $t \ge K_2$. Denote $b := 1-\gamma$. Then $1 > a_{\max} > b \ge 0$. For any $t \ge K_2$, an upper bound of $L(t)$ is

$$\begin{aligned} L(t) &\le a_{\max}L(t-1) + \delta_3 b^{t-1} \\ &\le a_{\max}^{t-K_2}L(K_2) + \delta_3(b^{t-1} + b^{t-2}a_{\max} + \cdots + b^{K_2}a_{\max}^{t-K_2-1}) \\ &= \left(a_{\max}^{-K_2}L(K_2) - \delta_3\frac{b^{K_2}a_{\max}^{-K_2}}{b-a_{\max}}\right)a_{\max}^t + \frac{\delta_3}{b-a_{\max}}b^t. \end{aligned}$$

Note that the coefficient of $b^t$ is negative. By choosing $\delta_4$ as the coefficient of $a_{\max}^t$, we get

$$L(t) \le \delta_4 a_{\max}^t = \delta_4(1-\gamma+\gamma\mu_{\max})^t.$$

□

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