A Comparison of Time-Sharing, DPC, and Beamforming for MIMO Broadcast Channels with Many Users *†

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Abstract

In this paper, we derive the scaling laws of the sum rate for fading MIMO Gaussian broadcast channels using time-sharing to the strongest user, dirty paper coding (DPC), and beamforming when the number of users (receivers) n is large. Throughout the paper, we assume a fix average transmit power and consider a block fading Rayleigh channel. First, we show that for a system with M transmit antennas and users equipped with N antennas, the sum rate scales like $M \log\log nN$ for DPC and beamforming when M is fixed and for any N (either growing to infinity or not). On the other hand, when both M and N are fixed, the sum rate of time-sharing to the strongest user scales like $\min(M,N)\log\log n$. Therefore, the asymptotic gain of DPC over time-sharing for the sum rate is $\frac{M}{\min(M,N)}$ when M and N are fixed. It is also shown that if M grows as $\log n$, the sum rate of DPC and beamforming will grow linearly in M, but with different constant multiplicative factors. In this region, the sum rate capacity of time-sharing scales like $N \log\log n$.

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1 Introduction

Using multiple antennas has been shown to increase the capacity of a point-to-point communication link linearly with $\min(M, N)$ for large signal to noise ratios, where M and N are the number of transmit and receiver antennas, respectively [1]. Recently, there has been a large amount of interest in the area of MIMO multiuser systems, more specifically, the capacity of MIMO Gaussian broadcast channels [2].

An example of a multiuser system is the broadcast channel which resembles the down-link communication in cellular systems. It is known that the single antenna broadcast channel is degraded so its capacity region is known and achieved by superposition coding [3, 4]. Furthermore, if the users are homogeneous and the transmitter has full channel state information (CSI), using time division multiplexing and transmitting to the best user maximizes the sum rate capacity (we call this scheduling time-sharing). However, if the transmitter and the receivers have full CSI, the MIMO broadcast channel is not degraded. In [6, 7, 8, 9], it is proved that the sum rate capacity with full CSI in both the transmitter and all the receivers is achieved by using dirty paper coding (DPC). In [5], it is further shown that the capacity region is in fact achieved by DPC.

On the other hand, traditionally beamforming has been used for the down-link scheduling in MIMO broadcast systems as a heuristic method to reduce the interference in the system. As pointed out in [11, 12, 13], even though the sum rate capacity of MIMO BC using DPC can be stated as a convex problem using duality, the sum rate (or throughput) achieved by optimal beamforming cannot be written as a convex optimization problem, and therefore numerically comparing the throughput of DPC and beamforming is computationally intensive, especially for a large number of users.

In this paper we investigate the scaling laws of the sum rate capacity of Gaussian MIMO broadcast channels with many users n using time-sharing, DPC, and beamforming and when the transmitter has M antennas and each receiver is equipped with N antennas. Previously, in [14, 12], asymptotic results for the sum rate of DPC and beamforming have been derived when n and M have the same growth rate. Furthermore, in [16], the asymptotic behavior of the throughput for DPC and time-sharing are obtained for large signal to noise ratios and large M when the other parameters of the system are fixed. However, motivated by a cellular system with large number of users (say 100) and having $M \leq 5$ which is about $\log n$, we consider a different region in which n is large and M is either fixed or growing to infinity with much less pace, i.e., logarithmically with n (see also [15]). This work also generalizes a result in [17] where the scaling laws of the sum rate of DPC is derived for the case where M is fixed and $N = 1$. Furthermore, we use the sum rate of the random beamforming proposed in [17] as a lower bound for the sum rate of DPC and beamforming which turns out to be tight for the regions considered in this paper.

In [16], it is conjectured that for MIMO BC the ratio of the sum rate using DPC over that of time-sharing is bounded by $\min\left\{\frac{M}{\min(M,N)}, n\right\}$ where M, N are the number of transmit/receive antennas and n denotes the number of users. In fact, we prove that the aforementioned ratio for a Rayleigh fading channel *is equal to* $\frac{M}{\min(M,N)}$ for large number of users and when M and N are fixed.

This paper is organized as follows: Section 2 introduces our notation and the channel model. Section 3, 4 and 5 deal with scaling laws of the sum rate for time-sharing, DPC, and beamforming, respectively. Section 6 compares the scaling laws for different scheduling schemes and Section 7 concludes the paper.

2 System Model

We consider a Gaussian broadcast channel with n homogeneous users, a transmitter with M antennas and receivers equipped with N antennas. We also assume a block fading model for the channel with coherence interval of T, so that the channel remains constant for T channel uses. Therefore we may write the received vector at the i'th receiver as,

$$Y_i = H_i S + N_i, \qquad i = 1, \ldots, n, \tag{1}$$

where H_i ($N \times M$) represents the channel, S ($M \times 1$) is the transmit symbol, N_i is the $N \times 1$ noise vector. Both H_i's and N_i's have independently and identically distributed (i.i.d.) complex Gaussian distribution with zero mean and the variance of one, $CN(0,1)$. Furthermore, the average power constraint of the input signal implies that $tr\left\{E(SS^*)\right\} \leq P$, where P is the total average transmit power which is assumed to be fixed throughout the paper. We further assume that the base station is subject to short term power constraint, i.e., the base station should satisfy the power constraint for each fading state [10].

Throughout the paper, we use $f(n) = O(g(n))$ to denote that $\lim_{n\to\infty} |\frac{f(n)}{g(n)}| \leq \alpha$ where α is a positive constant independent of n. Similarly, $f(n) = o(g(n))$ denotes that the limit of the ratio of $f(n)$ and $g(n)$ tends to zero as n grows.

3 Scaling Laws of Time-Sharing

In a single antenna broadcast system with full CSI in the transmitter, the sum-rate capacity can be achieved by using time-sharing and sending to the user with the largest capacity. However, for a multi-antenna broadcast system with full CSI in the transmitter, this is not the case. In this section, we derive the scaling laws for the sum rate of multi-antenna broadcast channel using time-sharing (to the strongest user) for large number of users.

It is clear that, by only sending to the strongest user, the sum-rate (denoted by $E\{R_{ts}\}$) can be written as [16, 11],

$$E\{R_{ts}\} = E\left\{\max_{i=1,\ldots,n} C(H_i, P)\right\} = E\left\{\max_{i=1,\ldots,n} \max_{P_i \geq 0, tr(P_i) \leq P} \log\det\left(I + H_i P_i H_i^*\right)\right\}. \quad (2)$$

where $C(H_i, P)$ is the capacity of the link between the transmitter and the i'th receiver with the channel matrix H_i, and P_i $(M \times M)$ is the optimal covariance matrix of the transmitted signal. Lemma 1 considers the case where M and N are fixed and n grows to infinity. Lemma 2 presents the result for the case that M is also growing to infinity but logarithmically with n.

Lemma 1. *For M, N, and P fixed, we have,*

$$\lim_{n\to\infty} \frac{E\{R_{ts}\}}{\min(M, N)\log\log n} = 1 \quad (3)$$

Proof: First, we assume $M \geq N$, the case where $N > M$ can be analyzed similarly. Using the inequality $\det(A) \leq \left(\frac{tr(A)}{N}\right)^N$ where A is an $N \times N$ matrix, we can bound $C(H_i, P)$ as,

$$C(H_i, P) \leq N \log\left(1 + \frac{1}{N} tr(H_i P_i H_i^*)\right) \quad (4)$$

Defining $H_i = [h_1^i \ \ldots \ h_M^i]$ (where h_j^i $(N \times 1)$ is the j'th column of H_i), we can use the inequality

$$tr(H_i P_i H_i^*) \leq \max_{1\leq j\leq M} h_j^{i*} h_j^i tr(P_i). \quad (5)$$

We can also find a lower bound by assigning equal power to N transmit antennas instead of M. Clearly, this leads to,

$$C(H_i, P) \geq N \log \left(1 + \frac{P}{N} \lambda_{\min}(H_i^{'} H_i^{'*})\right) \quad (6)$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of its argument and the matrix $H_i^{'}$ ($N \times N$) is a truncated version of H_i ($N \times M$) by omitting $M - N$ columns of H_i. Using (4) and (6), we may write the expected sum rate of the time-sharing scheme as,

$$E\left\{N \log \left(1 + \frac{P}{N^2} \max_{1 \leq i \leq n} \lambda_{\min}(H_i^{'} H_i^{'*})\right)\right\} \leq E\{R_{ts}\} \leq E\left\{\max_{i=1,\ldots,n} N \log \left(1 + \frac{P}{N} \max_{1 \leq j \leq M} h_j^{i*} h_j^i\right)\right\}. \quad (7)$$

It is worth noting that $h_j^{i*} h_j^i$'s have $\chi^2(2N)$ distribution. In [19], it is shown that $N\lambda_{\min}(H_i^{'} H_i^{'*})$ is exponentially distributed ([19], Theorem 5.5, p.62). Therefore, using the results in extreme value theory (see Appendix A of [17] and [20]), it can be shown that,

$$\Pr\left\{\log nM + (N-2) \log \log nM + O(\log \log \log n) \leq \max_{1 \leq i \leq nM, 1 \leq j \leq M} h_j^{i*} h_j^i \right.$$
$$\left. \leq \log nM + N \log \log nM + O(\log \log \log n)\right\} = 1 - O\left(\frac{1}{\log n}\right) \quad (8)$$

where $h_j^{i*} h_j^i$ for $i = 1, \ldots, n$ and $j = 1, \ldots, M$, are i.i.d. and have $\chi^2(2N)$ distribution.

Noting that $N\lambda_{\min}(H_i^{'} H_i^{'*})$ has $\chi^2(2)$ distribution, we can similarly prove that,

$$\Pr\left\{\log n - \log \log n \leq \max_{1 \leq i \leq n} N\lambda_{\min}(H_i^{'} H_i^{'*}) \leq \log n + \log \log n\right\} = 1 - O\left(\frac{1}{\log n}\right). \quad (9)$$

Defining $A = \log nM + N \log \log nM + O(\log \log \log n)$, we can now obtain an upper bound for $E\{R_{ts}\}$ as,

$$\begin{aligned} E\{R_{ts}\} &\leq E\left\{R_{ts}|_{\max_{1 \leq i \leq n, 1 \leq j \leq M}} h_j^{i*} h_j^i \leq A\right\} \Pr\left\{\max_{1 \leq i \leq n, 1 \leq j \leq M} h_j^{i*} h_j^i \leq A\right\} \\ &\quad + \int_A^\infty N \log(1 + \frac{P}{N} x) n f_N(x) (1 - F_N(x))^n \\ &\leq N \log\left(1 + \frac{P}{N} \log n\right) + \int_A^\infty N \log(1 + \frac{P}{N} x) n f_N(x) \end{aligned} \quad (10)$$

where $f_N(x)$ and $F_N(x)$ are the PDF and CDF of the a $\chi^2(2N)$ random variable. Denoting the second term in the right hand side of (10) by $G(n)$, it is straightforward to show that $\lim_{n \to \infty} \frac{G(n)}{\log \log n} = 0$ using L'Hopital's rule. Therefore, we can state that

$$E\{R_{ts}\} \leq N \log\left(1 + \frac{P}{N} \log n\right) + o(\log \log n). \quad (11)$$

Similarly, a lower bound can be written as

$$
\begin{aligned}
E\{R_{ts}\} &\geq E\left\{R_{ts} | \log n \leq \max N\lambda_{\min}(H_i^{'}H_i^{'*}) \leq \log n + \log\log n\right\} \times \\
&\quad \Pr\left\{\log n \leq \max N\lambda_{\min}(H_i^{'}H_i^{'*}) \leq \log n + \log\log n\right\} \\
&= N\log\left(1+\frac{P}{N^2}\log n\right)\left(1-\frac{1}{\log n}\right). \qquad (12)
\end{aligned}
$$

Eqs. (11) and (12) complete the proof and lead to (3). ∎

Lemma 2. *For $M = \beta_1 \log n$ where β_1 is a constant independent of n and for N and P fixed, we have,*

$$\lim_{n\to\infty} \frac{E\{R_{ts}\}}{N\log\log n} = 1 \qquad (13)$$

Proof: The proof is along the same line as the proof of Lemma 1. Clearly, assuming $M \geq N$, Eq. (7) holds for any M, N and P. Furthermore, the derivation of the upper and lower bounds in Lemma 1 was based on the distribution of $\max_{1\leq i\leq n, 1\leq j\leq M} h_j^{i*}h_j^i$ or $\max_{1\leq i\leq n} N\lambda_{\min}H_i^{'}H_i^{'*}$ where $h_j^{i*}h_j^i$'s and $H_iH_i^*$ have $\chi^2(2N)$ or $\chi^2(2)$ distributions for any M and N, respectively. As N is assumed fixed, both bounds both hold and therefore $E\{R_{ts}\}$ is growing like $N\log\log n$. ∎

4 Scaling Laws of DPC

In [17], assuming a transmitter with M antennas, single antenna receivers and total average transmit power of M, it is proved that the sum rate capacity of DPC scales like $M\log\log n$ for large values of n and when M is fixed. In this section, we first generalize this result to the case of having multiple antenna users, i.e., $N \geq 1$, and when the average total transmit power is fixed. Again, we further look into the scaling laws of the sum rate when M is also going to infinity logarithmically with n, i.e. with a much lower pace than n.

In the following Lemma, we show that when M is fixed the sum rate scales like $M\log\log nN$ as n grows to infinity and for any N no matter whether N grows to infinity or not.

Lemma 3. *For M and P fixed and any N, we have,*

$$\lim_{n\to\infty} \frac{E\{R_{DPC}\}}{M\log\log nN} = 1. \qquad (14)$$

Proof: The sum rate in MIMO BC channel has been recently addressed by several authors [6, 7, 8]. Using the duality between the broadcast channel and multiple access channel (MAC), the sum rate of MIMO BC, $E\{R_{DPC}\}$ is equal to [7, 8],

$$E\{R_{DPC}\} = E\left\{\max_{\{P_1\geq 0,\ldots,P_n\geq 0,\sum tr(P_i)\leq P\}} \log\det\left(I+\sum_{i=1}^{n} H_i^* P_i H_i\right)\right\} \tag{15}$$

where H_i are $N \times M$ channel matrices with i.i.d. $CN(0,1)$ distributions, P_i ($N \times N$) is the optimal power scheduling, and P is the total transmit power.

Using the inequality $\det(A) \leq \left(\frac{tr(A)}{M}\right)^M$ where A is an $M \times M$ matrix, we can write (15) as,

$$E\{R_{DPC}\} \leq ME\left\{\max_{\{P_1,\ldots,P_n,\sum_{i=1}^{n} tr(P_i)\leq P\}} \log\left(1+\frac{\sum tr(H_i^* P_i H_i)}{M}\right)\right\} \tag{16}$$

Denoting the matrix $H_i^* = \begin{bmatrix} g_1^{i*} & \ldots & g_N^{i*} \end{bmatrix}$ (where g_j^i ($1 \times M$) is the j'th row of H_i), we can state the following inequality,

$$tr(H_i^* P_i H_i) \leq \max_{1\leq j\leq N} g_j^i g_j^{i*} tr(P_i). \tag{17}$$

Using (17) and (16), we obtain,

$$\begin{aligned} E\{R_{DPC}\} &\leq ME\left\{\max_{\{P_1,\ldots,P_n,\sum tr(P_i)\leq P\}} \log\left(1+\frac{\sum_{i=1}^{n} \max\limits_{1\leq j\leq N} g_j^i g_j^{i*} tr(P_i)}{M}\right)\right\} \\ &\leq ME\left\{\max_{\{P_1,\ldots,P_n,\sum tr(P_i)\leq P\}} \log\left(1+\frac{\max\limits_{1\leq k\leq n}\max\limits_{1\leq j\leq N} g_j^k g_j^{k*} \sum_{i=1}^{n} tr(P_i)}{M}\right)\right\} \\ &= ME\left\{\log\left(1+\frac{P}{M}\max_{1\leq i\leq nN}\kappa_i\right)\right\}. \end{aligned} \tag{18}$$

where κ_i's are i.i.d. random variables with $\chi^2(2M)$ distribution. Eq. (8) states that with high probability the maximum of nN i.i.d. random variables with $\chi^2(2M)$ distribution behaves like $\log nN + O(\log\log n)$ [17] (see also Eq. (9)). Therefore similar to the argument in (10), we may write

$$E\{R_{DPC}\} \leq M\log\left(1+P\log nN\right) + o(\log\log n). \tag{19}$$

To prove that $M\log\log nN$ is achievable, we use the scheme proposed in [17] with partial side information that achieves $M\log\log nN$ when M is fixed. It is worth noting that in [17], the average transmit power was M ($P = M$), however, since M is fixed, it is easy to see that changing the average total transmit power from M to P (another constant) does not affect the scaling law of the sum rate. Therefore,

$$E\{R_{DPC}\} \geq M \log\log nN + O(\log\log\log n). \tag{20}$$

Eq. (19) and (20) complete the proof of the lemma. ■

The next Lemma considers a different region in which M is also logarithmically increasing with n.

Lemma 4. *For $M = \beta \log n$ and fixed N, P and β, we have,*

$$\lim_{n\to\infty} \frac{E\{R_{DPC}\}}{M} = \gamma \tag{21}$$

where γ is a constant independent of n. Furthermore, we can bound γ by $\gamma \leq \log(1+\alpha)$ where α is the unique solution to $\alpha - \beta\log\alpha = 1 + \beta - \beta\log\beta$.

Proof: As we stated in the proof of Lemma 1 (i.e. Eq. (18)), we can write the following upper bound for the sum rate capacity for any n and M,

$$E\{R_{DPC}\} \leq ME\left\{\log\left(1 + \frac{P}{M}\max_{1\leq i\leq nN}\kappa_i\right)\right\}. \tag{22}$$

where κ_i's are i.i.d. $\chi^2(2M)$ random variables, i.e., $M = \beta\log n$. The only difference here is that M is also a function of n and is going to infinity. In Appendix A, we prove that,

$$\Pr\left\{\max_{1\leq i\leq nN}\kappa_i \leq \alpha\log n + O(\log\log n)\right\} = 1 - O\left(\frac{1}{\log n}\right). \tag{23}$$

The upper bound in the Lemma follows by using the same technique as in Lemma 1.

In order to find a lower bound, we may use any suboptimal scheduling and show that its sum-rate is bigger than αM where α is a constant independent of n. This is in fact done in [17] using a random beamforming method. This completes the proof of the Lemma. ■

5 Scaling Laws of Beamforming

Traditionally, transmit beamforming has been used as a method in multiple transmit antenna systems to suppress the interference in the receivers. In this case, the transmitted signal is $\sum_{m=1}^{M}\phi_m s_m$ where

ϕ_m's are beams carrying information symbols s_m for M different users. The weight vectors ϕ_m should be chosen such that total transmit power is less than P and the sum-rate is maximized. We denote the resulting sum rate by $E\{R_{BF}\}$.

Clearly the sum rate of DPC is an upper bound for the sum-rate achieved by any beamforming scheme. In order to find a lower bound on the sum-rate of the optimal beamforming (that maximizes the sum-rate), we can use a random beamforming scheme as in [17], in which ϕ_m's are random orthonormal vectors, to find a lower bound for the throughput of the beamforming (see also [18]). It is shown in [17], that under average transmit power of M ($P = M$), the sum rate of random beamforming scales like $M \log\log n$ and $M\log(1+c)$ as M is fixed or logarithmically increases with n, respectively. In particular, when M, N, and the total average transmit power are fixed, it is shown that

$$\lim_{n\to\infty} \frac{E\{R_{BF}\}}{M \log\log nN} = 1, \tag{24}$$

Using the same technique as in [17], we can generalize the result to the case where P is fixed and M grows like logarithmically with n. We summarize the results in the following corollary:

Corollary 1. *Let $N$and P be fixed and $E\{R_{BF}\}$ denotes the sum-rate achieved by beamforming. If $M = \frac{\log n + 3\log\log n}{\frac{c}{P} + \log(1+c)}$ where c is a constant, then*

$$\lim_{n\to\infty} \frac{E\{R_{BF}\}}{M} = \gamma', \tag{25}$$

where γ' is a constant less than γ in Lemma 4 and larger than $\log(1+c)$.

Proof: The upper bound follows from the fact that $E(R_{BF}) \leq E(R_{DPC})$. As for the lower bound, we use the scheme of [17] to deduce that $E\{R_{BF}\} \geq \log(1+c)M + O(\log\log n)$. The proof is very similar to the proof for the case where the average transmit power per antenna is fixed ($P = M$) as in Theorem 1 and 2 of [17]. We omit the proof for the sake of brevity. ∎

6 Comparison of Time-Sharing, Beamforming, and DPC

Clearly, the scaling law of the sum rate is the same for beamforming and DPC when M is fixed and n grows to infinity. As the number of antennas is getting large and grows logarithmically with n, the sum rate of DPC and beamforming have the same growth rate, however, the beamforming is worse by a multiplicative constant.

On the other hand, in [16], the scaling laws of DPC and time-sharing are compared for the case of large P and large number of transmit antennas when all the other parameters are fixed. It is shown that in these cases, the ratio of the sum rate of DPC over that of time-sharing is equal to $\min\left(\frac{M}{\min(M,N)}, n\right)$. Based on the results in the previous sections, we can also compare the sum rate of DPC and time-sharing in a Rayleigh fading channel and for the case of large number of users and when the transmit antennas are fixed or grows logarithmically with n. Lemma 1 and 4 imply that for M and N fixed and when $M \geq N$, we have

$$\lim_{n\to\infty} \frac{E\{R_{DPC}\}}{E\{R_{ts}\}} = \lim_{n\to\infty} \frac{M \log\log n}{\min(M,N) \log\log n} = \frac{M}{\min(M,N)}. \tag{26}$$

Eq. (26) proves that the sum-rate of DPC (and beamforming) outperform that of the time-sharing if the transmitter is equipped with multiple antennas.

7 Conclusion

In this paper, we obtained the scaling laws of the sum rate of MIMO Gaussian broadcast channel using DPC, beamforming, and time-sharing. The focus of this work was in the case of large number of users and when the number of transmit antennas is fixed or growing logarithmically with n. It is shown that when M and N (number of transmit/receiver antennas) are fixed, the gain in using DPC over time-sharing is equal to $\frac{M}{min(M,N)}$.

A Proof of Eq. (23)

In this appendix, we investigate the behavior of the maximum of n i.i.d. random variable κ_i for $i = 1, \ldots, n$ with $\chi^2(2M)$ distribution where $M = \beta \log n$. Clearly the cumulative distribution function of κ_i can be written as,

$$F(x) = \Pr\{\kappa_i \leq x\} = 1 - e^{-x} \sum_{m=0}^{M-1} \frac{x^m}{m!} = 1 - \frac{\Gamma(M,x)}{\Gamma(M)} \tag{A.1}$$

In order to find the behavior of the maximum of κ_i's, we have to compute $F^n(x)$. Following the technique in [17], we initially solve the following equality,

$$F(x_l) = 1 - \frac{\Gamma(M, x_l)}{\Gamma(M)} = 1 - \frac{(\log n)^3}{n} \tag{A.2}$$

It is worth noting that both arguments of the incomplete Gamma function in Eq. (A.2) are going to infinity. The asymptotic expansion of the incomplete Gamma function has been studied by Tricomi [22, 23] and it is shown that

$$\Gamma(M, x) = \frac{e^{-x}x^M}{x-M+1}\left\{1 - \frac{M-1}{(x-M+1)^2} + \frac{2(M-1)}{(x-M+1)^3} + O\left(\frac{(M-1)^2}{(x-M+1)^4}\right)\right\} \tag{A.3}$$

as the modulus of $\sqrt{M}/(x-M)$ tends to zero. We can also write the asymptotic expansion of the Gamma function as [24] as,

$$\log\Gamma(M) = M\log M - M - \frac{1}{2}\log M + O(1). \tag{A.4}$$

Using the asymptotic expansions, we can solve (A.2) to get $x_l = \alpha\log n - \frac{5}{2}\log\log n + o(\log\log n)$ where α satisfies,

$$\alpha - \beta\log\alpha = 1 + \beta - \beta\log\beta. \tag{A.5}$$

Therefore the probability that the maximum of κ_i's is less than x_l can be written as,

$$\Pr\left\{\max_{1\le i\le n}\kappa_i \le x_l\right\} = (F(x_l))^n = \left(1 - \frac{(\log n)^3}{n}\right)^n = O\left(e^{-(\log n)^3}\right). \tag{A.6}$$

We can also find x_u such that $F(x_u) = 1 - \frac{1}{n\log n}$ as $x_u = \alpha\log n + \frac{3}{2}\log\log n + o(\log\log n)$. Therefore,

$$\Pr\left\{\max_{1\le i\le n}\kappa_i \le x_u\right\} = (F(x_u))^n = \left(1 - \frac{1}{n\log n}\right)^n = 1 - O(\frac{1}{\log n}). \tag{A.7}$$

Eq. (A.6) and (A.7) can be combined to get,

$$\Pr\left\{\alpha\log n - \frac{5}{2}\log\log n + o(\log\log n) \le \max_{1\le i\le n}\kappa_i \le \alpha\log n + \frac{3}{2}\log\log n + o(\log\log n)\right\}$$
$$= 1 - O\left(\frac{1}{\log n}\right). \tag{A.8}$$

that completes the proof for Eq. (23).

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