1 Claim

Consider an i.i.d Gaussian source sequence, so that \( x_i \sim \mathcal{N}(0, \sigma_x^2) \). Then average distortion of an \( N \) dimensional quantizer of rate \( R \) is a mapping is bounded from below by

\[
D > \sigma_x^2 \cdot 2^{-2R} \triangleq D^*.
\]

2 Proof

Let \( Q(x) \) be a VQ with rate \( R \) such that \( Q : \mathbb{R}^N \rightarrow \{ y_1, y_2, ..., y_{2^{RN}} \} \). Define the average distortion of the quantizer as

\[
D \triangleq \frac{1}{N} E[||x - Q(x)||^2].
\]

Assuming that \( Q(x) \) achieves a distortion of \( D < D^* \), let us define:

- \( D < D' < D^* \) which leads to the inequality: \( 2^{2R}D' < \sigma_x^2 \).
- A region of “good” points around each reconstruction vector

  \[
  B_i \triangleq \{ x : ||x - y_i||^2 \leq ND' \} : i = 1, 2, ..., 2^{RN}.
  \]

Thus, for any \( x \) that belongs to a \( B_i \), the VQ achieves a distortion no greater than \( D' \). Namely, \( x \in B_i \Rightarrow \frac{1}{N}||x - y_i||^2 < D' \).

- The union of “good” environments:

  \[
  G \triangleq \bigcup_{i=1}^{2^{RN}} B_i.
  \]

  Note that for points that are not in \( G \), the VQ distortion is greater then \( D' \).
Denote the volume of an $N$-dimensional ball of radius one by $\gamma_N$. It follows that the volume of an $N$-dimensional ball with radius $r$ is $\gamma_N r^N$, and therefore

$$\text{Vol}(B_i) = \gamma_N (ND'_i)^{\frac{N}{2}}.$$  

Define $B$ to be an $N$-dimensional ball centered at the origin, whose volume is given by:

$$\text{Vol}(B) = \sum_{i=1}^{2RN} \text{Vol}(B_i) \quad (1)$$

$$= \gamma_N (ND'_i)^{\frac{N}{2}} \cdot 2^{RN} \quad (2)$$

$$= \gamma_N (\sqrt{ND'_i} \cdot 2^R)^N. \quad (3)$$

It follows that $r_B = \sqrt{ND'_i} \cdot 2^R$. Also not that $\text{Vol}(G) \leq \sum_{i=1}^{2RN} \text{Vol}(B_i) = \text{Vol}(B)$, and equality occurs only when the balls are pairwise disjoint.

**Lemma 1** Given an i.i.d. Gaussian source $X_i$, $B$ as defined above and a region $S \subset \mathbb{R}^N$ such that $\text{Vol}(S) \leq \text{Vol}(B)$ holds, we have

$$\Pr(X \in S) \leq \Pr(X \in B).$$

It follows from the lemma that

$$\Pr(x \in G) \leq \Pr(x \in B) \quad (4)$$

$$= \Pr(||x||^2 < r_B^2) \quad (5)$$

$$= \Pr \left( \frac{1}{N} ||x||^2 \leq D'2^R \right) \quad (6)$$

$$= \Pr \left( \frac{1}{N} \sum_{i=1}^{N} x_i^2 < D'2^R \right) \quad (7)$$

But by the law of large numbers (LLN) we have:

$$\frac{1}{N} \sum_{i=1}^{N} x_i^2 \xrightarrow{\text{in probability}} E[x_i^2] = \sigma^2_x.$$

Recalling that $D' \cdot 2^R < \sigma^2_x$, we have:

$$\Pr(x \in G) \leq P(x \in B) \quad (8)$$

$$= \Pr \left( \frac{1}{N} ||x||^2 < D' \cdot 2^R \right). \quad (9)$$
Combining (2) and (9), it follows that
\[
\Pr(x \in G) \quad \overset{\text{in probability}}{\Rightarrow} \quad N \to \infty \to 0.
\]

To sum up,
\[
D = \frac{1}{N} E \left[ ||x - Q(x)||^2 \right] \tag{10}
\]
\[
= \frac{1}{N} \Pr(x \in G) \cdot E \left[ ||x - Q(x)||^2 | x \in G \right] \tag{11}
\]
\[
+ \frac{1}{N} \Pr(x \notin G) \cdot E \left[ ||x - Q(x)||^2 | x \notin G \right] \tag{12}
\]
\[
\geq 0 + \Pr(x \notin G) D' \tag{13}
\]
\[
\to 1 \cdot D' \tag{14}
\]
\[
> D. \tag{15}
\]

This leads to a contradiction to the assumption that the VQ attains a distortion of $D$. Thus, the minimal average distortion of a VQ, given an i.i.d Gaussian source, cannot be smaller than $D^* = \sigma^2 x \cdot 2^{-2R}$.

To complete the proof we must justify that why we may consider only the limit $N \to \infty$, i.e., why a finite and fixed cannot result in a lower distortion: Had there been an $N$-dimensional VQ with average distortion $D < D^*$, it is obvious that we can achieve $D$ for any $N' = Nk$, $k \in \mathbb{N}$ by concatenation. That is, define a VQ of dimension $N'$ by
\[
Q^{N'}(x_1^N, x_{N+1}^{2N}, \ldots, x_{N(k-1)+1}^{kN}) = (Q(x_1^N), Q(x_{N+1}^{2N}), \ldots, Q(x_{N(k-1)+1}^{kN})).
\]

Then it’s distortion is
\[
\frac{1}{N'} E[||x_1^{N'} - Q^{N'}(x_1^{N'})||^2] = \frac{1}{NK} E[||x_1^{kN} - Q^{N'}(x_1^{kN})||^2] \tag{16}
\]
\[
= \frac{1}{k} \sum_{m=1}^{k} \frac{1}{N} E[||x_{N(m-1)}^{mN} - Q(x_{N(m-1)}^{mN})||^2] \tag{17}
\]
\[
= D. \tag{18}
\]