

Due: Tuesday May 31, noon.

1. Let  $G = (V, E)$  be a graph with chromatic number  $\chi(G) = 1000$ . Let  $U \subset V$  be chosen uniformly at random. Let  $H = G[U]$  be the induced subgraph of  $G$  on  $U$ . Show that  $\mathbb{P}(\chi(H) \leq 400) < 1/100$ .
  
2. Find a threshold function for the property:  $G(n, p)$  contains at least  $n/6$  vertex disjoint triangles.
  
3. Let  $c$  be a constant,  $p = \frac{\log n + c}{n}$  and  $G \sim G(n, p)$ . Show that the number of isolated vertices in  $G(n, p)$  is asymptotically Poisson with mean  $e^{-c}$ . Deduce that  $G$  is connected with probability asymptotic to  $e^{-e^{-c}}$ .
  
4. The configuration model for a  $d$ -regular graph is defined as follows. Suppose  $dn$  is even, let  $V = [n]$  and  $W = [n] \times [d]$ . A configuration  $F$  is a partition of  $W$  into  $dn/2$  pairs. The image  $\pi(F)$  under projection onto the first co-ordinate is a  $d$ -regular multigraph on  $V$  (possibly with loops and parallel edges). Let  $G_d^*(n)$  denote the random multigraph obtained when  $F$  is chosen uniformly at random. Let  $G_d(n)$  denote the random  $d$ -regular graph obtained by sampling from  $G_d^*(n)$  and conditioning on the event that a simple graph is obtained.
  - (i) Show that  $G_d(n)$  has the uniform distribution over all  $d$ -regular graphs on  $[n]$ .
  - (ii) Let  $C_k(G)$  denote the number of cycles of length  $k$  in  $G$ , where if  $G$  is a multigraph  $C_1(G)$  is the number of loops and  $C_2(G)$  the number of pairs of parallel edges. Show that the random variables  $C_k(G_d^*(n))$  converge in distribution as  $n \rightarrow \infty$  to independent Poisson variables with parameters  $c_k = \frac{1}{2k}(d-1)^k$ .
  - (iii) Deduce that  $G_d^*(n)$  is simple with probability asymptotic to  $e^{-(d^2-1)/4}$  and that for  $k \geq 3$  the variables  $C_k(G_d(n))$  converge in distribution to independent Poisson variables with parameters  $c_k$ .