Combinatorial Analysis Ma121c 2005 **Homework 2**

Due: Tuesday May 3, noon.

1. Show that a k-uniform hypergraph with n vertices and $m \geq n$ edges contains an independent set [1] of size at least $\frac{1}{4}(n^k/m)^{1/(k-1)}$.

2. Given a graph $G = (V, E)$ let $c_t(G)$ denote the number of t-colourings of G, i.e. the number of functions $f : V \to \{1, \cdots, t\}$ such that $f(x) \neq f(y)$ for every edge $xy \in E$. If G has n vertices and m edges, show that $t^n - mt^{n-1} \leq c_t(G) \leq \frac{t-1}{m+t-1} t^n$. [2]

3. Show that there is a constant $c > 0$, so that if $a_1, \cdots, a_n$ are reals satisfying $\sum_{i=1}^n a_i^2 = 1$, and $X_1, \cdots, X_n$ are independent random variables each taking the value 1 or -1 with probability 1/2, then $\mathbb{P}(|\sum_{i=1}^n a_i X_i| \leq 1) \geq c$.

4. The crossing number $cr(G)$ of a graph G is the minimum number of pairs of intersecting edges (not counting intersections at a vertex) when G is drawn in the plane. Suppose G is a graph with n vertices and $m \geq 4n$ edges.

(i) Show that $cr(G) > m - 3n$.

(ii) By applying (i) to a random set of vertices, show that $cr(G) \geq \frac{m^3}{64n^2}$.

(iii) Suppose P is a set of points and L is a set of lines in the plane. An incidence between P and L is a pair (p, ℓ) so that $p \in P$, $\ell \in L$, $p \in \ell$. Let I be the number of incidences. By considering the graph with vertex set P and edge set those segments of lines from L that join two points of P, show that $I \leq 100(|P|^{2/3}|L|^{2/3} + |P| + |L|)$.

[1] set of vertices not containing an edge

[2] Hint: For the upper bound, show that any non-negative integral random variable X satisfies $\mathbb{P}(X = 0) \leq V(X)/\mathbb{E}(X^2)$.