Combinatorial Analysis Ma121c 2005 **Homework 1**

Due: Tuesday April 19, noon.

1. Show that there is a constant $c > 0$ so that if A is any $n \times n$ real matrix with distinct entries then one can permute its rows so that no column contains an increasing subsequence of length $\geq c\sqrt{n}$.

2. Let $A_1, \cdots, A_m$ and $B_1, \cdots, B_m$ be sets such that $|A_i| = r$, $|B_i| = s$, $A_i \cap B_i = \emptyset$ for all i, and for every $i \neq j$ at least one of $A_i \cap B_j$ and $A_j \cap B_i$ is non-empty. Show that $m \leq \frac{(r+s)^{r+s}}{r^r s^s}$.

3. Let C be a finite collection of binary strings with c_i strings of length i for each i.

(i) Suppose C is a prefix-free code, i.e. if C contains a string $x_1 \cdots x_i$ then it does not contain any other string that starts with $x_1 \cdots x_i$. Show that $\sum_i c_i 2^i \leq 1$.

(ii) Suppose C is a uniquely decipherable code, i.e. no two distinct concatenations of finite sequences of codewords results in the same string. Show that $\sum_i c_i 2^i \leq 1$.

4. Let $v_1, \cdots, v_m$ be vectors in $\mathbb{R}^n$ of length at most 1. Suppose $v = \sum_{i=1}^m p_i v_i$, where $0 \leq p_i \leq 1$ for each i. Show that there are numbers $a_1, \cdots, a_m$ each equal to 0 or 1 such that $\|v - \sum_{i=1}^m a_i v_i\| \leq \sqrt{n}/2$. [1]

[1] Hint: consider the cases $m \leq n$ and $m > n$ separately.