Homogeneous linear recurrences

Let $\{a_n\}_{n=0}^{\infty} = (a_0, a_1, a_2, \ldots)$ be a sequence of elements of a field F, and let $g(x) = \sum_n a_n x^n$ be the generating function of the sequence.

It is easy to see (check this) that the sequence $\{a_n\}$ satisfies a homogeneous linear recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_r a_{n-r} \quad \text{for } n \geq r \tag{1}$$

of degree r if and only if

$$g(x) = \frac{p(x)}{1 - c_1 x - c_2 x^2 - \ldots - c_r x^r} \tag{2}$$

where $p(x)$ is a polynomial of degree $< r$ over F.

Suppose

$$1 - c_1 x - c_2 x^2 - \ldots - c_r x^r = (1 - \alpha_1 x)^{e_1} (1 - \alpha_2 x)^{e_2} \cdots (1 - \alpha_k x)^{e_k}$$

in some extension of a field F.

Theorem. *Assume F has characteristic 0. Then $\{a_n\}$ satisfies (1) if and only if it is a linear combination of the r sequences*

$$\{n^{i-1} \alpha_j^n\}, \qquad j = 1, 2, \ldots, k; \quad i = 1, 2, \ldots, e_j. \tag{3}$$

Example. The sequence $\{a_n\}$satisfies

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} \quad \text{for } n \geq 3$$

if and only if

$$g(x) = \frac{p(x)}{1 - 7x + 16x^2 - 12x^3} = \frac{p(x)}{(1-3x)(1-2x)^2}$$

for $p(x)$ of degree at most 2 if and only if

$$a_n = c3^n + (sn + t)2^n$$

for some scalars c, s, t. The scalars c, s, t can be determined from a_0, a_1, a_2, or any three consecutive values of the sequence.

Corollary. *The sequence $\{a_n\}$ is the sequence of values $(f(0), f(1), f(2), \ldots)$ of a polynomial $f(n)$ of degree $\leq d$ if and only if*

$$g(x) = \frac{p(x)}{(1-x)^d}$$

for some polynomial $p(x)$ of degree $< d$.

Proof #1: First note that the sequences in (3) span the same space of sequences as

$$\{\binom{n+i-1}{i-1}\alpha_j^n\}, \qquad j = 1, 2, \ldots, k; \quad i = 1, 2, \ldots, e_j. \tag{4}$$

Recall (or, it will be explained in class) that

$$\frac{1}{(1-\beta x)^i} = \sum_{n=0}^{\infty} \binom{n+i-1}{i-1} \beta^n x^n.$$

That is, the generating function for the sequence $\binom{n+i-1}{i-1}\alpha_j^n$ is

$$g_{ij}(x) = \frac{1}{(1-\alpha_j x)^i}. \tag{5}$$

If $\{a_n\}$ is a linear combination of the sequences in (3), then $g(x)$ will be a linear combination of the power series in (5), and hence of the form (2).

For the converse, suppose $\{a_n\}$ satisfies (1) and use partial fractions:

$$\begin{aligned} g(x) &= \frac{p(x)}{(1-\alpha_1 x)^{e_1}(1-\alpha_2 x)^{e_2}\cdots(1-\alpha_k x)^{e_k}} \\ &= \sum_{j=1}^{k}\sum_{i=1}^{e_j} \frac{\gamma_{ij}}{(1-\alpha_j x)^i} \\ &= \sum_{j=1}^{k}\sum_{i=1}^{e_j} \gamma_{ij} \sum_{n=0}^{\infty} \binom{n+i-1}{i-1} \alpha_j^n x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=1}^{k}\sum_{i=1}^{e_j} \gamma_{ij} \binom{n+i-1}{i-1} \alpha_j^n \right) x^n, \end{aligned}$$

where the γ_{ij}'s are scalars. Compare the coefficients on x^n on the extremes to see that $\{a_n\}$ is a linear combination of the sequences $\{n^{i-1}\alpha_j^n\}$.

Proof #2: Since a sequence $(a_0, a_1, \ldots)$ satisfying (1) is uniquely determined by $(a_0, a_1, \ldots, a_{r-1})$, the vector space S of sequences $\{a_n\}$ that satisfy (1) has dimension r.

To show that the r sequences in (3) span S, it will suffice to show that they are linearly independent. We omit the demonstration.