The Number of Spanning Trees in K_n.

For the vertices of K_n, we take the ordered set $V = \{1, 2, 3, \ldots, n\}$. To every spanning tree T in K_n, we associate a 'name' $\mu(T) \in V^{n-2}$ in the manner described below.

Let $T_1 = T$. We generate a sequence of trees $T_1, T_2, \ldots, T_{n-1}$ and two sequences of vertices as follows: Given the tree T_i with $n - i + 1$ vertices, $i = 1, 2, \ldots, n-1$, let x_i be the least monovalent vertex of T_i and delete x_i and its incident edge $\{x_i, y_i\}$ from T_i to obtain a tree T_{i+1} on $n - i$ vertices. The *name* of T is to be

$$\mu(T) = (y_1, y_2, \ldots, y_{n-2}).$$

We claim that the mapping μ, from the set of all spanning trees in K_n to the set V^{n-2} of all possible names, is one-to-one and onto (bijective). This will prove that the number of spanning trees in K_n is n^{n-2}.

Example: For the tree on the left where $n = 10$, we have $(x_1, y_1) = (2, 5)$, $(x_2, y_2) = (4, 5)$, ..., $(x_9, y_9) = (8, 10)$. (The edges are the columns of the matrix below.)

$$\begin{bmatrix} 2 & 4 & 6 & 7 & 3 & 9 & 5 & 1 & 8 \\ 5 & 5 & 8 & 3 & 1 & 5 & 1 & 8 & 10 \end{bmatrix}$$

So $\mu(T) = (5, 5, 8, 3, 1, 5, 1, 8)$. Don't include $y_9 = 10$.

To show μ is bijective, we need be able to 'undo' it, informally speaking. How do we find a tree with a given name, and is there only one with that name?

Let us note some simple facts about the x_i's and y_i's. First, $y_{n-1} = n$, always. This is because every tree (with at least two vertices) has at least two monovalent vertices, so the vertex n will never be the least monovalent vertex. Second, $x_k, x_{k+1}, \ldots, x_{n-1}$ and n are the vertices of the tree T_k. Third, $\{x_i, y_i\}$, $k \le i \le n-1$, are exactly the edges of T_k, in some order.

We see that the number of times a vertex v occurs among $y_1, y_2, \ldots, y_{n-2}$ is $\deg_T(v) - 1$. This is because v occurs $deg_T(v)$ times among the edges $\{x_i, y_i\}$, $1 \le i \le n-1$, and exactly once among $x_1, x_2, \ldots, x_{n-1}, y_{n-1} = n$. Similarly, the number of times a vertex v of T_k occurs among $y_k, y_{k+1}, \ldots, y_{n-2}$ is its degree in the tree T_k less 1. In particular, *the monovalent vertices of T_k are those elements of V not in*

$$\{x_1, x_2, \ldots, x_{k-1}\} \cup \{y_k, y_{k+1}, \ldots, y_{n-1}\}.$$

Make sure you understand this. It means that x_k, the least monovalent vertex of T_k, is the least element of $\{1, 2, \ldots, n\}$ not in the above set. In particular, x_1 is the least element of V not in the name $\mu(T)$, and we can uniquely determine x_k from $\mu(T)$ and $x_1, \ldots, x_{k-1}$.

More details and more examples will be given in class.