Generating Functions and Applications

1. Some theory.

Formal power series are in one-to-one correspondence with infinite sequences:

$$(a_0, a_1, a_2, a_3 \ldots) \quad \leftrightarrow \quad f \text{ or } f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots. \qquad (*)$$

Here the 'x' is just a symbol and f or $f(x)$ is not a function; convergence plays no role. But the analogy with functions of a real or complex variable is important and useful. Addition and multiplication of formal power series is defined as 'usual' (see Biggs) and many conventions used in algebra or calculus are followed. We write, for example, just 1 for the formal power series $1 + 0x + 0x^2 + 0x^3 + \ldots$.

We use $f(0)$ to denote the 'constant term' a_0 of the power series f above. The x cannot be replaced by any other number, though it can be replaced by a formal power series $h(x)$ with no constant term. That is, with f as above, we may consider, for example,

$$f(x + x^2 - 2x^5) = a_0 + a_1(x + x^2 - 2x^5) + a_2(x + x^2 - 2x^5)^2 + a_3(x + x^2 - 2x^5)^3 + \ldots.$$

This is allowed because the computation of the coefficient of x^n is a finite procedure, for any $n = 0, 1, 2, \ldots$.

The notation $f - g = h$ means $f = h + g$, and $f/g = h$ means f, g and h are power series so that $f = hg$.

Some of the propositions below are in Biggs; others will be discussed in class. The statements should not surprise anyone. We will not check everything.

Proposition. *A power series f has an inverse $1/f$ (that is a power series, of course) if and only if $f(0) \neq 0$.*

The *formal derivative* f' for Df is defined, for f given as above, by

$$f' \text{ or } Df = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^4 + \ldots.$$

Proposition. *For a power series f as in $(*)$,*

$$a_n = \frac{1}{n!} f^{(n)}(0).$$

Proposition. *For power series f and g,*

$$(fg)' = fg' + gf'.$$

Proposition. *For a power series f and an integer m, positive, negative, or zero,*

$$(f^m)' = m f^{m-1} f'.$$

For a nonnegative integer k and any z, we define

$$\binom{z}{k} = \frac{1}{k!} z(z-1)(z-2)\cdots(z-k+1).$$

This definition coincides with that of the binomial coefficients when z is also a nonnegative integer.

Theorem (the Binomial Theorem for integral exponents). *For an integer m, positive, negative, or zero,*

$$(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k.$$

Proof:

$$\begin{aligned} D(1+x)^m &= m(1+x)^{m-1}, \\ D^2(1+x)^m &= (m-1)m(1+x)^{m-2}, \\ D^3(1+x)^m &= (m-2)(m-1)m(1+x)^{m-3}, \\ \cdots &= \cdots \\ D^k(1+x)^m &= (m-k+1)\cdots(m-2)(m-1)m(1+x)^{m-k}. \end{aligned}$$

This last expression has constant term $m(m-1)(m-2)\cdots(m-k+1)$. The coefficient of x^k in $(1+x)^m$ is thus $\binom{m}{k}$. □

Proposition. *Let f be a power series with $f(0) = 1$. Then there is a unique power series g such that $g(0) = 1$ and $g^2 = f$. (We write $g = f^{1/2}$.) More generally, given a positive integer m, there is a unique power series h so that $h(0) = 1$ and $h^m = f$ (and we write $f^{1/m}$ for this h).*

For f a power series with $f(0) = 1$ and any rational number r, we may then define f^r as $f^r = (f^{1/m})^n$ after we write $r = n/m$ with n and m integers. We should check that it does not matter whether the fraction n/m is in lowest terms or not.

Proposition. *For a power series f and a rational number r, positive, negative, or zero,*

$$(f^r)' = rf^{r-1}f'.$$

Theorem (the Binomial Theorem for rational exponents). *For a rational number r, positive, negative, or zero,*

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k.$$

The proof is the same as that of the previous form of the binomial theorem.

2. Catalan numbers.

The *Catalan number* c_n is the number of ways to bracket or parenthesize a 'product' of n letters, where the 'product' is a (possibly nonassociative) binary operation. For example, there are 5 ways to evaluate $abcd$, namely

$$(ab)(cd), \quad ((ab)c)d, \quad (a(bc))d, \quad a(b(cd)), \quad a((bc)d).$$

We have

$$c_2 = 1,\ c_3 = 3,\ c_4 = 5,\ c_5 = 14,\ c_6 = 42,\ \ldots.$$

We have seen in class that these numbers satisfy the recursion

$$c_n = c_{n-1} + c_2c_{n-2} + c_3c_{n-3} + \ldots + c_{n-3}c_3 + c_{n-2}c_2 + c_{n-1}$$

for $n = 2, 3, \ldots$. It will be convenient to let $c_0 = 0$ and $c_1 = 1$ because then we can write

$$c_n = c_0c_n + c_1c_{n-1} + c_2c_{n-2} + c_3c_{n-3} + \ldots + c_{n-3}c_3 + c_{n-2}c_2 + c_{n-1}c_1 + c_nc_0.$$

Let

$$f = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots = x + x^2 + 2x^3 + 5x_4 + 14x^5 + \ldots$$

be the generating function of the Catalan numbers. The above recursion means that the coefficients of x^n in f and f^2 are the same for all $n \geq 2$. In fact, $f^2 - f + x = 0$. By the quadratic formula (is this OK?),

$$f = \frac{1 \pm \sqrt{1-4x}}{2}.$$

The correct sign is '$-$' because f has constant term 0. Then by the binomial theorem,

$$f = \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{\infty}\binom{1/2}{k}(-4x)^k.$$

So for $n \geq 1$, comparing the coefficient of x^n on both sides of this equation gives

$$\begin{aligned} c_n &= -\frac{1}{2n!}(-4)^n\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n+1\right) \\ &= \frac{1}{n!}2^{n-1}\cdot 1\cdot 3\cdot 5\cdot 7\cdots(2n-3) = \frac{1}{n}\binom{2n-2}{n-1}. \end{aligned}$$

3. Average number of comparisons in QUICKSORT.

Let a_n denote the average number of comparisons the QUICKSORT algorithm requires to sort n distinct numbers. We have

$$a_n = n - 1 + \sum_{i=1}^{n}(a_{i-1} + a_{n-i}).$$

Then

$$na_n = n(n-1) + 2\sum_{j=0}^{n-1}a_j.$$

Multiply the above by x^{n-1} and sum over $n = 1, 2, 3, \ldots$ to get

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = x \sum_{n=2}^{\infty} n(n-1)x^{n-2} \; + \; 2\sum_{j=0}^{\infty}\left(a_j \sum_{n=j+1}^{\infty} x_{n-1}\right).$$

Let f be the generating function for the sequence $a_0, a_1, a_2, \ldots$ (we take a_0 to be 0). Then the above can be written

$$f' = 2x(1-x)^{-3} + 2(1-x)^{-1}f \qquad \text{or} \qquad f' - 2(1-x)^{-1}f = 2x(1-x)^{-3}. \qquad (**)$$

This is a first order linear differential equation.

The method of solving such an equation involves choosing a function (power series) u or $u(x)$ so that

$$\frac{u'}{u} = (\log u)' = -2(1-x)^{-3};$$

we may take $u = (1-x)^2$. Multiply both sides of $(**)$ by u to get

$$(1-x)^2 f' - 2(1-x)f = 2x(1-x)^{-1} = \frac{2}{1-x} - 2.$$

The left hand side is $((1-x)^2 f)'$, so we conclude

$$(1-x)^2 f = -2\log(1-x) - 2x$$

(there is no constant because $a_0 = 0$), and finally

$$f = -2(x + \log(1-x))(1-x)^{-2} = 2\left(\sum_{k=2}^{\infty} \frac{x^k}{k}\right)\left(\sum_{j=0}^{\infty}(j+1)x^j\right).$$

Comparing the coefficient of x^n on both sides of the above equation, we find

$$\begin{aligned} a_n &= 2\sum_{k=2}^{n} \frac{n-k+1}{k} = 2(n+1)\sum_{k=1}^{n}\frac{1}{k} - 4n \\ &\approx 2(n+1)(\gamma + \log n) - 4n = 2n\log n + O(n). \end{aligned}$$