Combinatorial Analysis Ma121a 2006 **Homework 8**

Due: Tuesday November 28, 5pm.

(There is extra time allotted for this assignment to accommodate Thanksgiving.)

Definitions: An r-graph G has a vertex set $V(G)$ and an edge set $E(G)$, where each edge is a subset of the vertex set of size r. $S \subset V(G)$ is independent if it does not contain an edge. A k-colouring of G is a function $f : V(G) \to \{1, \cdots, k\}$ which is not constant on any edge.

1. Show that an r-graph with less than 2^{r-1} edges has a 2-colouring. What is the smallest number of edges in a 3-graph that does not have a 2-colouring?

2. Let G be a r-graph and let $c_k(G)$ denote the number of k-colourings of G. If G has n vertices and m edges show that $1 - mk^{1-r} \leq c_k(G)/k^n \leq \frac{k^{r-1}-1}{k^{r-1}+m-1}$. (For the upper bound it will be helpful to show that any non-negative integral random variable X satisfies $\mathbb{P}(X = 0) \leq V(X)/\mathbb{E}(X^2)$.)

3. Show that an r-graph with n vertices and $m \geq n$ edges contains an independent set of size at least $\frac{1}{4}(n^k/m)^{1/(k-1)}$.

4. Suppose n points are chosen uniformly at random and independently on the unit circle. What is the probability that their convex hull contains the origin?

5. Let G be a random graph on n vertices, in which each pair is chosen to be an edge with probability $p(n)$, all choices being independent. Let $f(n)$ be any function such that $f(n) \to \infty$ as $n \to \infty$.

(i) If $p(n) = \frac{\log n - f(n)}{n}$ show that $\mathbb{P}(G \text{ is connected}) \to 0$ as $n \to \infty$.

(ii) If $p(n) = \frac{\log n + f(n)}{n}$ show that $\mathbb{P}(G \text{ is connected}) \to 1$ as $n \to \infty$.