Combinatorial Analysis Ma121a 2006 **Homework 4**

Due: Thursday October 26, 5pm.

1. The Bell number $B(n)$ is the number of partitions of a set of size n. For example, $B(3) = 5$ as $\{1,2,3\}$ can be partitioned as $\{1,2,3\}$, $\{1,2\} \cup \{3\}$, $\{1,3\} \cup \{2\}$, $\{2,3\} \cup \{1\}$ or $\{1\} \cup \{2\} \cup \{3\}$.

(i) Show that $B(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$.

(ii) Show that the exponential generating function is $\sum_n B(n)x^n/n! = e^{e^x-1}$.

(iii) Show that $B(n) = \frac{1}{e}\sum_k k^n/k!$.

2. Show that the number of partitions of n into parts not divisible by d equals the number of partitions in which no part occurs more than $d-1$ times.

3. Show that the number of self-conjugate partitions of n equals the number of partitions of n into unequal odd parts.

4. Let t_n be the number of labelled trees on $[n] = \{1, \cdots, n\}$. Let a_n be the number of *rooted* labelled trees on $[n]$, by which we mean a labelled tree in which one vertex is marked as the root. (Obviously $a_n = nt_n$.) Let f_n be the number of *rooted* labelled forests on $[n]$, by which we mean a labelled forest in which one vertex *in each component* is marked as a root. Let $T(x)$, $A(x)$, $F(x)$ be the corresponding exponential generating functions. Show that:

(i) $F(x) = e^{A(x)}$, (ii) $t_{n+1} = f_n$ and (iii) $x = A(x)e^{-A(x)}$.

Deduce Cayley's formula $t_n = n^{n-2}$, using the Lagrange inversion formula, which is as follows:

Lagrange inversion formula: Let $f(t) = \sum_{i\geq 0} a_i t^i$ be a formal power series with $a_0 \neq 0$. The equation $z = y/f(y)$ can be solved by a unique power series $y(z) = \sum_{n\geq 1} y_n z^n$, where y_n is obtained by taking the coefficient of t^{n-1} in $f(t)^n$ and dividing by n.