Combinatorial Analysis Ma121a 2006 **Homework 1**

Due: Thursday October 5, 5pm.

Preliminary remarks. All graphs are assumed to be simple and finite. Be aware that I will always use terminology as I define it in class, and occasionally it may differ from the textbook. Two important examples are 'path', which the textbook calls 'simple path', and 'cycle', which the textbook either calls 'simple closed path' (referring to the graph together a particular order of the vertices) or 'polygon' (referring to the graph itself).

1. Prove or disprove each of the following statements.

(i) Every simple graph with at least two vertices has two vertices with the same degree.

(ii) Every connected graph has a vertex that can be deleted to leave a connected graph.

(iii) If u, v, w are vertices of a graph G such that there is a path with an even number of edges from u to v and a path with an even number of edges from v to w, then then there is a path with an even number of edges from u to w.

2. Say that a graph G is bipartite if its vertex set can be partitioned as $V(G) = V_1 \cup V_2$ so that there are no edges that have both endpoints in V_1 and no edges that have both endpoints in V_2. Show that G is bipartite if and only if it does not have a subgraph that is a cycle with an odd number of edges.

3. Show that for any graph G there is a partition $V(G) = V_1 \cup V_2$ so that at least half of the edges of G have one endpoint in V_1 and the other in V_2.

4. Show that for any graph G there is a partition $V(G) = V_1 \cup V_2$ satisfying all of the following three conditions:

(i) At least half of the edges of G have one endpoint in V_1 and the other in V_2,

(ii) At most one third of the edges of G have both endpoints in V_1, and

(iii) At most one third of the edges of G have both endpoints in V_2.

Can this result be improved? Can you make a stronger statement if you assume that the number of edges in G is large? Can you think of any generalisations?