Combinatorial Analysis Ma121a 2004 **Homework 9**

Due: Thursday December 2, noon.

1. Let G be the electrical network formed by the edges of a cube, each having resistance 1. Find the effective resistance between any pair of points.

2. (i) Let G be an electrical network. Suppose that v is a degree 3 vertex where no current is allowed to enter or leave, and that the edges to its neighbours a, b, c have resistances A, B, C respectively. Show that an equivalent [1] network G^* is obtained by deleting v and joining a, b, c to each other, with $R(ab) = S/C$, $R(bc) = S/A$, $R(ca) = S/B$, where $S = AB + BC + CA$.

(ii) Let G be a network formed by the edges of a tetrahedron, with arbitrary resistances. Find an expression for the effective resistance between two points.

3. (i) Suppose G is a graph with $V(G) \subset \mathbb{R}^2$, so that any two vertices are at distance at least 1, and any two adjacent vertices are at distance at most 1000. Show that the unbiased random walk on G is recurrent.

(ii) Is there a subtree of the lattice $\mathbb{Z}^3$ on which the unbiased random walk is transient?

4. Let G be a non-bipartite finite electrical network with n vertices and m edges, all of resistance 1. Let $p_0 \in \mathbb{R}^V$ be a probability distribution on the vertices. The unbiased random walk from this distribution is a sequence of random variables $(X_i)_{i\geq 0}$ with distributions $(p_i)_{i\geq 0}$ given by $p_{i+1} = p_i P$, where the transition matrix P has entries P_{xy} equal to $1/d(x)$ if $y \in N(x)$, 0 otherwise.

(i) Define $\pi \in \mathbb{R}^V$ by $\pi_x = d(x)/2m$. Verify the 'detailed balance equations' $\pi_x P_{xy} = \pi_y P_{yx}$ and deduce that $\pi P = \pi$, i.e. π is a 'stationary distribution'.

(ii) Show that $p_i \to \pi$ pointwise as $i \to \infty$, for any p_0. [2]

(iii) The mean return time $H(x, x)$ is the expected length of time for the random walk starting at x to first return to x. By considering the random walk with $p_0 = \pi$ show that $\pi_x H(x, x) = 1$. [3]

(iv) The mean hitting time $H(x, y)$ of y from x is the expected length of time (non-zero) to first hit y starting at x. The mean commute time $K(x, y) = H(x, y) + H(y, x)$ is the expected length of a round-trip from x to y and back to x. Show that $K(x, y) = 2mR^*(x, y)$, where R^* denotes the effective resistance. Deduce that $\sum_{x\sim y} R^*(x, y) = n - 1$. [4]

[1] This means that any solution of Kirchoff's laws in G gives rise to a solution in G^* with the same currents and potentials apart from on the edges that were changed.

[2] Hint: Since G is not bipartite there is some k for which every entry of P^k is positive.

[3] Hint: Write $\pi_x H(x, x) = \sum_n \mathbb{P}(A_n)$ where A_n is the event that $X_0 = x$ and $X_i \neq x$ for $1 \leq i \leq n$. Show that $\mathbb{P}(A_n) = \mathbb{P}(B_{n-1}) - \mathbb{P}(B_n)$, where B_n is the event that $X_i \neq x$ for $0 \leq i \leq n$.

[4] Hint: Show that $K(x, y)\mathbb{P}(x \to y) = H(x, x) = 1 + \sum_y P_{xy} H(y, x)$.