Combinatorial Analysis Ma121a 2004 **Homework 8**

Due: Tuesday November 23, noon.

1. Show that the number of partitions of n into parts not divisible by d equals the number of partitions in which no part occurs more than $d-1$ times. [1]

2. Show that the number of self-conjugate partitions of n equals the number of partitions of n into unequal odd parts. [2]

3. The Bell number $B(n) = \sum_k \left\{ {n \atop k} \right\}$ is the number of partitions of $[n]$.

(i) Show that $B(n) = \sum_{k=0}^{n} \binom{n-1}{k} B_k$.

(ii) Show that the exponential generating function is $\sum_n B(n)x^n/n! = e^{e^x-1}$.

(iii) Show that $B(n) = \frac{1}{e}\sum_k k^n/k!$.

4. Let a_n be the number of possible final results in a competition among n players where ties are possible, i.e. the number of mappings $f : [n] \to [n]$ so that if f takes a value i it also takes all values $j \in [i]$.

(i) Determine the exponential generating function $\sum_n a(n)x^n/n!$.

(ii) Show that $a_n = \sum_{k=0}^{\infty} k^n/2^{k+1}$.

(iii) Show that $a_n = n!(1/\log 2)^n b_n$, where b_n satisfies $\limsup b_n^{1/n} = 1$. [3]

5. Show that the number of planted plane trees [4] on n nodes in which n_i nodes have i children is $\frac{1}{n}\binom{n}{n_0, n_1, \ldots}$, [5] provided that $n = \sum_i n_i$ and $n-1 = \sum_i i n_i$.

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[1] Problem 15H

[2] Problem 15F

[3] You may assume that the radius of convergence R of a power series $\sum_n c_n x^n$ satisfies $1/R = \limsup c_n^{1/n}$.

[4] A planted plane tree is a rooted unlabelled tree in which the children of any node are linearly ordered.

[5] Recall that the multinomial coefficient $\binom{n}{r_1, \cdots, r_k}$ is the coefficient of $\prod_{i=1}^{k} x_i^{r_i}$ in $(\sum_{i=1}^{k} x_i)^n$.