Combinatorial Analysis Ma121a 2004 **Homework 6**

Due: Monday November 8, noon.

1. Let $a_1, \cdots, a_n$ be real numbers with $|a_i| \geq 1$ for all i and let r be another real number. Show that there are at most $\binom{n}{\lfloor n/2 \rfloor}$ sets $I \subset [n]$ for which $r < \sum_{i \in I} a_i < r + 1$.

2. Given a set $T \subset [n]$ write $[n]^{(T)} = \cup_{i \in T} [n]^{(i)} = \{X \subset [n] : |X| \in T\}$. Let $f(n,t) = \max_{|T|=t} |[n]^{(T)}|$. Suppose that $\mathcal{A} \subset \mathcal{P}[n]$ contains no chain of length $t+1$. Show that $|\mathcal{A}| \leq f(n,t)$.

3. Suppose $\mathcal{A} \subset [n]^{(r)}$ with $|\mathcal{A}| \geq r+1$ and that any $r+1$ sets of $\mathcal{A}$ have a common element. Show that all the sets in $\mathcal{A}$ have a common element. [1]

4. Show that for any t, r there is n_0 so that if $n \geq n_0$ and $\mathcal{A} \subset [n]^{(r)}$ and every pair of sets from $\mathcal{A}$ have at least t common elements then $|\mathcal{A}| \leq \binom{n-t}{r-t}$. [2]

5. Given $\mathcal{A} \subset \mathcal{P}[n]$ and $X \subset [n]$ the trace of $\mathcal{A}$ on X is $\mathcal{A}[X] = \{A \cap X : A \in \mathcal{A}\}$. The VC-dimension of $\mathcal{A}$ is the largest d for which there is some set X of size d with $\mathcal{A}[X] = \mathcal{P}X$ consisting of all subsets of X. If $\mathcal{A}$ has VC-dimension d show that $|\mathcal{A}| \leq \sum_{i=0}^{d} \binom{n}{i}$. [3]

Notation: $[n] = \{1, \cdots, n\}$, $\mathcal{P}[n] = \{X : X \subset [n]\}$, $[n]^{(r)} = \{X : X \subset [n], |X| = r\}$.

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[1] Hint: It is enough to prove it for $|\mathcal{A}| = r+2$; this can be used to give an induction on $|\mathcal{A}|$.

[2] Hint: One can assume that $\mathcal{A}$ is maximal. It follows that there are $A_1, A_2 \in \mathcal{A}$ with $A_1 \cap A_2 = T$ where $|T| = t$. If all sets in $\mathcal{A}$ contain T we are done. Otherwise there is some $A_3 \in \mathcal{A}$ that does not contain T. Estimate the number of sets in A by considering their intersections with A_1, A_2, A_3.

[3] Hint: Consider the downshifting operators defined as follows. For $i \in [n]$ and $\mathcal{A} \subset \mathcal{P}[n]$ let $D_i(\mathcal{A}) = \{D_i(A) : A \in \mathcal{A}\}$, where $D_i(A)$ is equal to $A \backslash \{i\}$ if $i \in A$ and $A \backslash \{i\} \notin \mathcal{A}$, and equal to A otherwise. Show that $|D_i(\mathcal{A})| = |\mathcal{A}|$ and $|D_i(\mathcal{A})[X]| \leq |\mathcal{A}[X]|$ for any $X \subset [n]$.