Combinatorial Analysis Ma121a 2004 **Homework 4**

Due: Monday October 25, noon.

1. Let $x_1, \cdots, x_n$ be vectors of norm at least 1 in a Euclidean space. Show that at most $n^2/4$ of the vectors $\{x_i + x_j : i \neq j\}$ have norm less than 1.

2. Show that a graph with n vertices and e edges contains at least $\frac{e}{3n}(4e - n^2)$ triangles. [1]

3. Let V be an n-element set, write $V^{(r)} = \{X \subset V : |X| = r\}$ and suppose $\mathcal{H} \subset V^{(r)}$. [2] Show that for any $\epsilon, t > 0$ there is $n_0 > 0$ so that if $n \geq n_0$ and $|\mathcal{H}| \geq \epsilon n^r$ then we can find subsets $V_i \subset V$, $|V_i| = t$ for $1 \leq i \leq r$ so that for any choices of points $v_i \in V_i$, $1 \leq i \leq r$ we have $\{v_1, \cdots, v_r\} \in \mathcal{H}$. [3]

4. (i) Show that for any $\epsilon > 0$ and m there is n_0 so that if $n \geq n_0$ and G is a graph on n vertices with at least $(p + 3\epsilon)\binom{n}{2}$ edges then at least $\epsilon\binom{n}{m}$ subsets of size m contain at least $(p + \epsilon)\binom{m}{2}$ edges.

(ii) Show that for any $\epsilon > 0$ and r there are $\delta > 0$ and n_0 so that if $n \geq n_0$ and G is a graph on n vertices with at least $\left(\frac{r-2}{r-1} + \epsilon\right)\binom{n}{2}$ edges then G contains at least δn^r copies of K_r.

(iii) Suppose H is a graph with chromatic number r. Show that for any $\epsilon > 0$ there is n_0 so that if $n \geq n_0$ and G is a graph on n vertices with at least $\left(\frac{r-2}{r-1} + \epsilon\right)\binom{n}{2}$ edges then G contains a copy of H.

---

[1] This is problem 4C in the book. There is a hint in the back.

[2] We say that $\mathcal{H}$ is an r-uniform hypergraph

[3] Hint: For the case $r = 2$ we proved something stronger than this in class. The same idea can be used to give an induction on r.