Combinatorial Analysis Ma121a 2004 **Homework 3**

Due: Monday October 18, noon.

1. Show that $R(3, 4) = 9$ and $R(4, 4) = 18$. [1]

2. Show that there is a tournament on n vertices with at least $\frac{n!}{2^{n-1}}$ directed paths that use all the vertices.

3. For graphs G_1, G_2 let $R(G_1, G_2)$ be the smallest number n so that if the edges of K_n are all coloured red or blue then there must either be a copy of G_1 with all edges red or a copy of G_2 with all edges blue. [2]

(i) Show that any graph with chromatic number t contains a subgraph with chromatic number t and minimum degree at least $t - 1$.

(ii) Let T be a fixed tree with t vertices. Show that any graph with chromatic number t contains a subgraph isomorphic to T.

(iii) Let T be a fixed tree with t vertices. Show that $R(K_s, T) = (s - 1)(t - 1) + 1$.

4. Let $G = (V, E)$ be a graph and suppose that each vertex has a list of allowed colours. We say that G is colourable from these lists if we can assign each a vertex a colour from its list so that adjacent vertices get different colours. [3] The list chromatic number $\chi_l(G)$ of a graph G is the smallest k so that if all the lists have size k then G can be coloured from these lists.

Show that $\frac{1}{2}\log_2 n < \chi_l(K_{n,n}) < \log_2 n + 1$. [4]

Hints: For the lower bound, consider the case $n = \binom{2m+1}{m+1}$ and suppose that the lists in both parts consist of all $(m + 1)$-subsets of $\{1, \cdots, 2m + 1\}$. For the upper bound, randomly partition the set of all available colours into two groups, and assign each part of $K_{n,n}$ its own group of colours.

[1] For hints read page 29 of the textbook and the hints to problems 3C and 3D. Note that the book uses the notation $N(p, q; 2)$ for $R(p, q)$.

[2] These are often called graph Ramsey numbers. Since $R(K_p, K_q) = R(p, q)$ they generalise ordinary Ramsey numbers.

[3] If for example all lists are equal to $\{1, \cdots, k\}$ then G is colourable from the lists exactly when it has chromatic number at most k. Intuitively it seems that allowing the lists to be different will make colouring easier, but this exercise should dispel that idea!

[4] $K_{n,n}$ denotes the complete bipartite graph with n points in each part, i.e. the vertices are $\{a_1, \cdots, a_n, b_1, \cdots, b_n\}$ and the edges all a_ib_j with $1 \le i, j \le n$.