Notes on indirect utilities and the standard of living

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Recently, consumer theory has increasingly been attacked as cognitively implausible [...]. I want to argue that something like neoclassical consumer theory is a prerequisite for a measure of the standard of living. This is not a new idea. [...] Samuelson [10] [...]

1 Axioms for a numerical standard of living measure

Assume \( n \) “commodities,” some of which may be interpreted as instruments of saving. That is, we will take \( \mathbb{R}^n \) as the commodity space. Let us let us use the following notation for vector orders:

\[
x \geq y \text{ if } x_i \geq y_i \text{ for all } i; \quad x \gg y \text{ if } x_i > y_i \text{ for all } i;
\]

\[
\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x \geq 0 \} \quad \mathbb{R}^n_{++} = \{ x \in \mathbb{R}^n : x \gg 0 \}.
\]

We will consider the consumer’s consumption set \( X \) to be \( \mathbb{R}^n_+ \), and the set of admissible budgets to be \( \mathbb{B} = \mathbb{R}^n_{++} \times \mathbb{R}_+ \). A typical element of \( \mathbb{B} \) will be denoted \((p, m)\). A standard of living index

\[
v : \mathbb{B} \to \mathbb{R}
\]

assigns a real number to each budget\(^1\) \((p, m) \in \mathbb{B}\), where \( p \) is a vector of prices and \( m \) is money income. The following are natural requirements for a standard of living measure.

**Monotonicity in \( m \).** \( m' > m \implies v(p, m') > v(p, m) \).

**Monotonicity in \( p \).** \( (m > 0 \& p' \gg p) \implies v(p', m) < v(p, m) \).

**Homogeneity.** \( v(p, m) \) is homogeneous of degree zero in \((p, m)\).

There is another natural property that I do not believe follows from the others.

**Zero income.** \( v(p, 0) = v(p', 0) \) for all \( p, p' \in \mathbb{R}^n_{++} \).

We could if we wished, restrict the domain of \( v \) to require \( m > 0 \) and so dispense with the zero income requirement.

At the risk of being overly pedantic I note the following lemma, which says that regardless of prices, having zero income yields the worst standard of living.

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\(^1\)I shall use the term budget to refer to both the price-income pair \((p, m)\) and the budget set \( \{ x \in \mathbb{R}^n_+ : p \cdot x \leq m \} \). You should not get confused.
1 Lemma If \( v \) satisfies the above conditions, then for any \( p, p' \in \mathbb{R}^n_{++}, \) and \( m \geq 0, \)
\[
v(p, 0) \leq v(p', m).
\]

Proof: Since \( p \gg 0 \) and \( p' \gg 0 \), there is some \( \lambda > 0 \) such that
\[
\lambda p \gg p'.
\]
There are now two cases, \( m > 0 \) and \( m = 0 \). If \( m > 0 \), then
\[
 v(p', m) > v(\lambda p, m) > v(\lambda p, 0) = v(\lambda p, \lambda 0) = v(p, 0).
\]
The case \( m = 0 \) is covered by the zero income property.

Another property that a standard of living measure ought to incorporate what I shall call the Ville–Lancaster Principle. Kelvin Lancaster [6, p. 65] argued that

An individual’s welfare has unambiguously increased from situation I to situation II if his choice is expanded as a result of the change, that is, if, in situation II he can have—

1. what he chose in situation I, and
2. at least one choice not available to him in situation I.

A similar definition of an increase in the standard of living also appears in Ville [13, 14]. I argue that this principle implies the standard of living measure \( v \) has the following property.

Quasiconvexity in \( p \). For any \( p^0, p^1 \in \mathbb{R}^n_{++}, m \geq 0 \), \( \lambda \) satisfying \( 0 < \lambda < 1, \)
\[
v((1 - \lambda)p^0 + \lambda p^1, m) \leq \max\{v(p^0, m), v(p^1, m)\}.
\]

To see why the Ville–Lancaster Principle implies quasiconvexity, let \( p^\lambda = (1 - \lambda)p^0 + \lambda p^1 \). The standard of living \( v(p^\lambda, m) \) is attained by purchasing some \( x^\lambda \) in the budget \( (p^\lambda, m) \).

It follows that \( \min\{p^0 \cdot x^\lambda, p^1 \cdot x^\lambda\} \leq m \). To see this, suppose to the contrary that \( p^0 \cdot x^\lambda > m \) and \( p^1 \cdot x^\lambda > m \). Then the convex combination of these two yields \( p^\lambda \cdot x^\lambda > m \), which contradicts the fact that \( x^\lambda \) is affordable at \( (p^\lambda, m) \).

Without loss of generality let \( p^0 \cdot x^\lambda = \min\{p^0 \cdot x^\lambda, p^1 \cdot x^\lambda\} \leq m \). Unless we have one of the trivial cases \( p^0 = p^\lambda = p^1 \) or \( m = 0 \) (in which case quasiconvexity is trivially true, at least if one assumes the zero income property), I claim that the budget \((p^0, m)\) unambiguously increases welfare (standard of living) from the budget \((p^\lambda, m)\) in Ville and Lancaster’s sense. To see this, first note that \( x^\lambda \), the choice from budget \((p^\lambda, m)\), also belongs to the budget \((p^0, m)\). Now assume \( p^1 \neq p^0 \) and \( m > 0 \). Then there is some commodity \( i \) such that \( p^0_i < p^1_i < p^1_i \). (Do you see why?) Set \( \alpha = m/p^0_i \). Then
\[
m = p^0 \cdot (\alpha e^i) < p^\lambda \cdot (\alpha e^i),
\]
so \( \alpha e^i \) is in the budget \((p^0, m)\), but not in the budget \((p^\lambda, m)\).

Thus the Ville–Lancaster Principle implies \( v(p^\lambda, m) \leq v(p^0, m) \), and quasiconvexity is assured.

We now show that with one technical assumption (upper semicontinuity in income) any standard of living index satisfying the properties above must be the indirect utility for some utility function. (In fact, upper semicontinuity in income makes the zero income condition redundant.)
2 The indirect utility problem

It is well known that the indirect utility derived from a locally nonsatiated continuous utility will be continuous and satisfy properties \( P-S \) below. (Property \( N \) is a harmless normalization.) See, e.g., Diewert [3], Varian [12, Section 7.3], Mas-Colell, Whinston, and Green [8, Prop. 3.D.3]. Moreover \( u \) is determined on the range of the demand function by the inversion formula

\[
u(x) = \inf_p v(p, p \cdot x)
\]

The obvious question is: If \( v \) satisfies properties \( N-S \) is it an indirect utility? Surprisingly, the standard texts mentioned above do not answer this question. The inversion formula determines what \( u \) ought to be, but it remains to prove that \( v \) is the maximized value of \( u \) over the budget set. Diewert [3, Theorem 4, p. 558] comes close to stating Theorem 2, but he makes the ad hoc assumption the function \( u \) derived by the inversion formula is continuous. In general it is not. Since proving Theorem 2, I discovered that I was scooped by Krishna and Sonnenschein [5] who do prove Theorem 2, as part of a larger project, but do so less directly.

2 Theorem Let \( v: \mathcal{B} \to \mathbb{R} \) satisfy the following properties:

\[\begin{align*}
\text{N (Nonnegativity)}: \quad & v(p, m) \geq 0 \text{ for all } (p, m). \\
\text{P (Monotonicity in } p): \quad & (m > 0 \& p' \gg p) \implies v(p', m) < v(p, m). \\
\text{M (Monotonicity in } m): \quad & m' > m \implies v(p, m') > v(p, m). \\
\text{H (Homogeneity)}: \quad & v(\lambda p, \lambda m) = v(p, m) \text{ for all } \lambda > 0. \\
\text{Q (Quasiconvexity in } p): \quad & v(p, m) \text{ is quasiconvex in } p. \\
\text{S (Upper semicontinuity)}: \quad & v \text{ is upper semicontinuous on } \mathcal{B}. \\
\text{Z (The zero property)}: \quad & \text{For all } p, p', v(p, 0) = v(p', 0) = \min_{(p, m) \in \mathcal{B}} v(p, m).
\end{align*}\]

Then there is an upper semicontinuous monotone quasiconcave utility \( u: \mathbb{R}_+^n \to \mathbb{R} \) such that

\[v(p, m) = \max \{u(x) : x \in \mathbb{R}_+^n \text{ and } p \cdot x \leq m\}.
\]

Strictly speaking, Property \( Z \) is redundant, and Property \( S \) can be weakened to upper semicontinuity in \( m \). The proofs of these claims get in the way of the main flow of the argument, so I shall defer discussion until later.

This theorem leaves one loose end. Suppose I start with a utility \( \hat{u} \) having indirect utility \( v \). Theorem 2 asserts that the utility \( u \) generated by the inversion formula also has indirect utility \( v \), but Example 5 below provides an example where \( u \) is not equal to \( \hat{u} \), so the question remains, do \( u \) and \( \hat{u} \) generate the same demand correspondence? If \( v \) is differentiable, then Roy’s Law\(^2\) guarantees that the demand correspondence is actually a function and the demand function is determined by \( v \). But we know that if \( \hat{u} \) is not quasiconcave, then the recovered utility \( u \) may

\(^2\)I use the term Roy’s Law instead of the more familiar Roy’s Identity since John Chipman has objected that Roy’s Law is not an identity. He also points out that René Roy was French, so his name should be pronounced like the French “roi.”
have a demand correspondence that properly includes the demand of \( \hat{u} \), so to hope to prove that the demand depends only on \( v \) we need to restrict ourselves to quasiconcave utilities. Theorem 3 below provides a generalization of Roy’s Law that holds even if \( v \) is not differentiable.

Proof of Theorem 2: The proof is divided into several steps.

Step 1. Construction of \( u \):
Define \( u \) via the inversion formula
\[
u(x) = \inf \{v(p, p \cdot x) : p \in \mathbb{R}^n_+ \}. \tag{*}\]
Property \( N \) (nonnegativity) guarantees that this infimum is not \(-\infty\), and in fact \( u \) is a nonnegative real-valued function defined for all \( x \geq 0 \).

Step 2. \( v \) is the indirect utility for \( u \):
I first claim that for any \((p, m) \in B\),
\[
\bar{p} \cdot x \leq m \implies u(x) \leq v(\bar{p}, \bar{m}). \tag{1}
\]
To see this note that
\[
u(x) = \inf_p v(p, p \cdot x) \leq v(\bar{p}, \bar{p} \cdot x) \leq v(\bar{p}, \bar{m}),
\]
where the last inequality follows from \( M \) (monotonicity in \( m \)).

Inequality (1) shows that
\[
v(\bar{p}, \bar{m}) \geq \sup \{u(x) : \bar{p} \cdot x \leq \bar{m} \}.
\]
We now need to prove that the supremum is attained at some point, and that the inequality holds with equality.

There are two cases to be dealt with, \( m = 0 \) and \( m > 0 \). Start with the case \( m = 0 \). Then
\[
\max \{u(x) : x \in \mathbb{R}^n_+ \text{ and } \bar{p} \cdot x \leq 0 \} = u(0) = \min_{p'} \{v(p', p' \cdot 0) : v(p', \bar{m}) < v(p, \bar{m})\} = v(p, 0)
\]
where the last equality follows from \( Z \).

For the case \( m > 0 \), fix \((\bar{p}, \bar{m})\) with \( \bar{m} > 0 \) and define
\[
C = \{p \in \mathbb{R}^n_+ : v(p, \bar{m}) < v(\bar{p}, \bar{m})\}.
\]
Since \( \bar{m} > 0 \), assumption \( P \) (monotonicity in \( p \)) implies that the set \( C \) is nonempty. It also implies that \( \bar{p} \) belongs to the boundary of \( C \). Assumption \( Q \) (quasiconvexity in \( p \)) implies that \( C \) is convex. Therefore by the Supporting Hyperplane Theorem (see, for example, [1, Theorem 7.36, p. 239]) there is a nonzero vector \( x^* \) properly supporting the closure of \( C \) at \( \bar{p} \).
(See Figure 1.) That is,
\[
x^* \cdot \bar{p} \leq x^* \cdot p \text{ for all } p \in C = \{p : v(p, \bar{m}) < v(\bar{p}, \bar{m})\}. \tag{2}
\]
Further, \( P \) implies that \( C \) is an increasing set, that is, \( p \in C \) and \( p' \gg p \) imply that \( p' \in C \) so it follows that \( x^* \geq 0 \), and we may normalize it so that
\[
\mathbf{p} \cdot x^* = \mathbf{m}. \tag{3}
\]
Since \( v \) is upper semicontinuous, \( C \) is open, so in fact:
\[
v(p, \mathbf{m}) < v(\mathbf{p}, \mathbf{m}) \implies p \cdot x^* > \mathbf{p} \cdot x^*. \tag{4}
\]
Now by definition
\[
u(x^*) = \inf \{v(p, p \cdot x^*) : p \in \mathbb{R}^n_{++}\}.
\]
By \( \mathbf{H} \) (homogeneity of degree zero in \( p \)), since \( x^* \) is nonzero, we may normalize all prices without changing the infimum, so that
\[
u(x^*) = \inf \{v(p, \mathbf{m}) : p \in \mathbb{R}^n_{++} \text{ and } p \cdot x^* = \mathbf{m}\}. \tag{5}
\]
Now by the contrapositive of (4),
\[
p \cdot x^* = \mathbf{m} \implies v(p, \mathbf{m}) \geq v(\mathbf{p}, \mathbf{m}).
\]
But \( \mathbf{p} \) satisfies \( \mathbf{p} \cdot x^* = \mathbf{m} \), so
\[
v(\mathbf{p}, \mathbf{m}) = \min \{v(p, \mathbf{m}) : p \in \mathbb{R}^n_{++} \text{ and } p \cdot x^* = \mathbf{m}\} = u(x^*).
\]
Thus (1) implies
\[
\max \{u(x) : \mathbf{p} \cdot x \leq \mathbf{m}\} = u(x^*) = v(\mathbf{p}, \mathbf{m})
\]
This completes the proof that \( v \) is the indirect utility for \( u \).

\textit{Step 3.} \( u \) is monotonic:
That is,
\[
x' \gg x \implies u(x') > u(x). \tag{6}
\]
To see this, let \( P = \{ p : p \cdot x = 1 \} \) and \( P' = \{ p : p \cdot x' = 1 \} \). As we argued above, \( P \) implies
\[
u(x) = \inf \{ v(p, 1) : p \in P \}, \quad u(x') = \inf \{ v(p, 1) : p \in P' \}.
\]
Since \( x' \gg x \), in particular \( x' \gg 0 \), so the closure \( \overline{P'} \) of \( P' \) is compact. Moreover \( \overline{P'} \) is disjoint \( P \) and lies below it. It should be obvious from Figure 2 that \( u(x) = \inf \{ v(p, 1) : p \in P \} < \inf \{ v(p, 1) : p \in P' \} = u(x') \), but here is a proof anyhow: There is a sequence \( p'_n \in P' \) with \( v(p'_n, 1) \to u(x') \). Since \( \overline{P'} \) is compact, the sequence must have a subsequence that converges to \( p' \in \overline{P'} \). Then there is some \( \lambda > 0 \) such that the price vector \( p'' = p' + \lambda 1 \) satisfies \( p'' \cdot x = 1 \). Let \( p''' \) be halfway between \( p' \) and \( p'' \). Since \( p''' \gg p' \) there is some \( \varepsilon > 0 \) such that if \( d(p, p') < \varepsilon \), then \( p''' \gg p \). So for \( n \) large enough \( p''' \gg p'_n \), so \( v(p'_n, 1) > v(p''' \), \( 1) \). (See Figure 2.) Therefore
\[
u(x') = \lim_{n \to \infty} v(p'_n, 1) \geq v(p'', 1) > v(p''' \), 1) \geq \inf_{p \in P} v(p, 1) = u(x).
\]
This proves monotonicity.

**Step 4.** \( u \) is quasiconcave:
That is, for \( 0 \leq \lambda \leq 1 \),
\[
u(\lambda x + (1 - \lambda)x') \geq \min\{u(x), u(x')\}.
\]
This is certainly the case if either \( x \) or \( x' \) is zero, since zero is a global minimizer of \( u \). So assume that neither \( x \) nor \( x' \) is zero. Then, we have already seen that \( H \) implies that
\[
u(\lambda x + (1 - \lambda)x') = \inf \{ v(p, 1) : p \cdot (\lambda x + (1 - \lambda)x') = 1 \}
= \inf \{ v(p, 1) : p \cdot (\lambda x + (1 - \lambda)x') \leq 1 \},
\]

\( ^3 \)Note that \( x' \gg x \) implies that \( x' \neq 0 \), so that we may normalize prices by \( p \cdot x' = 1 \). If \( x = 0 \), we can’t have \( p \cdot 0 = 1 \), but a simple modification of the following argument shows that \( u(x') > u(0) \). I’ll leave that to you.
where the second equality follows from \( \mathbf{M} \). Now if \( p \cdot (\lambda x + (1 - \lambda)x') \leq 1 \) we must have \( p \cdot x \leq 1 \) or \( p \cdot x' \leq 1 \) (or both.) But \( p \cdot x \leq 1 \) implies \( v(p, 1) \geq u(x) = \inf \{ v(p, 1) : p \cdot x \leq 1 \} \). Similarly \( p \cdot x' \leq 1 \) implies \( v(p, 1) \geq u(x') \). Either way,
\[
p \cdot (\lambda x + (1 - \lambda)x') \leq 1 \implies v(p, 1) \geq \min \{u(x), u(x')\}.
\]
Therefore
\[
u(\lambda x + (1 - \lambda)x') = \inf \left\{ v(p, 1) : p \cdot (\lambda x + (1 - \lambda)x') \leq 1 \right\} \geq \min \{u(x), u(x')\}.
\]
So \( u \) is quasiconcave.

**Step 5. \( u \) is upper semicontinuous:**
Recall that \( u \) is upper semicontinuous if for each real \( \alpha \) the strict lower contour set \( \{ x : u(x) < \alpha \} \) is open. So fix \( \alpha \) and pick \( \mathbf{\bar{x}} \) such that \( u(\mathbf{\bar{x}}) < \alpha \). Since
\[
u(\mathbf{\bar{p}}, \mathbf{\bar{p}} \cdot \mathbf{\bar{x}}) < \alpha.
\]
By \( \mathbf{S} \) (upper semicontinuity of \( v \)), there is a neighborhood \( U \) of \( \mathbf{\bar{x}} \) such that \( x \in U \) implies
\[
u(\mathbf{\bar{p}}, \mathbf{\bar{p}} \cdot x) < \alpha.
\]
Thus for \( x \in U \),
\[
u(\mathbf{\bar{p}}, \mathbf{\bar{p}} \cdot x) \leq v(\mathbf{\bar{p}}, \mathbf{\bar{p}} \cdot x) < \alpha.
\]
Thus \( u \) is upper semicontinuous.

This completes the proof of the theorem.

**Normalization**
Theorem 2 may appear to have an unnecessarily restrictive \( \mathbf{N} \) that \( v \geq 0 \). Among other things this rules out the indirect utility from a Cobb-Douglas utility in logarithmic form. Of course, this utility has a problem with \( x = 0 \), so we really ought to allow \( v \) to be extended real-valued so that it may take on the values \( \pm \infty \). The theorem above can deal with extended-valued \( v \) via the continuous invertible transformation \( g(x) = e^x/(1 + e^x) \) with the convention \( g(-\infty) = 0 \) and \( g(\infty) = 1 \). Then \( g: [-\infty, \infty] \rightarrow [0, 1] \) is strictly increasing and has inverse \( g^{-1}(y) = \ln y/(1 - \ln y) \) for \( 0 < y < 1 \) and \( g^{-1}(0) = -\infty \) and \( g^{-1}(1) = \infty \). If \( v \) satisfies assumptions (1)–(5), then \( \hat{v} = g \circ v \) satisfies assumptions (0)–(5). Therefore \( \hat{v} \) is the indirect utility of some \( \hat{u} \). This implies that \( v \) is the indirect utility of \( u = g^{-1} \circ \hat{u} \).
A variant of Roy’s Law

The proof of Theorem 2 suggests the following variant of Roy’s law, which does not require differentiability. It may be found in Diewert [3, Corollary 4.1, p. 558].

3 Theorem Let \( v \) be an indirect utility function satisfying the hypotheses of Theorem 2. Let \( x^* \) be the demand correspondence derived from the recovered utility. Then for \( \vec{m} > 0 \) and any \( \vec{p} \in R^n_+ \),

\[
x^*(\vec{p}, \vec{m}) = \{ x \in R^n_+ : p \cdot x = m & x \text{ supports } \{ p \in R^n_+ : v(p, m) < v(\vec{p}, \vec{m}) \} \text{ at } \vec{p} \}.
\]

Proof: First assume that \( \vec{x} \in x^*(\vec{p}, \vec{m}) \). Then \( \vec{p} \cdot \vec{x} \leq \vec{m} \) and \( u(\vec{x}) = v(\vec{p}, \vec{m}) \). We wish to prove that \( v(p, m) < v(\vec{p}, \vec{m}) \) implies \( \vec{x} \cdot \vec{p} > \vec{p} \cdot \vec{p} \). We shall prove the contrapositive. So assume that \( \vec{x} \cdot \vec{p} \leq \vec{x} \cdot \vec{p} \leq \vec{m} \). The \( \vec{x} \) belongs to the budget \( (p, m) \), so \( v(p, m) \geq u(\vec{x}) = v(\vec{p}, \vec{m}) \).

For the converse, we have already proven (see (4) in the proof of Theorem 2) that if \( x \) supports the convex set \( \{ p : v(p, m) < v(\vec{p}, \vec{m}) \} \) at \( \vec{p} \), then \( x \) belongs to the demand set \( x^*(\vec{p}, \vec{m}) \).

We can translate this result as follows.

4 Corollary Let \( v \) be an indirect utility function satisfying the hypotheses of Theorem 2. Let \( x^* \) be the demand correspondence derived from the recovered utility. Then for \( \vec{m} > 0 \) and any \( \vec{p} \in R^n_+ \),

\[
x^*(\vec{p}, \vec{m}) = \{ x \in R^n_+ : \vec{p} \cdot x = \vec{m} \text{ and any } \vec{p} \in R^n_+ \} = \min \{ v(p, m) : p \cdot x \leq \vec{m} \} \}.
\]

Proof: First note that the monotonicity of \( v \) in \( m \) implies that \( \min \{ v(p, m) : p \cdot x \leq \vec{m} \} = \min \{ v(p, m) : p \cdot x = \vec{m} \} \).

Assume that \( x \in R^n_+ \) satisfies \( v(\vec{p}, \vec{m}) = \min \{ v(p, m) : p \cdot x \leq \vec{m} \} \). Then if \( p \in R^n_+ \) satisfies \( v(p, m) < v(\vec{p}, \vec{m}) \), we must have \( p \cdot x > \vec{m} \). In other words, \( x \) supports \( \{ p \in R^n_+ : v(p, \vec{m}) < v(\vec{p}, \vec{m}) \} \) at \( \vec{p} \). So by the theorem above, \( x \in x^*(\vec{p}, \vec{m}) \).

Now assume that \( x \in x^*(p, m) \). Then \( x \) satisfies \( \vec{p} \cdot x = \vec{m} \) and by the theorem above \( x \) supports \( \{ p \in R^n_+ : v(p, \vec{m}) < v(\vec{p}, \vec{m}) \} \). That is, \( v(p, m) < v(\vec{p}, \vec{m}) \) implies \( p \cdot x > \vec{p} \cdot x = \vec{m} \).

By contraposition, if \( p \cdot x \leq \vec{m} \), then \( v(p, m) \geq v(\vec{p}, \vec{m}) \).

Local nonsatiation vs. monotonicity

It is easy to show that if \( u \) is a locally nonsatiated continuous utility, then its indirect utility satisfies properties \( P - S \). Theorem 2 tells us that such an indirect utility is the indirect utility for a monotone quasiconcave utility. Does this mean that every locally nonsatiated utility is monotone and quasiconcave? Of course not. But it is true that if \( u \) is locally nonsatiated, then there is a monotone quasiconcave \( \hat{u} \) with the same indirect utility function. This should not come as a surprise. For instance, Richter [9, p. 50] shows that if a demand function has convex range and obeys the Strong Axiom of Revealed Preference, then it has a monotone utility. Here we do not use convexity of the range.
5 Example Consider the quasilinear utility function for two goods $x$ and $y$ defined by

$$u(x, y) = y - (1 - x)^2$$

which gives a linear demand function for $x$. It is locally nonsatiated but not monotone. It has the property that the demand for $x$ never exceeds 1. It has the same demand behavior as the monotone utility

$$\hat{u}(x, y) = \begin{cases} 
    y - (1 - x)^2 & x \leq 1 \\
    y & x \geq 1 
\end{cases}$$

See Figure 3.

In general, you can show the following: Let $u$ be a locally nonsatiated continuous utility with indirect utility $v$, and let $\hat{u}(x) = \inf_p v(p, p \cdot x)$. If $x^*$ is in the range of the demand function for $u$, then $\hat{u}(x^*) = u(x^*)$ and

$$\{x \in R^n_+ : \hat{u}(x) \geq \hat{u}(x^*)\}$$

is the increasing convex hull of $\{x \in R^n_+ : u(x) \geq u(x^*)\}$. (The increasing convex hull of a set $E$ is the smallest increasing convex set that includes $E$. A set $E$ is increasing if $x \in E$ and $x' \gg x$ imply $x' \in E$. I should write down the proof some time. If $x$ is not in the range of the demand, then all bets are off, see Example 5.

3 Continuity of $u$

Is $u$ continuous? Not necessarily. The next example shows that even if $v$ is fully continuous, the recovered function $u$ may fail to be lower semicontinuous. However, in the example, the restriction of the utility to the range of its demand is continuous. This leads me to conjecture that the recovered utility is continuous on the range of its demand. There are sufficient conditions for lower semi-continuity of $u$, see, e.g., Sonnenschein [11], but they don’t seem useful in this case, so I am not yet sure of the status of this conjecture.
6 Example  Consider $v: \mathbb{R}^2_{++} \times \mathbb{R}_+^2$ defined by

$$v(p_1, p_2, m) = \begin{cases} \frac{m}{p_2} & p_2 \geq m \\ 1 + \frac{m - p_2}{p_1} & p_2 < m. \end{cases}$$

Note that for $p \gg 0$ we have $v(p, 0) = 0$, and that $v$ is homogeneous of degree zero. So for $m = 1$ we have

$$v(p_1, p_2, 1) = \begin{cases} \frac{1}{p_2} & p_2 \geq 1 \\ 1 + \frac{1 - p_2}{p_1} & p_2 < 1. \end{cases}$$

If $p_2 < m$, then $v$ is 1 minus the slope of the line segment from $(0, m)$ to $(p_1, p_2)$ (the slope is negative). See Figure 4 for level curves of $v(\cdot, 1)$. The level curves are open half-lines or open line segments and $v$ is monotone decreasing in $p$, so $v$ is quasiconvex. While it looks like there is a discontinuity at $p = (0, 1)$, the function $v$ is not defined there, and is indeed continuous.

Now consider the sequence $x_n = ((n + 1)/n, (n + 1)/n)$. The set $\beta(x_n) = \{p : p \cdot x_n = 1\}$ is the line segment with slope $-1$ and intercept $n/(n + 1)$. Thus $u(x_n) = n/(n + 1)$. Now $x_n \to \overline{x} = (1, 1)$. But $v$ is constant on $\beta(\overline{x})$ and has the value $1 - (-1) = 2$. Thus

$$x_n \to \overline{x} \quad \text{but} \quad u(x_n) = n/(n + 1) \to 1 \neq 2 = u(\overline{x}).$$

So we cannot strengthen the theorem to conclude that $u$ must be continuous. But note that $u(x_n)$ is an infimum, not a minimum, so $x_n$ does not belong to the range of $u$’s demand. There may yet be some hope of continuity on the range of the demand.
The complete description of the utility function is

\[ u(x_1, x_2) = \begin{cases} 
  x_2 & x_2 < 1 \\
  1 + x_1 & x_2 \geq 1.
\end{cases} \]

To see this, refer to the pale blue lines in Figure 4. The set \( \beta(x) = \{p : p \cdot x = 1\} \) is the relative interior of line segment joining the points \((0, 1/x_2)\) and \((1/x_1, 0)\). If \( x_2 > 1 \), then \( p_2 = 1/x_2 > 1 \), so the infimum occurs at the vertical axis as \( p_2 \to 1/x_2 \), so \( v \to 1/p_2 \to x_2 \). If \( x_2 < 1 \), then the infimum occurs at the horizontal axis as \( p_2 \to 0 \), so \( v = 1 + (1 - p_2)/p_1 \to 1 + 1/(1/x_1) \to 1 + x_1 \). If \( x_2 = 1 \), then \( v \) is constant on the segment, so \( v = 1 \) − slope. But the slope of the segment from \((0, 1)\) to \((1/x_1, 0)\) is \(-x_1\).

Thus the indifference curves are horizontal half-lines for \( x_2 < 1 \) and vertical half-lines for \( x_2 \geq 1 \). This leads to the following demand behavior: Buy up to 1 unit of good 2, income permitting, and spend the rest on good 1:

\[
\begin{align*}
  x_1^*(p, m) &= \begin{cases} 
    0 & p_2 > m \\
    m - p_2 & p_2 \leq m
  \end{cases} \\
  x_2^*(p, m) &= \begin{cases} 
    m & p_2 > m \\
    1 & p_2 \leq m.
  \end{cases}
\end{align*}
\]

Figure 5. Preferences and demand for Example 6.
The range of the demand function is the union of vertical axis from (0, 0) to (0, 1) with the horizontal half-line $x_2 = 1$ (shown in red in Figure 5). Note that the utility is continuous on the range of its demand. Also note that the range is not a convex set. □

4 Application to the Composite Commodity Theorem

The indirect utility approach provides perhaps the simplest proof of the composite commodity theorem. This theorem states how to aggregate commodities and their prices and treat the demand as being derived from a utility maximization problem. This problem was first solved by Hicks [4] and Leontief [7].

Start by partitioning the $n$ commodities into $K$ groups, group $k$ having $n_k$ elements. Assume the commodities are numbered so that we may write $x \in \mathbb{R}^n$ as $(x_1, \ldots, x_K)$ where each $x_k \in \mathbb{R}^{n_k}$. Fix a price vector $p = (p_1, \ldots, p_K)$, where each $p_k \in \mathbb{R}^{n_k}$, and define $\varphi: \mathbb{R}^K \rightarrow \mathbb{R}^n$ by

$$\varphi(\pi) = (\pi_1 p_1, \ldots, \pi_K p_K),$$

and note that $\varphi$ is linear and monotonic. Define $\xi: \mathbb{R}^n \rightarrow \mathbb{R}^K$ by

$$\xi(x) = (p_1 \cdot x_1, \ldots, p_K \cdot x_K),$$

and observe that $\xi$ is linear and maps $\mathbb{R}^n$ onto $\mathbb{R}^K$.

Observe the following. For all $x \in \mathbb{R}^n_+$, and $\pi \in \mathbb{R}^K_+$,

$$\pi \cdot \xi(x) = \varphi(\pi) \cdot x. \tag{7}$$

Let $\hat{u}: \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$ be a utility function with indirect utility function $v$ that satisfies the hypotheses of Theorem 2. Let $u$ be the utility defined from $v$ by the inversion formula, and let $x^*: \mathcal{B} \rightarrow \mathbb{R}^n_+$ be its demand function. (For simplicity of notation, assume there is a demand function, not a correspondence.) Define $X^*: \mathbb{R}^K_+ \times \mathbb{R} \rightarrow \mathbb{R}^K_+$ by

$$X^*(\pi, m) = \xi(x^*(\varphi(\pi), m)).$$

We can think of $X^*$ as a demand function for the composite commodities $\xi_1, \ldots, \xi_K$.

7 Composite Commodity Theorem. Under the conditions above, there is an upper semi-continuous quasiconcave monotone utility function

$$U: \mathbb{R}^K_+ \rightarrow \mathbb{R}_+$$

that generates the demand $X^*$.

Proof: Define $V$ by

$$V(\pi, m) = v(\varphi(\pi), m).$$

It is trivial to verify that $V$ satisfies the hypotheses N, M, M, H, and S of Theorem 2. Quasiconvexity Q is also straightforward, as

$$\{\pi \in \mathbb{R}^K_+: V(\pi, m) < \alpha\} = \varphi^{-1}\{\{p \in \mathbb{R}^n_+: v(p, m) < \alpha\}\}. $$
and the linear inverse of a convex set is convex.

Theorem 2 implies that the function $U: \mathbb{R}_+^K \to \mathbb{R}_+$ defined by $U(\xi) = \inf_\pi V(\pi, \pi \cdot \xi)$ is monotone, quasiconcave, upper semiconcave, and upper semicontinuous in $\pi$, $m > p$.

$$V(\pi, m) = \max \{ U(\xi) : \pi \cdot \xi \leq m \}.$$ It remains to show that $V(\pi, m) = U(X^*(\pi, m)) = U(\xi(x^*(\pi, m)))$.

So fix $(\pi, m)$, and to simplify notation, let $p = \varphi(\pi)$, $x^* = x^*(p, m)$, and $X^* = X^*(\pi, m) = \xi(x^*)$. Then

$$v(p, m) = u(x^*) \quad \text{Theorem 2}$$

$$= \inf_p v(p, p \cdot x^*) \quad \text{construction of } u$$

$$\leq \inf_\pi v(\varphi(\pi), \varphi(\pi) \cdot x^*) \quad \text{infimum over a smaller set of prices}$$

$$= \inf_\pi V(\pi, \pi \cdot x^*) \quad \text{construction of } V \text{ and equation (7)}$$

$$= U(\xi(x^*)) \quad \text{construction of } U$$

$$= U(X^*) \quad \text{construction of } X^*$$

$$\leq V(\pi, m) \quad \text{since } \pi \cdot X^*(\pi, m) \leq m$$

$$= v(\varphi(\pi), m) \quad \text{construction of } V$$

$$= v(p, m) \quad \text{construction of } p.$$ Thus all the inequalities are equalities, and we are done. \[\square\]

**A Appendix: Weakening the hypotheses.**

Earlier I asserted that in Theorem 2, we can drop hypothesis Z and weaken S to

$S'$ (Upper semicontinuity in $m$): $v(p, \cdot)$ is upper semicontinuous for each $p$.

Here is the step showing Z is redundant: Step 0. $v(p, 0)$ is independent of $p$ (the zero income property now follows from the others):

Since $v(p, \cdot)$ is upper semicontinuous in $m$ and decreasing in $m$, it is therefore continuous at $m = 0$ for each $p$. Pick $p, p' > 0$. For $\lambda > 0$ large enough, we have $\lambda p \gg p'$. Then for each $n \geq 1$,

$$v(p, 1/n) = v(\lambda p, \lambda/n) > v(p', \lambda/n),$$ where the equality follows from H, and the inequality follows from P. Letting $n \to \infty$, continuity at 0 implies $v(p, 0) \geq v(p', 0)$. But the roles of $p$ and $p'$ are symmetrical, so the reverse inequality also holds, implying

$$v(p, 0) = v(p', 0) \quad \text{for all } p, p' \quad (8)$$

Monotonicity now implies that $v(p', 0) = \min_{(p, m) \in B} v(p, m)$.

Now I show that in the presence of P and H, Property S' implies that $v$ is upper semicontinuous on $B$. To see this, suppose $v(p, m) < \alpha$.

First consider the case where $\overline{m} > 0$. By $S'$, for $\lambda > 1$ small enough, $v(p, \lambda \overline{m}) < \alpha$. Then setting $\gamma = 1/\lambda < 1$ we have

$$v(\gamma p, \overline{m}) = v(p, \lambda \overline{m}) < \alpha.$$ Thus $G = \{ p \in \mathbb{R}_+^m : p \ll \overline{p} \} \times \{ m \in \mathbb{R}_+ : m < \lambda \overline{m} \}$ is a neighborhood of $(\overline{p}, \overline{m})$ and for any $(p, m) \in G$, we have $v(p, m) < \alpha$. 

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For the case $m = 0$ implies that there is $m' > 0$ with $v(p, m') < \alpha$, and the argument above applies with $m'$ replacing $m$.
Thus $v$ is upper semicontinuous on $\mathcal{B}$.

References


