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Sums of sets, etc.*

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If E and F are subsets of $\boldsymbol{R}^{\mathrm{m}}$, define the **sum**

$$E + F = \{x + y : x \in E;\ y \in F\}.$$

More generally the sum $E_1 + \cdots + E_n$ is the set of vectors of the form $x_1 + \cdots + x_n$, where each $x_i \in E_i$.

The next result may be found for instance in [4]. It relies on the simple fact that

$$p \cdot (x_1 + \cdots + x_n) = p \cdot x_1 + \cdots + p \cdot x_n.$$

1 Lemma *Let $E_1, \ldots, E_n$ be sets in $\boldsymbol{R}^{\mathrm{m}}$, and put $E = E_1 + \cdots + E_n$. Let $x_i \in E_i$, $i = 1, \ldots, n$, and $x = x_1 + \cdots + x_n$. Then*

$$x \text{ maximizes } p \text{ over } E \iff (x_i \text{ maximizes } p \text{ over } E_i \text{ for each } i = 1, \ldots, n).$$

Proof: ($\implies$) Suppose by way of contradiction that for some j, $z \in E_j$ and $p \cdot z > p \cdot x_j$. Then $x' = x_1 + \cdots + x_{j-1} + z + x_{j+1} + \cdots + x_n \in E$, and $p \cdot x' > p \cdot x$, a contradiction.

($\impliedby$) Let $z \in E$. Then $z = z_1 + \cdots + z_n$, where each $z_i \in E_i$. By hypothesis, $p \cdot z_i \leqslant p \cdot x_i$ for each i, so summing we have $p \cdot z = p \cdot (z_1 + \cdots + z_n) \leqslant p \cdot (x_1 + \cdots + x_n) = p \cdot x$, so x maximizes p over E. ∎

1 Is a sum of closed sets closed?

An important question is whether the sum of closed sets is itself closed. The next example shows that it is not automatic.

2 Example The sum $E + F$ may fail to be closed even if E and F are closed. For instance, set

$$E = \{(x, y) \in \boldsymbol{R}^2 : y \geqslant 1/x \text{ and } x > 0\} \quad \text{and} \quad F = \{(x, y) \in \boldsymbol{R}^2 : y \geqslant -1/x \text{ and } x < 0\}$$

*These notes are largely based on Border [1], and provide some proofs omitted from Debreu [2].

Then E and F are closed, but

$$E + F = \{(x, y) \in \boldsymbol{R}^2 : y > 0\}$$

is not closed. □

To state sufficient conditions for the sum of closed sets to be closed we must make a fairly long digression.

2 Asymptotic cones

A **cone** is a nonempty subset of $\boldsymbol{R}^{\mathrm{m}}$ closed under multiplication by nonnegative scalars. That is, C is a cone if whenever $x \in C$ and $\lambda \in \boldsymbol{R}_+$, then $\lambda x \in C$. A cone is **nontrivial** if it contains a point other than zero.

3 Definition *Let $E \subset \boldsymbol{R}^{\mathrm{m}}$. The **asymptotic cone** of E, denoted $\boldsymbol{A}E$ is the set of all possible limits z of sequences of the form $(\lambda_n x_n)_n$, where each $x_n \in E$, each $\lambda_n > 0$, and $\lambda_n \to 0$. Let us call such a sequence a **defining sequence for $\boldsymbol{z}$**.*

This definition is equivalent to that in Debreu [2], and generalizes the notion of the recession cone of a convex set. This form of the definition was chosen because it makes most properties of asymptotic cones trivial consequences of the definition.

The **recession cone** 0^+F of a closed convex set F is the set of all directions in which F is unbounded, that is, $0^+F = \{z \in \boldsymbol{R}^{\mathrm{m}} : \big(\forall x \in F\big) \big(\forall \alpha \geqslant 0\big) \big[x + \alpha z \in F\big]\}$. (See Rockafellar [5, Theorem 8.2].)

4 Lemma **(a)** *$\boldsymbol{A}E$ is indeed a cone.*

(b) *If $E \subset F$, then $\boldsymbol{A}E \subset \boldsymbol{A}F$.*

(c) *$\boldsymbol{A}(E + x) = \boldsymbol{A}E$ for any $x \in \boldsymbol{R}^{\mathrm{m}}$.*

(cc) *$0^+E \subset \boldsymbol{A}E$.*

(d) *$\boldsymbol{A}E_1 \subset \boldsymbol{A}(E_1 + E_2)$.*

(e) *$\boldsymbol{A}\prod_{i \in I} E_i \subset \prod_{i \in I} \boldsymbol{A}E_i$.*

(f) *$\boldsymbol{A}E$ is closed.*

(g) *If E is convex, then $\boldsymbol{A}E$ is convex.*

(h) *If E is closed and convex, then $\boldsymbol{A}E = 0^+E$. (The asymptotic cone really is a generalization of the recession cone.)*

(i) *If C is a cone, then $\boldsymbol{A}C = \overline{C}$.*

(j) $\boldsymbol{A}\bigcap_{i\in I} E_i \subset \bigcap_{i\in I} \boldsymbol{A}E_i$. *The reverse inclusion need not hold.*

(k) *If* $E+F$ *is convex, then* $\boldsymbol{A}E+\boldsymbol{A}F \subset \boldsymbol{A}(E+F)$.

(l) *A set* $E \subset \boldsymbol{R}^{\mathrm{m}}$ *is bounded if and only if* $\boldsymbol{A}E=\{0\}$.

Proof: Here are proofs of selected parts. The others are easy, and should be treated as an exercise.

(cc) $0^+E \subset \boldsymbol{A}E$.

Let $z \in 0^+E$. Then for any $n>0$ and any $x \in E$, we have $x+nz \in E$. But $\frac{1}{n}(x+nz) \to z$, so $z \in \boldsymbol{A}E$.

(d) $\boldsymbol{A}E_1 \subset \boldsymbol{A}(E_1+E_2)$.

For $x_2 \in E_2$, by definition $E_1+x_2 \subset E_1+E_2$, so by (b), $\boldsymbol{A}(E_1+x_2) \subset \boldsymbol{A}(E_1+E_2)$, so by (c), $\boldsymbol{A}E_1 \subset \boldsymbol{A}(E_1+E_2)$.

(f) $\boldsymbol{A}E$ is closed.

Let x_n be a sequence in $\boldsymbol{A}E$ with $x_n \to x$. For each m there is a sequence $\lambda_{n,m}x_{n,m}$ with $\lim_m \lambda_{n,m}x_{n,m}=x_n$, $\lambda_{n,m}\to 0$ as $m\to\infty$, $x_{n,m}\in E$, and each $\lambda_{n,m}>0$. Then for each k there is N_k such that for all $n \geqslant N_k$, $\|x_n-x\|<1/k$, and M_k such that for all $m \geqslant M_k$, $\|\lambda_{N_k,m}x_{N_k,m}-x_{N_k}\|<1/k$, and L_k such that for all $m\geqslant L_k$, $\lambda_{N_k,m}<1/k$. Set $P_k=\max\{M_k,L_k\}$, $y_k=x_{N_k,P_k}$, and $\lambda_k=\lambda_{N_k,P_k}$. Then each $\lambda_k>0$, $\lambda_k\to 0$ and $\|\lambda_k y_k - x\|<2/k$, so $x\in\boldsymbol{A}E$.

(g) If E is convex, then $\boldsymbol{A}E$ is convex.

Let $x,y\in\boldsymbol{A}E$ and $\alpha\in[0,1]$. Since $\boldsymbol{A}E$ is a cone, $\alpha x\in\boldsymbol{A}E$ and $(1-\alpha)y\in\boldsymbol{A}E$. Thus there are defining sequences $\lambda_n x_n\to\alpha x$ and $\gamma_n y_n \to (1-\alpha)y$. Since E is convex, $z_n=\frac{\lambda_n}{\gamma_n+\lambda_n}x_n+\frac{\gamma_n}{\gamma_n+\lambda_n}y_n\in E$ for each n. Set $\delta_n=\gamma_n+\lambda_n>0$. Then $\delta_n\to 0$ and $\delta_n z_n=\lambda_n x_n+\gamma_n y_n\to\alpha x+(1-\alpha)y$. Thus $\alpha x+(1-\alpha)y\in\boldsymbol{A}E$.

(h) If E is closed and convex, then $\boldsymbol{A}E=0^+E$.

In light of (cc), it suffices to prove that $\boldsymbol{A}E\subset 0^+E$, so let $z\in\boldsymbol{A}E$, $x\in E$, and $\alpha\geqslant 0$. We wish to show that $x+\alpha z\in E$. By definition of $\boldsymbol{A}E$ there is a sequence $\lambda_n z_n\to z$ with $z_n\in E$, $\lambda_n>0$, and $\lambda_n\to 0$. Then for n large enough $0\leqslant\alpha\lambda_n<1$, so $(1-\alpha\lambda_n)x+\alpha\lambda_n z_n\in E$ as E is convex. But $(1-\alpha\lambda_n)x+\alpha\lambda_n z_n\to x+\alpha z$. Since E is closed, $x+\alpha z\in E$.

(i) If C is a cone, then $\boldsymbol{A}C=\overline{C}$.

It is easy to show that $C\subset\boldsymbol{A}C$, as $\frac{1}{n}nx\to x$ is a defining sequence. Since $\boldsymbol{A}C$ is closed by (f), we have $\overline{C}\subset\boldsymbol{A}C$. On the other hand if $\lambda_n\geqslant 0$ and $x_n\in C$, then $\lambda_n x_n\in C$, as C is a cone, so $\boldsymbol{A}C\subset\overline{C}$.

(j) $\boldsymbol{A}\bigcap_{i\in I} E_i \subset \bigcap_{i\in I} \boldsymbol{A}E_i$. The reverse inclusion need not hold.

By (b), $\boldsymbol{A}\bigcap_{i\in I} E_i \subset \boldsymbol{A}E_j$ for each j, so $\boldsymbol{A}\bigcap_{i\in I} E_i \subset \bigcap_{i\in I} \boldsymbol{A}E_i$.

For a failure of the reverse inclusion, consider the even nonnegative integers $E_1 = \{0, 2, 4, \ldots\}$ and the odd nonnegative integers $E_2 = \{1, 3, 5, \ldots\}$. Then $E_1 \cap E_2 = \varnothing$, so $\boldsymbol{A}(E_1 \cap E_2) = \varnothing$, but $\boldsymbol{A}E_1 = \boldsymbol{A}E_2 = \boldsymbol{A}E_1 \cap \boldsymbol{A}E_2 = \boldsymbol{R}_+$.

(k) If $E + F$ is convex, then $\boldsymbol{A}E + \boldsymbol{A}F \subset \boldsymbol{A}(E + F)$.

Let z belong to $\boldsymbol{A}E + \boldsymbol{A}F$. Then there exist defining sequences $(\lambda_n x_n) \subset E$ and $(\alpha_n y_n) \subset F$ with $\lambda_n x_n + \alpha_n y_n \to z$. Let $x' \in E$ and $y' \in F$. (If either E or F is empty, the result is trivial.) Then $\big(\lambda_n(x_n + y')\big) \subset E + F$ and $\big(\alpha_n(x' + y_n)\big) \subset E + F$, so

$$(\lambda_n + \alpha_n)\left(\frac{\lambda_n}{\lambda_n + \alpha_n}(x_n + y') + \frac{\alpha_n}{\lambda_n + \alpha_n}(x' + y_n)\right) \to z,$$

is a defining sequence for z in $E + F$.

(l) A set $E \subset \boldsymbol{R}^{\mathrm{m}}$ is bounded if and only if $\boldsymbol{A}E = \{0\}$.

If E is bounded, clearly $\boldsymbol{A}E = \{0\}$. If E is not bounded, let $\{x_n\}$ be an unbounded sequence in E. Then $\lambda_n = \|x_n\|^{-1} \to 0$ and $(\lambda_n x_n)$ is a sequence on the unit sphere, which is compact. Thus there is a subsequence converging to some x in the unit sphere. Such an x is a nonzero member of $\boldsymbol{A}E$.

■

5 Example The asymptotic cone of a non-convex set need not be convex. Let $E = \{(x, y) \in \boldsymbol{R}^2 : y = \frac{1}{x},\ x > 0\}$. This hyperbola is not convex and its asymptotic cone is the union of the nonnegative x- and y-axes. But the asymptotic cone of a non-convex set may be convex. Just think of the integers in $\boldsymbol{R}^1$. □

6 Example It need not be the case that $\boldsymbol{A}(E + F) \subset \boldsymbol{A}E + \boldsymbol{A}F$, even if E and F are closed and convex. For instance, let E be the set of points lying above a standard parabola:

$$E = \{(x, y) : y \geqslant x^2\}.$$

The asymptotic cone of E, which is the same as its recession cone, is just the positive y-axis:

$$\boldsymbol{A}E = \{(0, y) : y \geqslant 0\}.$$

So $\boldsymbol{A}E + \boldsymbol{A}(-E)$ is just the y-axis. Now observe that $E + (-E) = \boldsymbol{R}^2$, so $\boldsymbol{A}\big(E + (-E)\big) = \boldsymbol{R}^2$. Thus

$$\boldsymbol{A}E + \boldsymbol{A}(-E) \subsetneq \boldsymbol{A}\big(E + (-E)\big).$$

□

3 When a sum of closed sets is closed

We now turn to the question of when a sum of closed sets is closed. The following definition may be found in Debreu [2, 1.9. m., p. 22].

7 Definition *Let $C_1, \ldots, C_n$ be cones in $\boldsymbol{R}^{\mathrm{m}}$. We say that they are* ***positively semi-independent*** *if whenever $x_i \in C_i$ for each $i = 1, \ldots, n$,*

$$x_1 + \cdots + x_n = 0 \quad \Longrightarrow \quad x_1 = \cdots = x_n = 0.$$

Clearly, any subcollection of a collection of semi-independent cones is also semi-independent. Note that in Example 6, $\boldsymbol{A}(-E) = -\boldsymbol{A}(E)$, so these nontrivial asymptotic cones are not positively semi-independent.

8 Theorem (Closure of the sum of sets) *Let $E, F \subset \boldsymbol{R}^{\mathrm{m}}$ be closed and nonempty. Suppose that $\boldsymbol{A}E$ and $\boldsymbol{A}F$ are positively semi-independent. (That is, $x \in \boldsymbol{A}E$, $y \in \boldsymbol{A}F$ and $x + y = 0$ together imply that $x = y = 0$.) Then $E + F$ is closed, and $\boldsymbol{A}(E + F) \subset \boldsymbol{A}E + \boldsymbol{A}F$.*

The proof relies on the following simple lemma, which is closely related to Lemma 1 in Gale and Rockwell [3].

9 Lemma *Under the hypotheses of Theorem 8, if (λ_n) is a bounded sequence of real numbers with each $\lambda_n > 0$, (x_n) is a sequence in E, and (y_n) is a sequence in F, and if $\lambda_n(x_n + y_n)$ converges to some point, then there is a common subsequence along which both $(\lambda_k x_k)$ and $(\lambda_k y_k)$ converge.*

Proof: It suffices to prove that both $(\lambda_n x_n)$ and $(\lambda_n y_n)$ are bounded sequences. Suppose by way of contradiction that $\lambda_n(x_n + y_n)$ converges to some point, but say $(\lambda_n x_n)$ is unbounded. Since (λ_n) is bounded, it must be the case that both $\|\lambda_n x_n\| \to \infty$ and $\|x_n\| \to \infty$, so for large enough n we have $\|\lambda x_n\| > 0$. Thus for large n we may divide by $\|\lambda_n x_n\|$ and define

$$\hat{x}_n = \frac{\lambda_n}{\|\lambda_n x_n\|} x_n, \quad \hat{y}_n = \frac{\lambda_n}{\|\lambda_n x_n\|} y_n, \quad \hat{z}_n = \frac{\lambda_n}{\|\lambda_n x_n\|}(x_n + y_n),$$

and observe that

$$\hat{z}_n = \hat{x}_n + \hat{y}_n.$$

But $\big(\lambda_n(x_n + y_n)\big)$ is convergent, and hence bounded, so $\hat{z}_n \to 0$. In addition the sequence $(\hat{x}_n)$ lies on the unit sphere, so it has a convergent subsequence, say $\hat{x}_k \to \hat{x}$, where $\|\hat{x}\| = 1$. Then

$$\hat{y}_k = \hat{z}_k - \hat{x}_k \to -\hat{x}.$$

But $\hat{y}_k = (\lambda_k / \|\lambda_k x_k\|) y_k$, and $\lambda_k / \|\lambda_k x_k\| \to 0$, so $(\lambda_k / \|\lambda_k x_k\|) y_k$ is a defining sequence that puts $-\hat{x} \in \boldsymbol{A}F$. But a simialr argument shows that $\hat{x} \in \boldsymbol{A}E$. Since $\boldsymbol{A}E$ and $\boldsymbol{A}F$ are positively semi-independent, it follows that $\hat{x} = 0$, contradicting $\|\hat{x}\| = 1$.

Thus $(\lambda_n x_n)$, is a bounded sequence, and by a similar argument so is $(\lambda_n y_n)$, so they have common subsequence on which they both converge. ∎

Proof of Theorem 8: First, $E + F$ is closed: Let $x_n + y_n \to z$ with $\{x_n\} \subset E$, $\{y_n\} \subset F$. By Lemma 9 (with $\lambda_n = 1$ for all n) there is a common subsequence with $x_k \to x$ and $y_k \to y$. Since E and F are closed, $x \in E$ and $y \in F$. Therefore $z = x + y \in E + F$, so $E + F$ is closed.

To see that $\boldsymbol{A}(E + F) \subset \boldsymbol{A}E + \boldsymbol{A}F$, let $z \in \boldsymbol{A}(E + F)$. That is, z is the limit of a defining sequence $\big(\lambda_n(x_n + y_n)\big)$, where $x_n \in E$ and $y_n \in F$. Since $\lambda_n \to 0$, it is a bounded sequence. Thus by Lemma 9 there is a common convergent subsequence, and by definition $\lim_k \lambda_k x_k \in \boldsymbol{A}E$ and $\lim_k \lambda_k y_k \in \boldsymbol{A}F$, so $z \in \boldsymbol{A}E + \boldsymbol{A}F$. ∎

10 Corollary *Let $E_i \subset \boldsymbol{R}^{\mathrm{m}}$, $i = 1, \dots, n$, be closed and nonempty. If $\boldsymbol{A}E_i$, $i = 1, \dots, n$, are positively semi-independent, then $\sum_{i=1}^n E_i$ is closed, and $\boldsymbol{A}\sum_{i=1}^n E_i \subset \sum_{i=1}^n \boldsymbol{A}E_i$.*

Proof: This follows from Theorem 8 by induction on n. ∎

11 Corollary *Let $E, F \subset \boldsymbol{R}^{\mathrm{m}}$ be closed and let F be compact. Then $E + F$ is closed.*

Proof: A compact set is bounded, so by Lemma 4(l) its asymptotic cone is $\{0\}$. Apply Theorem 8. ∎

4 When is an intersection of closed sets bounded?

12 Proposition *Let $E_i \subset \boldsymbol{R}^{\mathrm{m}}$, $i = 1, \dots, n$, be nonempty. If $\bigcap_{i=1}^n \boldsymbol{A}E_i = \{0\}$, then $\bigcap_{i=1}^n E_i$ is bounded.*

Proof: By Lemma 4(l), $\bigcap_{i=1}^n E_i$ is bounded if and only if $\boldsymbol{A}\left(\bigcap_{i=1}^n E_i\right) = \{0\}$. But by Lemma 4(j), $\boldsymbol{A}\left(\bigcap_{i=1}^n E_i\right) \subset \bigcap_{i=1}^n \boldsymbol{A}E_i$, and the proposition follows. ∎

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