Topic 9: Support Functions

9.1 Support functions

The Separating Hyperplane Theorem 8.3.1 is the basis for a number of results concerning closed convex sets. Given any set \( A \) in \( \mathbb{R}^m \), its closed convex hull \( \overline{A} \) is by definition the intersection of all closed convex sets that include \( A \). But Theorem 8.3.4 sharpens this result to

\[
\overline{A} = \bigcap \{ H : A \subset H \text{ and } H \text{ is a closed half space} \}.
\]

So an already closed convex set is the intersection of all the closed half spaces that include it.

The support function of a set \( A \) is a handy way to summarize all the closed half spaces that include \( A \). There are two ways to define support functions, and I will give them names inspired by economics.\(^1\)

The cost function \( \mu_A \) of \( A \) is defined by

\[
\mu_A(p) = \inf \{ p \cdot x : x \in A \}.
\]

The profit function \( \pi_A \) of \( A \) is defined by

\[
\pi_A(p) = \sup \{ p \cdot x : x \in A \}.
\]

Clearly

\[
\pi_A(p) = -\mu_A(-p).
\]

We allow for the case that \( \mu_A(p) = -\infty \) or \( \pi_A(p) = \infty \). The set of points where the support function of a nonempty set is finite is a convex cone. Note that \( \mu_{\emptyset} \) is the improper concave function \( +\infty \). Also note that the infimum or supremum may not actually be attained even if it is finite. For instance, consider the closed convex set \( A = \{ (x, y) \in \mathbb{R}^2_+ : xy \geq 1 \} \), and let \( p = (0, 1) \). Then \( \mu_A(p) = 0 \) even though \( p \cdot (x, y) = y > 0 \) for all \( (x, y) \in A \). If \( A \) is compact, then of course \( \mu_A \) and \( \pi_A \) are always finite, and the infimum and supremum are achieved as a maximum and minimum.

It is possible to restate Corollary 8.3.2 as follows.

9.1.1 Corollary Let \( C \) be a nonempty closed convex subset of a Hilbert space. Assume that the point \( x \) does not belong to \( C \). Then there exists a nonzero \( p \) such that

\[ p \cdot x < \mu_C(p) \]

\(^1\)Fenchel [3] and Roko and I [1, p. 288] call the profit function the support function, while Mas-Colell, Whinston, and Green [4] use the cost function.
and letting $p' = -p$, we have

$$p' \cdot x > \pi_C(p').$$

Theorem 8.3.4 yields the following description of $\overline{\partial} A$ in terms of $\mu_A$ and $\pi_A$.

**9.1.2 Theorem** For any set $A$ in $\mathbb{R}^m$,

$$\overline{\partial} A = \left\{ x \in \mathbb{R}^m : (\forall p \in \mathbb{R}^m) \ [p \cdot x \geq \mu_A(p)] \right\},$$

and

$$\overline{\partial} A = \left\{ x \in \mathbb{R}^m : (\forall p \in \mathbb{R}^m) \ [p \cdot x \leq \pi_A(p)] \right\},$$

Moreover, $\mu_A = \mu_{\overline{\partial} A}$ and $\pi_A = \pi_{\overline{\partial} A}$.

**Proof:** I shall just prove the results about the cost function $\mu$. Observe that

$$C := \left\{ x \in \mathbb{R}^m : (\forall p \in \mathbb{R}^m) \ [p \cdot x \geq \mu_A(p)] \right\} = \bigcap_{p \in \mathbb{R}^m} \{ p \geq \mu_A(p) \}$$

is an intersection of closed half spaces. By definition, if $x \in A$, then $p \cdot x \geq \mu_A(p)$, that is, $A \subset \{ p \geq \mu_A(p) \}$. Thus by Theorem 8.3.4, $\overline{\partial} A \subset C$.

For the reverse inclusion, suppose $x \notin \overline{\partial} A$. By Corollary 9.1.1 there is a nonzero $p$ such that $\mu_A(p) > p \cdot x$, so $x \notin C$.

To see that $\mu_A = \mu_{\overline{\partial} A}$ first note that $\mu_A \geq \mu_{\overline{\partial} A}$ since $A \subset \overline{\partial} A$. The first part of the theorem implies $\mu_{\overline{\partial} A} \geq \mu_A$.

**9.1.3 Theorem** Let $A$ be a nonempty closed convex set in $\mathbb{R}^m$.

Then the profit function $\pi_A$ is a regular (proper and lower semicontinuous) convex and homogenous function on $\mathbb{R}^m$.

The cost function $\mu_A$ is a regular (proper and upper semicontinuous) concave and homogenous function on $\mathbb{R}^m$.

**Proof:** Homogeneity of the cost and profit functions is obvious.

Each $x$ defines a linear (and therefore both concave and convex) function $\ell_x$ via $\ell_x : p \mapsto p \cdot x$. Moreover each $\ell_x$ is continuous, and therefore both upper and lower semicontinous.

Now $\mu_A = \inf_{x \in A} \ell_x$, so by Exercise 1.3.3 (4), it is concave and by Proposition A.8.4 it is upper semicontinuous. Similarly, $\pi_A$ is convex and lower semicontinuous.

Let $x$ belong to $A$ (it is nonempty). Then for any $p$, $\pi_A(p) \geq p \cdot x$, so $\pi_A(p)$ is never $-\infty$. And $pi_A(0) = 0$, so dom $\pi_A$ is nonempty. Therefore $\pi_A$ is a proper convex function. Similarly, $\mu_A$ is a proper concave function.

Theorem 9.1.2 asserts that we can recover the $A$ (more precisely its closed convex hull) from the profit function $\pi_A$. Theorem 9.1.3 asserts that $\pi_A$ is a homogeneous regular convex function. Suppose we take an arbitrary homogeneous regular convex function $\mathbb{R}^m$, is it the support function of some nonempty closed convex set? Yes. (And there is an analogous result for regular concave functions.)
9.1.4 Theorem Let \( f : \mathbb{R}^m \rightarrow \mathbb{R}^# \) be a homogeneous regular convex function. Then \( f \) is the profit function \( \pi_{C_f} \) of the nonempty closed convex set 

\[
C_f = \{ x \in \mathbb{R}^m : (\forall p \in \mathbb{R}^m) [ p \cdot x \leq f(p)] \}
\]

\[
= \{ x \in \text{dom } f : (\forall p \in \mathbb{R}^m) [ p \cdot x \leq f(p)] \}.
\]

Proof: The first thing is to note that if \( p \notin \text{dom } f \), then \( f(p) = \infty \), so \( p \cdot x < f(p) \) for all \( x \), so the two definitions of \( C_f \) agree.

Now we show that \( C_f \) is closed and convex. Since \( f \) is proper, \( \text{dom } f \) is a nonempty convex cone, so it contains 0 and by homogeneity \( f(0) = 0 \). Now \( \{ x : 0 \cdot x \leq 0 \} = \mathbb{R}^m \). For each nonzero \( p \in \text{dom } f \), the set \( \{ x : p \cdot x \leq f(p) \} \) is a closed hyperplane. Since \( C_f \) is the intersection of these sets, it is closed and convex.

The proof that \( C_f \) is nonempty is subtle, and is a byproduct of the following argument.

Fix some \( \bar{p} \in \text{dom } f \). If \( x \in C_f \), then by definition \( \bar{p} \cdot x \leq f(\bar{p}) \), so 

\[
\pi_{C_f}(\bar{p}) = \sup_{x \in C_f} \bar{p} \cdot x \leq f(\bar{p}).
\]

We need to show that the supremum is actually equal to \( f(\bar{p}) \). This means that for every \( \varepsilon > 0 \), we wish to find some \( y \in C_f \) so that \( \bar{p} \cdot y \) is within \( \varepsilon \) of \( f(p) \), that is,

\[
f(\bar{p}) - \varepsilon \leq \bar{p} \cdot y \leq f(\bar{p}).
\]

Since \( f \) is proper and convex, its epigraph is nonempty and convex. Since \( f \) is homogeneous, its epigraph is a cone. And since \( f \) is lower semicontinuous, its epigraph is closed. Now the point \( (\bar{p}, f(\bar{p}) - \varepsilon) \) does not belong to \( \text{epi } f \), which is nonempty, closed, and convex, so by the Strong Separating Hyperplane Theorem 8.3.1 we can separate it from the epigraph. That is, there exists some \( (x, \lambda) \in \mathbb{R}^{m+1} \) satisfying

\[
(x, \lambda) \cdot (\bar{p}, f(\bar{p}) - \varepsilon) < (x, \lambda) \cdot (p, \beta) \quad \text{for all } (p, \beta) \in \text{epi } f.
\]

Expanding this and evaluating at \( \beta = f(p) \), gives

\[
\bar{p} \cdot x + \lambda f(\bar{p}) - \lambda \varepsilon < 0 \leq p \cdot x + \lambda f(p) \quad \text{for all } p.
\]

Evaluating the right-hand side at \( p = \bar{p} \) implies \(-\lambda \varepsilon < 0 \), so

\[
\lambda > 0.
\]

Taking just the left-hand inequality gives

\[
\bar{p} \cdot x < \lambda \varepsilon - \lambda f(\bar{p}),
\]
so dividing both sides by $-\lambda < 0$ reverses the inequality, and setting $y = -(1/\lambda)x$ gives

$$\bar{p} \cdot y > f(\bar{p}) - \varepsilon.$$ 

It remains to show that $y$ belongs to $C_f$. Since $x = -\lambda y$, the right-hand side of (2) says that

$$0 \leq -\lambda p \cdot y + \lambda f(p) \quad \text{for all } p,$$

which, since $\lambda > 0$, implies $f(p) \geq p \cdot y$ for all $p$. In other words, $y \in C_f$. This does two things. It shows that $f = \pi_{C_f}$ and also that $C_f$ is nonempty. 

The technique of separating a point from an epigraph will appear later on in connection with regular convex functions and subgradients.

### 9.2 Sublinear functions

#### 9.2.1 Definition

A function $f$ from a convex cone $C$ in a real vector space into $\mathbb{R}$ is

**positively homogeneous of degree 1** if for every vector $x \in C$ and every real $\lambda > 0$,

$$f(\lambda x) = \lambda f(x).$$

We usually shorten this by saying simply that $f$ is **homogeneous**.

**subadditive** if for all vectors $x$ and $y$ in $C$,

$$f(x + y) \leq f(x) + f(y).$$

**superadditive** if for all vectors $x$ and $y$ in $C$,

$$f(x + y) \geq f(x) + f(y).$$

**sublinear** if it is both homogeneous and subadditive.

#### 9.2.2 Remark

- There are other notions of homogeneity. More generally, $f$ is positively homogeneous of degree $k$ if for every $\lambda > 0$, we have $f(\lambda x) = \lambda^k f(x)$. If I ever mean anything other than of homogeneity of degree one, I will make it explicit.

- By these definitions we ought to say that $f$ is **superlinear** if it is both homogeneous and superadditive, but I’ve never heard the term.

- Note the definition of homogeneity restricts attention to $\lambda > 0$, not $\lambda \geq 0$. This avoids the question of deciding how to interpret $0 \cdot \infty$ for extended real valued functions. (If $\lambda > 0$, then $\lambda \cdot \infty = \infty$.)
• Note that for a homogeneous function defined at 0, we have \( f(0) = f(\lambda 0) = \lambda f(0) \) for any \( \lambda > 0 \), so \( f(0) = 0 \).

• So if \( f \) is homogenous on the punctured cone \( C \setminus \{0\} \), where \( C \) is a true cone, it can be extended to be homogenous on all of \( C \) simply by setting \( f(0) = 0 \).

• Note that \( f \) defined by \( f(0) = 0 \), and \( f(x) = \infty \) for \( x \neq 0 \) is homogeneous.

• If \( f \) is a homogeneous function on a cone \( C \), we can extend it to be homogenous on the entire vector space by setting \( f(x) = \infty \) for any nonzero \( x \) not in \( C \).

9.2.3 Exercise A homogeneous function is subadditive if and only it is convex. It is superadditive if and only if it is concave.

The epigraph of a sublinear function is a convex cone. The hypograph of a homogeneous concave function is a convex cone.

\[ \square \]

9.3 Gauge functions

9.3.1 Definition The **gauge function**, or more simply the **gauge**, \( p_A \) of a subset \( A \) of a vector space is defined by

\[ p_A(x) = \inf\{\alpha > 0 : x \in \alpha A\}, \]

where, you may recall, \( \inf \emptyset = \infty \).

Note that the gauge of \( A \) is always nonnegative.

9.3.2 Example

• The most important example of a gauge is a norm. If \( U \) is the closed unit ball in a normed space,

\[ U = \{x \in X : \|x\| \leq 1\}, \]

then

\[ p_\mathbb{U}(x) = \|x\|. \]

• The gauge of \( \mathbb{R} \) is always zero. So is the gauge of the integers. Thus two distinct sets may have the same gauge.

\[ \square \]

We shall see in a moment (Lemma 9.3.6) that gauges are the nonnegative sublinear functions.

9.3.3 Definition
• A set \( A \) is **star-shaped about zero** if \( x \in A \) implies that the line segment \([0, x]\) is a subset of \( A \).

• A set \( A \) is **absorbing** if it is star-shaped about zero and its gauge \( p_A \) is everywhere finite. This means that for every \( x \in X \), there is some \( \lambda_x \) so that for \( 0 < \lambda \leq \lambda_x \), we have \( \lambda x \in A \).

• A set \( A \) is **circled**, or **balanced**, if for each \( x \in A \) the line segment \([-x, x]\) lies in \( A \).

**9.3.4 Remark**

• A balanced set is star-shaped about zero.

• Circled sets have the property that the gauge \( p_A \) satisfies \( p_A(x) = p_A(-x) \).

• The unit ball in any normed space is convex, absorbing, and circled.

**9.3.5 Definition**

• A **seminorm** is a subadditive function \( p : X \to \mathbb{R} \) (not \( \mathbb{R}^! \)) on a vector space satisfying

\[
p(\alpha x) = |\alpha| p(x)
\]

for all \( \alpha \in \mathbb{R} \) and all \( x \in X \).

• A seminorm \( p \) that satisfies \( p(x) = 0 \) if and only if \( x = 0 \) is called a **norm**.

**9.3.6 Lemma** For a nonnegative extended real function \( f : X \to \mathbb{R}^\ast \) on a vector space, we have the following.

1. \( f \) is homogeneous if and only if it is a gauge function, in which case it is the gauge of the set

\[
U_f = \{ x \in X : f(x) \leq 1 \}.
\]

2. \( f \) is sublinear if and only if it is the gauge of \( U_f \) and \( U_f \) is convex.

3. \( f \) is a seminorm if and only if it is the gauge of \( U_f \) and \( U_f \) is circled, convex, and absorbing. The set \( U_f \) is called the **unit ball** of the seminorm.

*Proof:* First observe that for any set \( A \) its gauge has the property that

\[
x \in \alpha A \implies p_A(x) \leq \alpha.
\]

Next observe that if \( A \) is star-shaped about zero, the converse is true, so

\[
x \in \alpha A \iff p_A(x) \leq \alpha. \tag{3}
\]
Now note that if $f$ is nonnegative and homogeneous, then the set
\[ U_f = \{ x \in X : f(x) \leq 1 \} \]
is star-shaped about zero.

1. We want to show that if $f$ is nonnegative and homogeneous, then it is the gauge $p_{U_f}$ of the star-shaped set $U_f$. From (3) and the definition of $U_f$ we have
\[ p_{U_f}(x) \leq \alpha \iff x \in \alpha U_f \iff f(x) \leq \alpha, \]
so $f = p_{U_f}$.

2. We want to show that if $f$ is nonnegative and homogeneous, then $f$ is subadditive if and only if $U_f$ is convex. We already know that $f = p_{U_f}$.

So first assume that $U_f$ is convex. Let $\alpha, \beta > 0$ satisfy $x \in \alpha U_f$ and $y \in \beta U_f$. Then $x + y \in \alpha U_f + \beta C = (\alpha + \beta)C$, so $p_{U_f}(x + y) \leq \alpha + \beta$. Taking infima yields $p_{U_f}(x + y) \leq p_{U_f}(x) + p_{U_f}(y)$, so $p_{U_f} = f$ is subadditive.

For the converse, assume that $f$ is subadditive. We need to show that $U_f$ is convex. So let $x, y \in U_f$ and let $0 < \lambda < 1$. If $f$ is subadditive, then
\[ f((1 - \lambda)x + \lambda y) \leq f((1 - \lambda)x) + f(\lambda y) = (1 - \lambda)f(x) + \lambda f(y) \leq 1, \]
where the first inequality is subadditivity, the equality is homogeneity, and the last inequality is just $x, y \in U_f$. But this just asserts that $(1 - \lambda)x + \lambda y \in U_f$ as desired.

3. The proof is very similar to the above and is left as an exercise. \(\square\)

### 9.4 Gauge functions and support functions

Profit functions are sublinear (Theorem 9.1.3), and by Lemma 9.3.6 every nonnegative sublinear function is a gauge. Thus every nonnegative profit function is also a gauge. How do we guarantee that $\pi_A$ is nonnegative? One simple way is to require that $0 \in A$. Then $\sup x \in A p \cdot x \geq p \cdot 0 = 0$.

In fact, if $A \subset \mathbb{R}^m$ is a closed convex set that contains $0$, then by Lemma 9.3.6 its profit function $\pi_A$ is the gauge of
\[ \{ p \in \mathbb{R}^m : \pi_A(p) \leq 1 \} = \left\{ p : (\forall x \in A) \ [p \cdot x \leq 1] \right\}. \]

This suggest the following definition.\(^2\)

**9.4.1 Definition** Given a nonempty set $A$ in $\mathbb{R}^m$, define the **polar** of $A$, denoted $A^\circ$ by
\[ A^\circ = \{ p \in \mathbb{R}^m : (\forall x \in A) \ [p \cdot x \leq 1] \}. \]

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\(^2\)This definition is what Roko and I called the **one-sided polar** in [1, p. 215].
9.4.2 Exercise The polar $A^\circ$ is a closed convex set that contains 0. □

9.4.3 Exercise (The polar of cone) If $C$ is cone, then $C^\circ = C^*$, where $C^* = \{ p : (\forall x \in C) \ [p \cdot x \leq 0] \}$ is the dual cone of $C$. □

9.4.4 Bipolar Theorem For a nonempty set $A$,

$$A^{\circ\circ} = \overline{w}(A \cup \{0\}).$$

Consequently, if $C$ is a closed convex set that contains 0, then $C = C^{\circ\circ}$.

Proof: By definition

$$A^{\circ\circ} = \{ x : (\forall p \in A^\circ) \ [p \cdot x \leq 1] \}.$$

Now if $x \in A$ and $p \in A^\circ$ we have $p \cdot x \leq 1$, so $A \subset A^{\circ\circ}$. Thus by Exercise 9.4.2, we have

$$\overline{w}(A \cup \{0\}) \subset A^{\circ\circ}.$$

For the reverse inclusion, we shall prove that $y \notin \overline{w}(A \cup \{0\}) \implies y \notin A^{\circ\circ}$. So assume $y \notin \overline{w}(A \cup \{0\})$. By the Strong Separating Hyperplane Theorem 8.3.1 there exists a nonzero $p$ and a real $\alpha$ with

$$p \cdot y > \alpha > p \cdot x$$

for all $x \in \overline{w}(A \cup \{0\})$. Taking $x = 0$ we see that, $\alpha > 0$, so we can multiply the inequality (*) by $1/\alpha$. Setting $p' = (1/\alpha)p$, we have $p' \cdot y > 1 > p' \cdot x$ for all $x \in A$. So $p' \in A^\circ$. But $p' \cdot y > 1$, so $y \notin A^{\circ\circ}$. This completes the proof. □

9.4.5 Proposition Let $C$ be a closed convex set that contains 0. Then:

1. $\pi_C = p_{C^\circ}$.

2. $p_C = \pi_{C^\circ}$.

Proof: This is not really hard. I just have a hard dealing with suprema. I’m never sure what’s obvious and what isn’t so I will probably spell this out in too much detail.

1. By Exercise 9.4.2, $C^\circ$ is convex and contains 0. Then for any $p$, the set $\{ \alpha : p \in \alpha C^\circ \}$ is an interval with lower bound $p_{C^\circ}(p)$. So,

$$\beta > p_{C^\circ}(p) \implies p \in \beta C^\circ, \implies (\forall x \in C) \ [p \cdot x \leq \beta] \implies \pi_C(p) \leq \beta.$$
That is, $\beta > p_{C^\circ}(p) \implies \beta \geq \pi_C(p)$. So

$$\pi_C(p) \leq p_{C^\circ}(p).$$

For the reverse inequality,

$$\beta > \pi_C(p) \implies (\forall x \in C) \ [p \cdot x \leq \pi_C(p) < \beta]$$
$$\implies p \in \beta C^\circ$$
$$\implies \beta \geq p_{C^\circ}(p).$$

Thus

$$\pi_C(p) \geq p_{C^\circ}(p).$$

These two inequalities imply that

$$\pi_C = p_{C^\circ}.$$

2. From part (1) applied to $C^\circ$ we see that $\pi_{C^\circ} = p_{C^{\circ\circ}} = p_C$, where the last equality is the Bipolar Theorem.

References


