General Relativity Ph236b
Problem Set 8
Due: Hand in To Tristan, by March 13, 2007

Preview: Problems 1 and 2 are technical exercises to be sure you understand some of the technical aspects of the derivations in Ch. 9. Problems 3 and 4 are intended to be pretty cool—in them, you will use a neat trick to calculate the Hawking temperature of a black hole and of de Sitter space. You should be able to guess the solution to Problem 5, but filling in the details might be a bit of work. Problem 6 should be fun.

1. **Conservation of particle number for inertial observers:** Suppose that $f_{\vec{k}}(x^\mu) \propto e^{ik_\mu x^\mu}$ are an orthonormal set of plane waves in Minkowski space. We can perform a Lorentz transformation to a new set of coordinates x' construct a new set of plane waves $g_{\vec{k}}(x'^\mu) \propto e^{ik_\mu x'^\mu}$. Show that the number of particles in the g basis is the same as in the f basis.

2. **Derivation of Unruh radiation:** Justify carefully the claim, made on p. 410 of Carroll's book, that the $h_k^{(1)}$ modes are analytic and bounded for complex (t, x) so long as $\mathrm{Im}(t - x) \leq 0$.

3. **Hawking radiation:** In a tour-de-force calculation, Hawking showed that, through quantum-mechanical processes, a Schwarzchild black hole emits a blackbody spectrum of radiation at a temperature $k_B T = \hbar/8\pi M$. This result can be understood in a fairly simple way. Recall that in the Feynman formulation of quantum mechanics, the amplitude for a system to take a path with an action S is $e^{iS/\hbar}$. Recall also that in statistical mechanics, the probability for a system at a temperature T to be found in a state with free energy S is $e^{-S/k_B T}$. This leads to the concept of finite-temperature field theory, which resembles quantum field theory with an imaginary time that is periodic with period $t = (iT)^{-1}$. So all you have to do is the following. Start with the Schwarzchild metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \, d\Omega^2.$$

   Then make a transformation to a *Euclidean* Schwarzchild metric (with a positive signature) by making a change of variables to $\tau = it$. Your problem is then to find another change of variables that makes it clear that the imaginary-time coordinate τ in this Euclidean Schwarzchild metric is periodic with period $\tau = 8\pi M$.

4. **The temperature of de Sitter space:** Let's now try to apply the result in problem 3 to determine the temperature in de Sitter space. Consider the de Sitter metric in static coordinates:

$$ds^2 = -\left(1 - \frac{\Lambda}{3} r^2\right) dt^2 + \left(1 - \frac{\Lambda}{3} r^2\right)^{-1} dr^2 + r^2 d\Omega^2.$$

Consider the Euclidean-signature version of this spacetime by making the replacement $t \to i\tau$. Show that a coordinate transformation can be made to make the Euclidean metric regular at the horizon, so long as τ is made periodic. What quantum-mechanical temperature do you then infer?

5. **Accelerated mirrors:** Show that an accelerated mirror in (1+1)-dimensional Minkowski space emits quantum-mechanical radiation. Find the temperature of the radiation.

6. **Acoustic Hawking radiation:** The existence of Hawking radiation is associated with the existence of an event horizon, a surface beyond which light signals cannot escape. Consider now fluid mechanics. Think up a physical situation in which a fluid flow might exist in such a way that there may be a sound horizon, a surface beyond which sound waves cannot escape. Estimate the flux of sound waves for your configuration.