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A GEOMETRIC PROOF OF THE EXISTENCE OF WHITNEY STRATIFICATIONS

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Dedicated to 60th birthday of my teacher Yulij Sergeevich Ilyashenko

Abstract. In this paper we give a simple geometric proof of existence of so-called Whitney stratification for (semi)analytic and (semi)algebraic sets. Roughly, stratification is a partition of a singular set into manifolds so that these manifolds fit together "regularly". The proof presented here does not use analytic formulas only qualitative considerations. It is based on a remark that if there are two manifolds of the partition V and W of different dimension and $V \subset \overline{W}$, then *irregularity* of the partition at a point x in V corresponds to the existence of *nonunique* limits of tangent planes T_yW as y approaches x.

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1. Introduction

A stratification of a set, e. g., an analytic variety, is, roughly, a partition of it into manifolds so that these manifolds fit together "regularly". Stratification theory was originated by Thom and Whitney for algebraic and analytic sets. It was one of the key ingredients in Mather's proof of the topological stability theorem [14] (see [4] and [3] for the history and further applications of stratification theory). My interest in stratification theory was stimulated while I have been studying the fundamental paper of Ilyashenko and Yakovenko [6]. This paper initiated my research in bifurcation theory. Enthusiastic support of Yulij Sergeevich and invaluable advices have been absolutely indispensable to carry out the project of improving the main result from [6] about finiteness of the number of limit cycles bifurcating from an elementary polycycle to an explicit upper bound [8]. This was my first serious project in mathematics.

In this paper, given a partition of a singular set (which we know always exists), we prove that there is a "regular" partition. Our proof is based on a remark that if there are two parts of the partition V and W of different dimension and $V \subset \overline{W}$,

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then *irregularity* of the partition at a point x in V corresponds to the existence of *nonunique* limits of tangent planes T_yW as y approaches x.

Consider either the category of (semi)analytic (or (semi)algebraic) sets. Call a subset $V \subset \mathbb{R}^m$ (or $\mathbb{C}^m$) a *semivariety* if locally at each point $x \in \mathbb{R}^m$ (or $\mathbb{C}^m$) it is a finite union of subsets defined by equations and inequalities

$$f_1 = \cdots = f_k = 0 \quad \begin{cases} g_1 \neq 0, \ldots, g_l \neq 0 & \text{(complex case)}, \\ g_1 > 0, \ldots, g_l > 0 & \text{(real case)}, \end{cases} \tag{1}$$

where f_i's and g_j's are real (or complex) analytic (or algebraic) depending on the case under consideration.

In the real algebraic case semivarieties are usually called semialgebraic sets; in the complex algebraic case they are called constructible, and in either analytic case they are called semianalytic sets. Semivarieties are closed under Boolean operations.

Definition 1 (Whitney)**.** Let V_i, V_j be disjoint manifolds in $\mathbb{R}^m$ (or $\mathbb{C}^m$), $\dim V_j > \dim V_i$, and let $x \in V_i \cap \overline{V}_j$. A triple (V_j, V_i, x) is called *a*- (resp. *b*-) regular if

(A) when a sequence $\{y_n\} \subset V_j$ tends to x and $T_{y_n}V_j$ tends in the Grassmanian bundle to a subspace τ_x of $\mathbb{R}^m$ (or $\mathbb{C}^m$), then $T_xV_i \subset \tau_x$;

(B) when sequences $\{y_n\} \subset V_j$ and $\{x_n\} \subset V_i$ each tends to x, the unit vector $(x_n - y_n)/|x_n - y_n|$ tends to a vector v, and $T_{y_n}V_j$ tends to τ_x, then $v \in \tau_x$[1].

V_j is called *a*- (resp. *b*-) regular over V_i if each triple (V_j, V_i, x) is *a*- (resp. *b*-) regular.

Definition 2 (Whitney)**.** Let V be a semivariety in $\mathbb{R}^m$ (or $\mathbb{C}^m$). A disjoint decomposition

$$V = \bigsqcup_{i \in I} V_i, \qquad V_i \cup V_j = \varnothing \text{ for } i \neq j, \tag{2}$$

into smooth semivarieties $\{V_i\}_{i \in I}$, called strata, is called an *a*- (resp. *b*-) regular stratification if

(1) each point has a neighborhood intersecting only finitely many strata;
(2) the frontier $\overline{V}_j \setminus V_j$ of each stratum V_j is a union of other strata $\bigsqcup_{i \in J(i)} V_i$;
(3) any triple (V_j, V_i, x) such that $x \in V_i \subset \overline{V}_j$ is *a*- (resp. *b*-) regular.

Theorem 1 [20], [18], [11]**.** *For any semivariety V in $\mathbb{R}^m$ (or $\mathbb{C}^m$) there is an a- (resp. b-) regular stratification.*

Existence of stratifications in the complex analytic case was proved by Whitney [20]. Later Thom published a sketch of a proof [18]. Then Lojasiewicz [11] extended these results to the semianalytic case. Later other proofs were found. In [5] Hironaka found a nice proof using his resolution of singularities. J. Bochnak, M. Coste, and M.-F. Roy [1] and Z. Denkowska, K. Wachta [2] follow the classical route of Whitney [20] via the wing lemma. [1] uses a Nash wing lemma and [2] apply the parameterized Puiseux Theorem of W. Pawlucki [17]. T. Wall [19] and S. Lojasiewicz, J. Stasica, and K. Wachta [13] found proofs which use on Milnor's curve selection lemma [15]. The latter proof also uses the subanalyticity of

[1] This way of defining *b*-regularity is due to Mather [14]. Whitney's definition [20] is equivalent to this one provided of *a*-regularity.

the tangent map (for which an elementary proof was given by Z. Denkowska and K. Wachta [2]). For a nice exposition of the theory of semianalytic and subanalytic sets see [12].

Our proof does not use any analytic tools. It is consists of pure geometric consideration. In [9] using very similar idea we prove existence of so called a_p-stratification of Thom for polynomial functions. This is a theorem of Hironaka [5]. These ideas have been used to obtain an explicit estimate of the number of limit cycles [8].

Proof of Theorem 1. A semivariety V has well-defined dimension, say $d \le m$. Denote by V_{reg} the set of points, where V is locally a real (or complex) analytic submanifold of $\mathbb{R}^m$ (or $\mathbb{C}^m$) of dimension d. V_{reg} is a semivariety, moreover, $V_{\text{sing}} = V \setminus V_{\text{reg}}$ is a semivariety of positive codimension in V, i. e., $\dim V_{\text{sing}} < \dim V$. In the analytic case all these results may be found in Lojasiewicz [11]; in the algebraic case they are not difficult (see, e. g., [15]).

Step 1. There is a filtration of V by semivarieties

$$V^0 \subset V^1 \subset \cdots \subset V^d = V, \tag{3}$$

where for each $k = 1, \ldots, d$ the set $V^k \setminus V^{k-1}$ is a manifold of dimension k. This follows from the Lojasiewicz result. Indeed, consider $V_{\text{sing}} \subset V$, then $V \setminus V_{\text{sing}}$ is a manifold of dimension d and $\dim V_{\text{sing}} < d$. Inductive application of these arguments completes the proof.

A *refinement* of a decomposition $V = \bigsqcup_{i\in I} V_i$ is a decomposition $V = \bigsqcup_{i'\in I'} V_{i'}$ such that any stratum V_j of the first decomposition is a union of some strata of the second one, i. e., there is a set $I'(j) \subset I'$ such that $V_j = \bigsqcup_{i'\in I'(j)} V_{i'}$.

Step 2. Let $V \subset \mathbb{R}^m$ (or $\mathbb{C}^m$) be a manifold and $W \subset V$ be a semivariety. Denote by $\mathrm{Int}_V(W)$ the set of interior points of W in V with respect to the induced from $\mathbb{R}^m$ (resp. $\mathbb{C}^m$) topology. Let V_i and V_j be a pair of distinct strata. For each point $x \in V_i \cap \overline{V}_j$ denote by $V_j^{\text{con},x}$ a local connected component of V_j at x, i. e., a connected component of intersection of V_j with a ball centered at x and call it *essential* if the closure of $V_j^{\text{con},x}$ has x in the interior, $x \in \mathrm{Int}_{V_i}(V_i \cap \overline{V}_j^{\text{con},x})$. Denote by $V_j^{\text{ess},x}$ the union of all local essential components of V_j. Lojasiewicz [11] showed that V_j has only finitely many local connected components at each point x. $\square$

Theorem 2. *For any two disjoint strata V_j and V_i the set of points*

$$\mathrm{Sing}_{a\ (\text{resp.} b)}(V_j, V_i) = \{x \in V_i \cap \overline{V}_j \colon (V_j^{\text{ess},x}, V_i, x) \text{ is not } a\text{- (resp. } b\text{-) regular}\},$$

is a semivariety in V_i and $\dim \mathrm{Sing}_{a\ (\text{resp. } b)}(V_j, V_i) < \dim V_i$.

Remark 1. The set $\mathrm{Sing}_{a\ (\text{resp. } b)}(V_j, V_i)$ is a priori empty, if the intersection $\overline{V}_j \cap V_i$ has dimension smaller than $\dim V_i$ (in particular, if $\dim V_j \le \dim V_i$), so, the only interesting case is when dimensions are equal.

Let us show that this theorem is sufficient to prove Theorem 1. Consider a decomposition $V = \bigsqcup_{i\in I} V_i$ and split the strata into two groups: the first group consists of strata of dimension at least k and the second group is of the rest. Suppose that each stratum from the first group is a- (resp. b-) regular over each stratum from the second group. Then by definition of a- (resp. b-) regularity any refinement of a stratum from the second group preserves this a- (resp. b-) regularity.

Now apply this refinement inductively. Consider strata in $V^d \setminus V^{d-1}$ of dimension d. Using Theorem 2 and the result of Lojasiewicz [11] that a frontier of a semivariety has dimension less than the semivariety itself, refine V^{d-1} so that each d-dimensional stratum is a- (resp. b-) regular over each stratum in V^{d-1}. The above remark shows that any further refinement of the strata in V^{d-1} preserves the a- (resp. b-) regularity of strata from $V^d \setminus V^{d-1}$ over it. This reduces the problem of the existence of stratification for d-dimensional semivarieties to the same problem for $(d-1)$-dimensional semivarieties. Induction on dimension completes the proof of Theorem 1.

Our proof is based on the observation that if $V_i \subset \overline{V}_j$ are a pair of strata a- (resp. b-) regularity of V_j over V_i at x in V_i is closely related to whether the limit of tangent planes $T_y V_j$ is unique or not as y from V_j tends to x. The rest of the paper is devoted to the proof of Theorem 2 which consists of two steps. In section 2 we relate a- (resp. b-) regularity with (non)uniqueness of limits of tangent planes $T_y V_j$, then based on it and Rolle's lemma in section 4 we prove Theorem 2.

2. The Key Definitions

Let V_i and V_j be a pair of distinct strata. Define

$$\begin{aligned}\mathrm{Un}_a(V_j, V_i) = \{x \in V_i \cap \overline{V}_j \colon &\text{ for any } V_j^{\mathrm{con},x} \text{ there exists } \tau_x \text{ which} \\ &\text{belongs to an appropriate Grassmanian of } T_x\mathbb{R}^m \text{ (resp. } T_x\mathbb{C}^m\text{) such} \\ &\text{that for any } \{y_n\} \subset V_j^{\mathrm{con},x} \text{ tending to } x \text{ we have } T_{y_n}V_j \to \tau_x\}.\end{aligned} \tag{4}$$

Since a- (resp. b-) regularity is a local property, without loss of generality we can assume that locally V_i is an s-plane with a basis of unit vectors $e_1, \dots, e_s$. Using an idea of Kuo [10] (see also [19]) we define *a Kuo map* $\mathcal{P}^{a \text{ (resp. } b)} \colon V_j \to \mathbb{R}$ which measures non a- (resp. b-) regularity in terms of an angle between a vector or a plane and the tangent plane to V_j. Denote by $p_i \colon \mathbb{R}^m \to V_i^\perp$ (resp. $p_i^\perp \colon \mathbb{R}^m \to V_i$) the orthogonal projection along V_i (resp. $V_i^\perp$) onto the complement $V_i^\perp$ (resp. V_i) with x being the origin of $\mathbb{R}^m$. Let the tangent plane T_yV_j to V_j at y be naturally embedded into $\mathbb{R}^m$. Denote

$$\begin{gathered}\pi_j \colon V_j \times \mathbb{R}^m \to \mathbb{R}^m, \quad \pi_{j,t} \colon V_j \to \mathbb{R}^m \text{ for } t = 1, \dots, s+1 \text{ defined by} \\ \pi_j(y, v) = \pi_{T_yV_j}(v), \quad \pi_{j,t}(y) = \pi_j(y, e_t), \quad \pi_{j,s+1}(y) = \pi_j\left(y, \frac{p_i(y)}{|p_i(y)|}\right),\end{gathered} \tag{5}$$

where $\pi_j(y, v)$ is the orthogonal projection of v along the embedded plane T_yV_j. Define analytic functions $\mathcal{P}^{a \text{ (resp. } b)} \colon V_j \to \mathbb{R}$ by $\mathcal{P}^a(y) = \sum_{t=1}^{s} |\pi_{j,t}(y)|^2$ (resp. $\mathcal{P}^b(y) = \sum_{t=1}^{s+1} |\pi_{j,t}(y)|^2$). By the definition the level sets of $\mathcal{P}^{a \text{ (resp. } b)}$ are semivarieties.

Notice that the first s terms of the function $\mathcal{P}^a(y)$ measure the angle between $T_xV_i = V_i$ and T_yV_j and the last term measures the angle between the $V_i^\perp$- component of $(y-x)/|y-x|$ and T_yV_j. Since any vector can be decomposed into V_i and $V_i^\perp$ components, this proves the following

There is only one Fact in the paper — OK?

Fact 1. *For any pair of distinct strata V_j and V_i existence of a sequence $\{y_n\} \subset V_j$ tending to x with a nonzero limit of $\mathcal{P}^{a \text{ (resp. } b)}(y_n)$ is equivalent to a- (resp. b-) irregularity of V_j over V_i at x.*

Denote

$$\begin{aligned}&\mathrm{Un}_b(V_j,\, V_i) = \{x \in \mathrm{Un}_a(V_j,\, V_i)\colon \text{ for any } V_j^{\mathrm{con},x}, \text{ there exists}\\ &\epsilon \in \mathbb{R} \text{ such that for any } \{y_n\} \subset V_j^{\mathrm{con},x} \text{ tending to } x,\, \mathcal{P}^b(y_n) \to \epsilon\}.\end{aligned} \tag{6}$$

Lemma 1. *Let V_i and V_j be a pair of disjoint strata in $\mathbb{R}^m$ (or $\mathbb{C}^m$) with $V_i \cap \overline{V}_j \neq \varnothing$. Then $\mathrm{Sing}_{a \text{ (resp. } b)}(V_j,\, V_i)$ and $\mathrm{Un}_{a \text{ (resp. } b)}(V_j,\, V_i)$ are semivarieties and*

$$\mathrm{Sing}_a(V_j,\, V_i) \subset \mathrm{Sing}_b(V_j,\, V_i), \quad \mathrm{Sing}_{a \text{ (resp. } b)}(V_j,\, V_i) \subset V_i \setminus \mathrm{Un}_{a \text{ (resp. } b)}(V_j,\, V_i).$$

Remark 2. The new result here is that

$$\mathrm{Sing}_{a \text{ (resp. } b)}(V_j,\, V_i) \subset V_i \setminus \mathrm{Un}_{a \text{ (resp. } b)}(V_j,\, V_i).$$

The other inclusion may be found in [20], [14], [11].

Proof. Let's first prove that $\mathrm{Sing}_a(V_j,\, V_i)$ is a semivariety. Consider $V_i \times TV_j = \{(x,\, y,\, T_yV_j)\colon x \in V_i,\, y \in V_j\}$. It is a semivariety in an appropriate Grassmanian bundle over $\mathbb{R}^m \times \mathbb{R}^m$ (resp. $\mathbb{C}^m \times \mathbb{C}^m$) and so is its closure. The condition $T_xV_i \not\subset \tau_x$ is semialgebraic and a projection of a semivariety is a semivariety. In the real (resp. complex) algebraic case it is called the Tarski–Seidenberg Principle [7] (resp. elimination theory [16]). In the real analytic case it depends on a generalization due to Lojasiewicz [11] to varieties analytic in some variables and algebraic in others. In the complex analytic case, a proof may be found in [20]. Similar arguments show $\mathrm{Sing}_b(V_j,\, V_i)$ and $\mathrm{Un}_{a \text{ (resp. } b)}(V_j,\, V_i)$ are semivarieties.

Now let's see that $\mathrm{Sing}_a(V_j,\, V_i) \subset \mathrm{Sing}_b(V_j,\, V_i)$. For any sequence $\{y_n\} \subset V_j$ such that $T_{y_n}V_j$ has a limit τ_x as y_n tends to x and any $v \in T_xV_i$ there is a sequence $\{x_n\} \subset V_i$ such that x_n tends to x slower than the sequence $\{y_n\}$, i. e., $|y_n - x|/|x_n - x| \to 0$ and the unit vectors $(x_n - y_n)/|x_n - y_n|$ tends to v as $n \to \infty$[2]. If $x \notin \mathrm{Sing}_b(V_j,\, V_i)$, then v belongs to τ_x. Since any $v \in T_xV_i$ belongs to τ_x, T_xV_i also belongs to τ_x.

To see that $\mathrm{Sing}_a(V_j,\, V_i) \subset V_i \setminus \mathrm{Un}_a(V_j,\, V_i)$, suppose $x \in \mathrm{Sing}_a(V_j,\, V_i) \cap \mathrm{Un}_a(V_j,\, V_i)$. Fix an a-irregular essential local connected component $V_j^{\mathrm{con},x}$ of V_j at x. There is a $\dim V_j$-plane τ_x such that for any sequence $\{y_n\} \subset V_j^{\mathrm{con},x}$ tending to x we have $T_{y_n}V_j \to \tau_x$. Since $x \in \mathrm{Sing}_a(V_j,\, V_i)$, we have $T_xV_i \not\subset \tau_x$, i. e., there is a unit vector $v \in T_xV_i$ which has a positive angle with τ_x, i. e., $\angle(v,\, \tau_x) = 2\delta > 0$. Denote by $C_{\delta,v}(x) = \{y \in \mathbb{R}^m\colon \big(\frac{y-x}{|y-x|},\, v\big) > 1 - \delta\}$ the δ-cone around v centered at x and by $l_v(x)$ the ray starting at x in the direction of v. The intersection $V_j^{\mathrm{con},x} \cap C_{\delta,v}(x) = V_{j,\delta,v}^{\mathrm{con},x}$ is a semivariety and $l_v(x)$ is in its closure. By the Lojasiewicz result $V_{j,\delta,v}^{\mathrm{con},x}$ consists of a finite number of connected components. So one can choose a connected component $W_{j,\delta,v}^{\mathrm{con},x} \subset V_{j,\delta,v}^{\mathrm{con},x}$ which contains $l_v(x)$ in the closure. By Milnor's curve selection lemma [15], [19] there is an analytic curve γ which belongs to $W_{j,\delta,v}^{\mathrm{con},x} \cup \{x\}$. Since γ is analytic, it has a limiting tangent vector w at x which is by our construction should belong to τ_x and $C_{\delta,v}(x)$. This is a contradiction with $\angle(v,\, \tau_x) = 2\delta$.

[2] This was first noticed by J. Mather [14].

To see that $\mathrm{Sing}_b(V_j, V_i) \subset V_i \setminus \mathrm{Un}_b(V_j, V_i)$ it is sufficient to prove $\mathrm{Sing}_b(V_j, V_i) \cap \mathrm{Un}_a(V_j, V_i) \subset V_i \setminus \mathrm{Un}_b(V_j, V_i)$. Let $x \in \mathrm{Sing}_b(V_j, V_i) \cap \mathrm{Un}_a(V_j, V_i)$ and $V_j^{\mathrm{con},x}$ be a b-irregular essential local connected component at x. Since $x \in \mathrm{Un}_a(V_j, V_i)$, there is a unique limiting tangent plane $\tau_x = \lim T_{y_n} V_j$ independent of $\{y_n\} \subset V_j^{\mathrm{con},x}$ tending to x and by the previous passage x is a-regular, i.e., $V_i \subset \tau_x$. By Fact 1 and b-irregularity of x there is a sequence $\{y_n\} \in V_j^{\mathrm{con},x}$ such that $|\mathcal{P}^b(y_n)| \to 2\delta \neq 0$. Let's prove existence of a sequence $\{y'_n\} \in V_j^{\mathrm{con},x}$ such that $|\mathcal{P}^b(y_n)| \to \epsilon < \delta$ which shows that $x \notin \mathrm{Un}_b(V_j, V_i)$.

For each $\tilde{x} \in V_i$ close to x consider the "level" set $V_j^{\mathrm{con},x}(\tilde{x}) = V_j^{\mathrm{con},x} \cap (V_i^\perp + \{\tilde{x}\})$ over $\tilde{x}$. Transversality of τ_x with $V_i^\perp$ and uniqueness of $\lim T_{y_n} V_j^{\mathrm{con},x}$ imply that $V_j^{\mathrm{con},x}(\tilde{x})$ is a manifold and $\tau_j(y) = T_y V_j \cap V_i^\perp$ depends continuously on y in $V_j^{\mathrm{con},x}$. Consider the set of $\tilde{x} \in V_i$ for which the corresponding "level" set $V_j^{\mathrm{con},x}(\tilde{x})$ has $\tilde{x}$ in the closure, i.e., $\tilde{x} \in \overline{V}_j^{\mathrm{con},x}(\tilde{x})$. Since $V_j^{\mathrm{con},x}$ is essential, the set of such $\tilde{x}$'s is everywhere dense in a neighborhood of x in V_i. Moreover, the "angle" function $\mathcal{P}^b$ is bounded in absolute value by δ on each local connected "level" component of $V_j^{\mathrm{con},x}(\tilde{x})$ having $\tilde{x}$ in its closure. Thus, one can find a sequence of points $\{y_n\} \subset V_j^{\mathrm{con},x}$ tending to x each point y_n of which belongs to a "level" connected component of $V_j^{\mathrm{con},x}(\pi_i^\perp(y_n))$, having $\pi_i^\perp(y_n) \in V_i$ in the closure. By construction $|\mathcal{P}^b(y_n)| < \delta$ for all n. □

3. Separation of planes

Consider the real case. The complex case can be done in a similar way. Let τ_0 and τ_1 be two distinct orientable k-dimensional planes in $\mathbb{R}^m$. An orientable $(m-k)$-dimensional plane l in $\mathbb{R}^m$ *separates* τ_0 and τ_1 if l is transversal to τ_0 and τ_1 and the orientations induced by $\tau_0 + l$ and $\tau_1 + l$ in $\mathbb{R}^m$ are different. Notice that there always exists an open set of orientable $(m-k)$-planes separating any two distinct orientable k-plane.

Rolle's lemma. *If a continuous family of orientable k-planes $\{\tau_t\}_{t\in[0,1]}$ connects τ_0 and τ_1 and an orientable $(m-k)$-plane l separates τ_0 and τ_1. Then for some $t^* \in (0, 1)$ transversality of τ_{t^*} and l fails.*

In what follows we use the transversality theorem [4] which says: *if $V \subset \mathbb{R}^m$ is a manifold, then almost every plane of dimension k is transversal to V.*

4. A Reduction Lemma

Lemma 2. *Let V_j and V_i be a distinct strata and $\dim V_j > \dim V_i$. Then there is a set of strata $\{V_j^p\}_{p\in\mathbb{Z}}$ (resp. $\{V_i^p\}_{p\in\mathbb{Z}}$) in V_j (resp. in V_i) each of positive codimension in V_j (resp. in V_i) such that*

$$\mathrm{Sing}_{a\ (\mathrm{resp.}\ b)}(V_j, V_i) \subset \bigcup_{p\in\mathbb{Z}} \mathrm{Sing}_{a\ (\mathrm{resp.}\ b)}(V_j^p, V_i) \cup \bigcup_{p\in\mathbb{Z}} V_i^p. \tag{7}$$

Remarks. 1. Inductive application of this lemma to the right-hand side of (7) reduces dimensions of V_j^p's up to $\dim V_i$.

2. Lojasiewicz [11] proved that the frontier $\mathrm{Sing}_{a\ (\text{resp. } b)}(V_j^p, V_i) \subset V_i \cap \overline{V}_j^p$ of a semivariety has dimension strictly smaller than the semivariety itself.

3. By Lemma 1 the set $\mathrm{Sing}_{a\ (\text{resp. } b)}(V_j, V_i)$ is a semivariety. Since a countable union of semivarieties of positive codimension in V_i contains $\mathrm{Sing}_{a\ (\text{resp. } b)}(V_j, V_i)$, $\mathrm{Sing}_{a\ (\text{resp. } b)}(V_j, V_i)$ has a positive codimension in V_i which proves Theorem 2.

Proof. If $x \in \mathrm{Sing}_{a\ (\text{resp. } b)}(V_j, V_i)$, then by the construction of $\mathcal{P}^{a\ (\text{resp. } b)}$, for some $\epsilon > 0$ there is a sequence $\{y_n\} \subset V_j^{\mathrm{con},x}$ with $\mathcal{P}^{a\ (\text{resp. } b)}(y_n) \to \epsilon$. Denote $\dim V_j$ by k and notice that the Kuo function takes values in $[0, k+1]$. There are two cases:

(1) there are different limits: $\mathcal{P}^{a\ (\text{resp. } b)}(y_n') \to \epsilon'$, $\mathcal{P}^{a\ (\text{resp. } b)}(y_n) \to \epsilon''$, and $\epsilon' \neq \epsilon''$;

(2) the limit $\mathcal{P}^{a\ (\text{resp. } b)}(y_n)$ is unique, positive, and independent of $\{y_n\}$ [3].

Consider case (1). By Sard's lemma there is a regular value $\epsilon^* \in (\epsilon', \epsilon'')$ of $\mathcal{P}^{a\ (\text{resp. } b)}$. By the rank theorem $V_j^{\epsilon^*} = (\mathcal{P}^{a\ (\text{resp. } b)})^{-1}(\epsilon^*)$ is a smooth semivariety of codimension 1 in V_j. Let's show that $x \in \overline{V}_j^{\epsilon^*}$. Consider a local connected component $V_j^{\mathrm{con},x}$ and two sequences $\{y_n'\}$ and $\{y_n''\}$ in $V_j^{\mathrm{con},x}$ converging to x such that $\mathcal{P}^{a\ (\text{resp. } b)}(y_n') \to \epsilon'$ and $\mathcal{P}^{a\ (\text{resp. } b)}(y_n'') \to \epsilon''$ as $n \to \infty$. $\mathcal{P}^{a\ (\text{resp. } b)}$ is continuous and $V_j^{\mathrm{con},x}$ is connected, thus we can connect each y_n' and y_n'' in $V_j^{\mathrm{con},x}$ by a curve and find a sequence $\tilde{y}_n \to x$ for which $\mathcal{P}^{a\ (\text{resp. } b)}(\tilde{y}_n) = \epsilon^*$. Thus $x \in \overline{V}_j^{\epsilon^*}$. Consider a countable dense set $\{\epsilon_p\}_{p\in\mathbb{Z}_+}$ in $[0, k+1]$ of regular values of $\mathcal{P}^{a\ (\text{resp. } b)}$ so that for any two $\epsilon' \neq \epsilon''$, there is a separating $\epsilon_p \in (\epsilon', \epsilon'')$. Define $V_j^p = (\mathcal{P}^{a\ (\text{resp. } b)})^{-1}(\epsilon_p)$. Thus any b-irregular point x is in the closure of the union $\bigcup_{p\in\mathbb{Z}_+} V_j^p$. After consideration of case (2), we will prove that V_j^p is b-irregular over V_i at those x.

Consider case (2). By Lemma 1 in this case if $x \in \mathrm{Sing}_{a\ (\text{resp. } b)}(V_j, V_i)$, then x belongs to $V_i \setminus \mathrm{Un}_a(V_j, V_i)$. Therefore, there are two sequences $\{y_n^0\}$, $\{y_n^1\}$ in a local connected component $V_j^{\mathrm{con},x}$ tending to x such that $T_{y_n^0}V_j \to \tau_0$, $T_{y_n^1}V_j \to \tau_1$, and $\tau_0 \neq \tau_1$. Choose an orientation of $T_{y_0^0}V_j$. By connecting y_0^0 locally with all other points $\{y_n^s\}$ one can induce an orientation on all other $T_{y_n^s}V_j$ so that the orientations of τ_0 and τ_1 coincide with the orientations of the limits.

There is an orientable $(m-k)$-plane l_j separating τ_0 and τ_1 and transversal to V_j (by the transversality theorem). Consider the orthogonal projection π_{l_j} along l_j onto its orthogonal complement $l_j^\perp$. Denote by $p_{l_j,j}$ its restriction to V_j, $p_{l_j,j} = \pi_{l_j}|_{V_j} \colon V_j \to l_j^\perp$. Denote by $\mathrm{Crit}(l_j, V_j)$ the set of critical points of $p_{l_j,j}$ in V_j where the rank of $p_{l_j,j}$ is not maximal. Then $\mathrm{Crit}(l_j, V_j)$ is a semivariety in V_j and $\dim \mathrm{Crit}(l_j, V_j) < \dim V_j$. Connect two points y_n^0 and y_n^1 by a curve in V_j, then $T_{y_n^0}V_j$ deformates continuously to $T_{y_n^0}V_j$. Then by Rolle's lemma there is a critical point of $p_{l_j,j}$ in $V_j^{\mathrm{con},x}$ arbitrarily close to x. Thus $x \in \overline{\mathrm{Crit}(l_j, V_j)}$.

By the transversality theorem there is a countable dense set of orientable $(m-k)$-planes $\{l_j^r\}_{r\in\mathbb{Z}_+}$ transversal to V_j such that they separate any two distinct orientable

[3] One can show that this case is impossible.

k-planes τ_0 and τ_1. Therefore, we have

$$V_i \setminus \mathrm{Un}_a(V_j, V_i) \subset \bigcup_{r\in\mathbb{Z}_+} \{\overline{\mathrm{Crit}(l_j^r, V_j)} \setminus \mathrm{Crit}(l_j^r, V_j)\}. \tag{8}$$

By Lemma 1 we know that $V_i \setminus \mathrm{Un}_a(V_j, V_i)$ is a semivariety. We know that $\mathrm{Crit}(l_j, V_j) \subset V_j$ is a semivariety and $\dim \mathrm{Crit}(l_j, V_j) < \dim V_j$. Thus we can decompose it into strata $\mathrm{Crit}(l_j, V_j) = \bigsqcup_{p\in L_j} V_j^p$. Renumerate these V_j^p's to have $\{V_j^p\}_{p\in\mathbb{Z}_-}$.

Consider strata $\{V_j^p\}_{p\in\mathbb{Z}} \subset V_j$ which we constructed in the cases 1 and 2. Then

$$\mathrm{Sing}_{a\ (\text{resp. } b)}(V_j, V_i) \subset \bigcup_{p\in\mathbb{Z}} \{\overline{V}_j^p \setminus V_j^p\}. \tag{9}$$

Notice that if $\overline{V}_j^p \cap V_i$ has dimension less than $\dim V_i$, then there is "nothing to do", so, the only case to study is when the former dimension is larger.

The definitions of $\mathcal{P}^{a\ (\text{resp. } b)}$ and $\pi_{j,s}$ explicitly imply that (7) is satisfied, because $\mathcal{P}^{a\ (\text{resp. } b)}(y_n)$ has a positive limit point for any $\{y_n\} \subset V_j^p$. If one projects along a smaller plane ($T_{y_n}V_j^p \subset T_{y_n}V_j$), then the size of the projection is larger. Thus for the Kuo map $\mathcal{P}_{j^p,i}^{a\ (\text{resp. } b)} \colon V_j^p \to \mathbb{R}$, defined in (5), the sequence $\mathcal{P}_{j^p,i}^{a\ (\text{resp. } b)}(y_n)$ also has a positive limit point. Now to separate interior and boundary points of the closures $\overline{V}_j^p$ in V_i define the set $V_i^p = (V_i \cap \overline{V}_j^p) \setminus \mathrm{Int}_{V_i}(V_i \cap \overline{V}_j^p)$. This completes the proof of the lemma and Theorem 2. □

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