A GEOMETRIC PROOF OF THE EXISTENCE OF WHITNEY STRATIFICATIONS

V. YU. KALOSHIN

Dedicated to 60th birthday of my teacher Yulij Sergeevich Ilyashenko

ABSTRACT. In this paper we give a simple geometric proof of existence of so-called Whitney stratification for (semi)analytic and (semi)algebraic sets. Roughly, stratification is a partition of a singular set into manifolds so that these manifolds fit together “regularly”. The proof presented here does not use analytic formulas only qualitative considerations. It is based on a remark that if there are two manifolds of the partition $V$ and $W$ of different dimension and $V \subset \overline{W}$, then irregularity of the partition at a point $x$ in $V$ corresponds to the existence of nonunique limits of tangent planes $T_yW$ as $y$ approaches $x$.


Key words and phrases. Stratifications, (semi)algebraic sets, (semi)analytic sets, Wing lemma.

1. Introduction

A stratification of a set, e.g., an analytic variety, is, roughly, a partition of it into manifolds so that these manifolds fit together “regularly”. Stratification theory was originated by Thom and Whitney for algebraic and analytic sets. It was one of the key ingredients in Mather’s proof of the topological stability theorem [14] (see [4] and [3] for the history and further applications of stratification theory). My interest in stratification theory was stimulated while I have been studying the fundamental paper of Ilyashenko and Yakovenko [6]. This paper initiated my research in bifurcation theory. Enthusiastic support of Yulij Sergeevich and invaluable advices have been absolutely indispensable to carry out the project of improving the main result from [6] about finiteness of the number of limit cycles bifurcating from an elementary polycycle to an explicit upper bound [8]. This was my first serious project in mathematics.

In this paper, given a partition of a singular set (which we know always exists), we prove that there is a “regular” partition. Our proof is based on a remark that if there are two parts of the partition $V$ and $W$ of different dimension and $V \subset \overline{W}$, Received June 10, 2003.

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then irregularity of the partition at a point \( x \) in \( V \) corresponds to the existence of nonunique limits of tangent planes \( T_yW \) as \( y \) approaches \( x \).

Consider either the category of (semi)analytic (or (semi)algebraic) sets. Call a subset \( V \subset \mathbb{R}^m \) (or \( \mathbb{C}^m \)) a semivariety if locally at each point \( x \in \mathbb{R}^m \) (or \( \mathbb{C}^m \)) it is a finite union of subsets defined by equations and inequalities

\[
  f_1 = \cdots = f_k = 0 \quad \begin{cases} 
  g_1 \neq 0, \ldots, g_l \neq 0 & \text{(complex case)}, \\
  g_1 > 0, \ldots, g_l > 0 & \text{(real case)},
\end{cases}
\]

where \( f_i \)'s and \( g_j \)'s are real (or complex) analytic (or algebraic) depending on the case under consideration.

In the real algebraic case semivarieties are usually called semialgebraic sets; in the complex algebraic case they are called constructible, and in either analytic case they are called semianalytic sets. Semivarieties are closed under Boolean operations.

**Definition 1** (Whitney). Let \( V_i, V_j \) be disjoint manifolds in \( \mathbb{R}^m \) (or \( \mathbb{C}^m \)), \( \dim V_j > \dim V_i \), and let \( x \in V_i \cap V_j \). A triple \( (V_j, V_i, x) \) is called a-(resp. b-) regular if

\( (A) \) when a sequence \( \{y_n\} \subset V_j \) tends to \( x \) and \( T_{y_n}V_j \) tends in the Grassmanian bundle to a subspace \( \tau_x \) of \( \mathbb{R}^m \) (or \( \mathbb{C}^m \)), then \( T_x V_i \subset \tau_x \);

\( (B) \) when sequences \( \{y_n\} \subset V_j \) and \( \{x_n\} \subset V_i \) each tend to \( x \), the unit vector \( (x_n - y_n)/|x_n - y_n| \) tends to a vector \( v \), and \( T_{y_n}V_j \) tends to \( \tau_x \), then \( v \in \tau_x \).

\( V_j \) is called a-(resp. b-) regular over \( V_i \) if each triple \( (V_j, V_i, x) \) is a-(resp. b-) regular.

**Definition 2** (Whitney). Let \( V \) be a semivariety in \( \mathbb{R}^m \) (or \( \mathbb{C}^m \)). A disjoint decomposition

\[
  V = \bigcup_{i \in I} V_i, \quad V_i \cup V_j = \emptyset \text{ for } i \neq j,
\]

into smooth semivarieties \( \{V_i\}_{i \in I} \), called strata, is called an a-(resp. b-) regular stratification if

1. each point has a neighborhood intersecting only finitely many strata;
2. the frontier \( V_i \setminus V_j \) of each stratum \( V_j \) is a union of other strata \( \bigcup_{i \in J(i)} V_i \);
3. any triple \( (V_j, V_i, x) \) such that \( x \in V_i \subset V_j \) is a-(resp. b-) regular.

**Theorem 1** [20], [18], [11]. For any semivariety \( V \) in \( \mathbb{R}^m \) (or \( \mathbb{C}^m \)) there is an a-(resp. b-) regular stratification.


\footnote{This way of defining b-regularity is due to Mather [14]. Whitney’s definition [20] is equivalent to this one provided of a-regularity.}
the tangent map (for which an elementary proof was given by Z. Denkowska and K. Wachta [2]). For a nice exposition of the theory of semianalytic and subanalytic sets see [12].

Our proof does not use any analytic tools. It is consists of pure geometric consideration. In [9] using very similar idea we prove existence of so called $a_p$-stratification of Thom for polynomial functions. This is a theorem of Hironaka [5]. These ideas have been used to obtain an explicit estimate of the number of limit cycles [8].

**Proof of Theorem 1.** A semivariety $V$ has well-defined dimension, say $d \leq m$. Denote by $V_{\text{reg}}$ the set of points, where $V$ is locally a real (or complex) analytic submanifold of $\mathbb{R}^m$ (or $\mathbb{C}^m$) of dimension $d$. $V_{\text{reg}}$ is a semivariety, moreover, $V_{\text{sing}} = V \setminus V_{\text{reg}}$ is a semivariety of positive codimension in $V$, i.e., $\dim V_{\text{sing}} < \dim V$. In the analytic case all these results may be found in Lojasiewicz [11]; in the algebraic case they are not difficult (see, e.g., [15]).

**Step 1.** There is a filtration of $V$ by semivarieties

$$V^0 \subset V^1 \subset \cdots \subset V^d = V,$$

where for each $k = 1, \ldots, d$ the set $V^k \setminus V^{k-1}$ is a manifold of dimension $k$. This follows from the Lojasiewicz result. Indeed, consider $V_{\text{sing}} \subset V$, then $V \setminus V_{\text{sing}}$ is a manifold of dimension $d$ and $\dim V_{\text{sing}} < d$. Inductive application of these arguments completes the proof.

A refinement of a decomposition $V = \bigsqcup_{i \in I} V_i$ is a decomposition $V = \bigsqcup_{i' \in I'} V_{i'}$ such that any stratum $V_j$ of the first decomposition is a union of some strata of the second one, i.e., there is a set $I'(j) \subset I'$ such that $V_j = \bigsqcup_{i' \in I'(j)} V_{i'}$.

**Step 2.** Let $V \subset \mathbb{R}^m$ (resp. $\mathbb{C}^m$) be a manifold and $W \subset V$ be a semivariety. Denote by $\text{Int}_V(W)$ the set of interior points of $W$ in $V$ with respect to the induced from $\mathbb{R}^m$ (resp. $\mathbb{C}^m$) topology. Let $V_i$ and $V_j$ be a pair of distinct strata. For each point $x \in V_i \cap \bar{V}_j$ denote by $V_{j, x}^{\text{con}}$ a local connected component of $V_j$ at $x$, i.e., a connected component of intersection of $V_j$ with a ball centered at $x$ and call it essential if the closure of $V_{j, x}^{\text{con}}$ has $x$ in the interior, $x \in \text{Int}_V(V_i \cap \bar{V}_j^{\text{con}})$. Denote by $V_{j, x}^{\text{ess}}$ the union of all local essential components of $V_j$. Lojasiewicz [11] showed that $V_j$ has only finitely many local connected components at each point $x$. \hfill $\Box$

**Theorem 2.** For any two disjoint strata $V_j$ and $V_i$ the set of points

$$\text{Sing}_{a_0} (\text{resp. } b_0)(V_j, V_i) = \{ x \in V_i \cap \bar{V}_j : (V_j^{\text{ess}, x}, V_i, x) \text{ is not } a_0- \text{ (resp. } b_0- \text{) regular},$$

is a semivariety in $V_i$ and $\dim \text{Sing}_{a_0} (\text{resp. } b_0)(V_j, V_i) < \dim V_i$.

**Remark 1.** The set $\text{Sing}_{a_0} (\text{resp. } b_0)(V_j, V_i)$ is a priori empty, if the intersection $\bar{V}_j \cap V_i$ has dimension smaller than $\dim V_i$ (in particular, if $\dim V_j \leq \dim V_i$), so, the only interesting case is when dimensions are equal.

Let us show that this theorem is sufficient to prove Theorem 1. Consider a decomposition $V = \bigsqcup_{i \in I} V_i$ and split the strata into two groups: the first group consists of strata of dimension at least $k$ and the second group is of the rest. Suppose that each stratum from the first group is $a_0$- (resp. $b_0$-) regular over each stratum
from the second group. Then by definition of \( a \)- (resp. \( b \)) regularity any refinement of a stratum from the second group preserves this \( a \)- (resp. \( b \))- regularity.

Now apply this refinement inductively. Consider strata in \( V^d \setminus V^{d-1} \) of dimension \( d \). Using Theorem 2 and the result of Lojasiewicz [11] that a frontier of a semivariety has dimension less than the semivariety itself, refine \( V^{d-1} \) so that each \( d \)-dimensional stratum is \( a \)- (resp. \( b \)) regular over each stratum in \( V^{d-1} \). The above remark shows that any further refinement of the strata in \( V^{d-1} \) preserves the \( a \)- (resp. \( b \))- regularity of strata from \( V^d \setminus V^{d-1} \) over it. This reduces the problem of the existence of stratification for \( d \)-dimensional semivarieties to the same problem for \( (d-1) \)-dimensional semivarieties. Induction on dimension completes the proof of Theorem 1.

Our proof is based on the observation that if \( V_i \subset V_j \) are a pair of strata \( a \)- (resp. \( b \))- regularity of \( V_j \) over \( V_i \) at \( x \in V_i \) is closely related to whether the limit of tangent planes \( T_y V_j \) is unique or not as \( y \) from \( V_j \) tends to \( x \). The rest of the paper is devoted to the proof of Theorem 2 which consists of two steps. In section 2 we relate \( a \)- (resp. \( b \))- regularity with (non)uniqueness of limits of tangent planes \( T_y V_j \), then based on it and Rolle’s lemma in section 4 we prove Theorem 2.

2. The Key Definitions

Let \( V_i \) and \( V_j \) be a pair of distinct strata. Define

\[
U_{\alpha}(V_j, V_i) = \{x \in V_i \cap V_j : \text{for any } V_{\alpha, x} \text{ there exists } \tau_x \text{ which belongs to an appropriate Grassmanian of } T_x \mathbb{R}^m \text{ (resp. } T_x \mathbb{C}^m) \text{ such that for any } \{y_n\} \subset V_{\alpha, x} \text{ tending to } x \text{ we have } T_{y_n} V_j \to \tau_x \}. \tag{4}
\]

Since \( a \)- (resp. \( b \))- regularity is a local property, without loss of generality we can assume that locally \( V_i \) is an \( s \)-plane with a basis of unit vectors \( e_1, \ldots, e_s \). Using an idea of Kuo [10] (see also [19]) we define a Kuo map \( \mathcal{P}^a \) (resp. \( \mathcal{P}^b \)): \( V_j \to \mathbb{R} \) which measures non- \( a \)- (resp. \( b \))- regularity in terms of an angle between a vector or a plane and the tangent plane to \( V_j \). Denote by \( p_i: \mathbb{R}^m \to V_i^+ \) (resp. \( p_i^-: \mathbb{R}^m \to V_i \)) the orthogonal projection along \( V_i \) (resp. \( V_i^+ \)) onto the complement \( V_i^- \) (resp. \( V_i \)) with \( x \) being the origin of \( \mathbb{R}^m \). Let the tangent plane \( T_y V_j \) to \( V_j \) at \( y \) be naturally embedded into \( \mathbb{R}^m \). Denote

\[
\pi_j: V_j \times \mathbb{R}^m \to \mathbb{R}^m, \quad \pi_{j,t}: V_j \to \mathbb{R}^m \text{ for } t = 1, \ldots, s+1 \text{ defined by } \quad \pi_j(y, v) = \pi_{T_y V_j}(v), \quad \pi_{j,t}(y) = \pi_j(y, e_t), \quad \pi_{j,s+1}(y) = \pi_j \left( y, \frac{p_i(y)}{|p_i(y)|} \right), \quad \tag{5}
\]

where \( \pi_j(y, v) \) is the orthogonal projection of \( v \) along the embedded plane \( T_y V_j \). Define analytic functions \( \mathcal{P}^a \) (resp. \( \mathcal{P}^b \)): \( V_j \to \mathbb{R} \) by \( \mathcal{P}^a(y) = \sum_{t=1}^s |\pi_{j,t}(y)|^2 \) (resp. \( \mathcal{P}^b(y) = \sum_{t=1}^{s+1} |\pi_{j,t}(y)|^2 \)). By the definition the level sets of \( \mathcal{P}^a \) (resp. \( \mathcal{P}^b \)) are semivarieties.

Notice that the first \( s \) terms of the function \( \mathcal{P}^a(y) \) measure the angle between \( T_x V_i = V_i \) and \( T_y V_j \) and the last term measures the angle between the \( V_i^- \)- component of \( (y-x)/|y-x| \) and \( T_y V_j \). Since any vector can be decomposed into \( V_i \) and \( V_i^+ \) components, this proves the following.
Fact 1. For any pair of distinct strata $V_j$ and $V_i$ existence of a sequence $\{y_n\} \subset V_j$ tending to $x$ with a nonzero limit of $P^a$ (resp. $b$)$(y_n)$ is equivalent to a- (resp. b-) irregularity of $V_j$ over $V_i$ at $x$.

Denote

$$U_a(V_j, V_i) = \{x \in \text{Un}_a(V_j, V_i) : \text{for any } V_j^{\text{con}, x}, \text{ there exists } \epsilon \in \mathbb{R} \text{ such that for any } \{y_n\} \subset V_j^{\text{con}, x} \text{ tending to } x, P^a(y_n) \to \epsilon\}. \quad (6)$$

Lemma 1. Let $V_i$ and $V_j$ be a pair of disjoint strata in $\mathbb{R}^m$ (or $\mathbb{C}^m$) with $V_i \cap V_j \neq \emptyset$. Then $\text{Sing}_a (\text{resp. } b) (V_j, V_i)$ and $U_a (\text{resp. } b) (V_j, V_i)$ are semivarieties and

$$\text{Sing}_a (V_j, V_i) \subset \text{Sing}_b (V_j, V_i), \quad \text{Sing}_a (\text{resp. } b) (V_j, V_i) \subset V_i \setminus \text{Un}_a (\text{resp. } b) (V_j, V_i).$$

Remark 2. The new result here is that

$$\text{Sing}_a (\text{resp. } b) (V_j, V_i) \subset V_i \setminus \text{Un}_a (\text{resp. } b) (V_j, V_i).$$

The other inclusion may be found in [20], [14], [11].

Proof. Let’s first prove that $\text{Sing}_a (V_j, V_i)$ is a semivariety. Consider $V_i \times TV_j = \{(x, y, T_y V_j) : x \in V_i, y \in V_j\}$. It is a semivariety in an appropriate Grassmanian bundle over $\mathbb{R}^m \times \mathbb{R}^m$ (resp. $\mathbb{C}^m \times \mathbb{C}^m$) and so is its closure. The condition $T_x V_i \not\subset \tau_x$ is seminalgebraic and a projection of a semivariety is a semivariety. In the real (resp. complex) algebraic case it is called the Tarski–Seidenberg Principle [7] (resp. elimination theory [16]). In the real analytic case it depends on a generalization due to Łojasiewicz [11] to varieties analytic in some variables and algebraic in others. In the complex analytic case, a proof may be found in [20]. Similar arguments show $\text{Sing}_a (V_j, V_i)$ and $U_a (\text{resp. } b) (V_j, V_i)$ are semivarieties.

Now let’s see that $\text{Sing}_a (V_j, V_i) \subset \text{Sing}_b (V_j, V_i)$. For any sequence $\{y_n\} \subset V_j$ such that $T_{y_n} V_j$ has a limit $\tau_x$ as $y_n$ tends to $x$ and any $v \in T_x V_i$ there is a sequence $\{x_n\} \subset V_i$ such that $x_n$ tends to $x$ slower than the sequence $\{y_n\}$, i.e., $|y_n - x_n|/|x_n - x| \to 0$ and the unit vectors $(x_n - y_n)/|x_n - y_n|$ tends to $v$ as $n \to \infty^2$.

If $x \notin \text{Sing}_b (V_j, V_i)$, then $v$ belongs to $\tau_x$. Since any $v \in T_x V_i$ belongs to $\tau_x$, $T_x V_i$ also belongs to $\tau_x$.

To see that $\text{Sing}_a (V_j, V_i) \subset V_i \setminus \text{Un}_a (V_j, V_i)$, suppose $x \in \text{Sing}_a (V_j, V_i) \cap \text{Un}_a (V_j, V_i)$. Fix an $a$-irregular essential local connected component $V_j^{\text{con}, x}$ of $V_j$ at $x$. There is a $d$-plane $\tau_x$ such that for any sequence $\{y_n\} \subset V_j^{\text{con}, x}$ tending to $x$ we have $T_{y_n} V_j \to \tau_x$. Since $x \in \text{Sing}_a (V_j, V_i)$, we have $T_x V_i \not\subset \tau_x$, i.e., there is a unit vector $v \in T_x V_i$ which has a positive angle with $\tau_x$, i.e., $\angle (v, \tau_x) = 2\delta > 0$.

Denote by $C_{\delta,v}(x) = \{y \in \mathbb{R}^m : (\frac{y - x}{|y - x|}, v) > 1 - \delta\}$ the $\delta$-cone around $v$ centered at $x$ and by $l(x)$ the ray starting at $x$ in the direction of $v$. The intersection $V_j^{\text{con}, x} \cap C_{\delta,v}(x) = V_j^{\text{con}, x}$ is a semivariety and $l(x)$ is in its closure. By the Łojasiewicz result $V_j^{\text{con}, x}$ consists of a finite number of connected components. So one can choose a connected component $W_j^{\text{con}, x} \subset V_j^{\text{con}, x}$ which contains $l(x)$ in the closure. By Milnor’s curve selection lemma [15], [19] there is an analytic curve $\gamma$ which belongs to $W_j^{\text{con}, x} \cup \{x\}$. Since $\gamma$ is analytic, it has a limiting tangent vector $w$ at $x$ which is by our construction should belong to $\tau_x$ and $C_{\delta,v}(x)$. This is a contradiction with $\angle (v, \tau_x) = 2\delta$.

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This was first noticed by J. Mather [14].
To see that $\text{Sing}_b(V_j, V_i) \subset V_i \setminus \text{Un}_b(V_j, V_i)$ it is sufficient to prove $\text{Sing}_b(V_j, V_i) \cap \text{Un}_a(V_j, V_i) \subset V_i \setminus \text{Un}_b(V_j, V_i)$. Let $x \in \text{Sing}_b(V_j, V_i) \cap \text{Un}_a(V_j, V_i)$ and $V^\cong_j$ be a $b$-irregular essential local connected component at $x$. Since $x \in \text{Un}_a(V_j, V_i)$, there is a unique limiting tangent plane $\tau_x = \lim_{y \to x} V_j$ independent of $\{y_n\} \subset V^\cong_j$ tending to $x$ and by the previous passage $x$ is $a$-regular, i.e., $V_i \ni \tau_x$. By Fact 1 and $b$-irregularity of $x$ there is a sequence $\{y_n\} \in V^\cong_j$ such that $|P^b(y_n)| \to 2\delta \neq 0$.

Let’s prove existence of a sequence $\{y'_n\} \in V^\cong_j$ such that $|P^b(y'_n)| \to \epsilon < \delta$ which shows that $x \notin \text{Un}_b(V_j, V_i)$.

For each $\tilde{x} \in V_i$ close to $x$ consider the “level” set $V^\cong_j(\tilde{x}) = V^\cong_j \cap (V^\bot_j + \{\tilde{x}\})$ over $\tilde{x}$. Transversality of $\tau_x$ with $V^\bot_j$ and uniqueness of $\lim_{y \to x} V^\cong_j$ imply that $V^\cong_j(\tilde{x})$ is a manifold and $\tau_j(y) = T_yV_j \cap V^\bot_j \text{ depends continuously on } y \in V^\cong_j$. Consider the set of $\tilde{x} \in V_i$ for which the corresponding “level” set $V^\cong_j(\tilde{x})$ has $\tilde{x}$ in the closure, i.e., $\tilde{x} \in V^\cong_j(\tilde{x})$. Since $V^\cong_j$ is essential, the set of such $\tilde{x}$’s is everywhere dense in a neighborhood of $x$ in $V_i$. Moreover, the “angle” function $P^b$ is bounded in absolute value by $\delta$ on each local connected “level” component of $V^\cong_j(\tilde{x})$ having $\tilde{x}$ in its closure. Thus, one can find a sequence of points $\{y_n\} \subset V^\cong_j$ tending to $x$ each point $y_n$ of which belongs to a “level” connected component of $V^\cong_j(\pi^+_j(y_n))$, having $\pi^+_j(y_n) \in V_i$ in the closure. By construction $|P^b(y_n)| < \delta$ for all $n$. □

3. Separation of planes

Consider the real case. The complex case can be done in a similar way. Let $\tau_0$ and $\tau_1$ be two distinct orientable $k$-dimensional planes in $\mathbb{R}^m$. An orientable $(m-k)$-dimensional plane $l$ in $\mathbb{R}^m$ separates $\tau_0$ and $\tau_1$ if $l$ is transversal to $\tau_0$ and $\tau_1$ and the orientations induced by $\tau_0 + l$ and $\tau_1 + l$ in $\mathbb{R}^m$ are different. Notice that there always exists an open set of orientable $(m-k)$-planes separating any two distinct orientable $k$-plane.

Rolle’s lemma. If a continuous family of orientable $k$-planes $\{\tau_t\}_{t \in [0,1]}$ connects $\tau_0$ and $\tau_1$ and an orientable $(m-k)$-plane $l$ separates $\tau_0$ and $\tau_1$. Then for some $t^* \in (0, 1)$ transversality of $\tau_{t^*}$ and $l$ fails.

In what follows we use the transversality theorem [4] which says: if $V \subset \mathbb{R}^m$ is a manifold, then almost every plane of dimension $k$ is transversal to $V$.

4. A Reduction Lemma

Lemma 2. Let $V_j$ and $V_i$ be a distinct strata and $\dim V_j > \dim V_i$. Then there is a set of strata $\{V^p_j\}_{p \in \mathbb{Z}}$ (resp. $\{V^p_i\}_{p \in \mathbb{Z}}$) in $V_j$ (resp. in $V_i$) each of positive codimension in $V_j$ (resp. in $V_i$) such that

$$\text{Sing}_a(\text{resp. } b)(V_j, V_i) \subset \bigcup_{p \in \mathbb{Z}} \text{Sing}_a(\text{resp. } b)(V^p_j, V_i) \cup \bigcup_{p \in \mathbb{Z}} V^p_i.$$  (7)

Remarks. 1. Inductive application of this lemma to the right-hand side of (7) reduces dimensions of $V^p_j$’s up to $\dim V_i$. 
2. Lojasiewicz [11] proved that the frontier $\text{Sing}_a(\text{resp. } b)(V^p_j, V_i) \subset V_i \cap V^p_j$ of a semivariety has dimension strictly smaller than the semivariety itself.

3. By Lemma 1 the set $\text{Sing}_a(\text{resp. } b)(V^p_j, V_i)$ is a semivariety. Since a countable union of semivarieties of positive codimension in $V_i$ contains $\text{Sing}_a(\text{resp. } b)(V^p_j, V_i)$, $\text{Sing}_a(\text{resp. } b)(V^p_j, V_i)$ has a positive codimension in $V_i$ which proves Theorem 2.

Proof. If $x \in \text{Sing}_a(\text{resp. } b)(V^p_j, V_i)$, then by the construction of $\mathcal{P}^a(\text{resp. } b)$, for some $\epsilon > 0$ there is a sequence \( \{y_n\} \subset V^\text{con,x}_j \) with \( \mathcal{P}^a(\text{resp. } b)(y_n) \to \epsilon \). Denote $\dim V^p_j$ by $k$ and notice that the Kuo function takes values in $[0, k+1]$. There are two cases:

1. there are different limits: $\mathcal{P}^a(\text{resp. } b)(y_n') \to \epsilon'$, $\mathcal{P}^a(\text{resp. } b)(y_n) \to \epsilon''$, and $\epsilon' \neq \epsilon''$;

2. the limit $\mathcal{P}^a(\text{resp. } b)(y_n)$ is unique, positive, and independent of $\{y_n\}$.

Consider case (1). By Sard’s lemma there is a regular value $\epsilon^* \in (\epsilon', \epsilon'')$ of $\mathcal{P}^a(\text{resp. } b)$. By the rank theorem $V^p_j = (\mathcal{P}^a(\text{resp. } b))^{-1}(\epsilon^*)$ is a smooth semivariety of codimension 1 in $V^p_j$. Let’s show that $x \in V^p_j \cap T^p_j$. Consider a local connected component $V^\text{con,x}_j$ and two sequences $\{y'_n\}$ and $\{y''_n\}$ in $V^\text{con,x}_j$ converging to $x$ such that $\mathcal{P}^a(\text{resp. } b)(y'_n) \to \epsilon'$ and $\mathcal{P}^a(\text{resp. } b)(y''_n) \to \epsilon''$ as $n \to \infty$. $\mathcal{P}^a(\text{resp. } b)$ is continuous and $V^\text{con,x}_j$ is connected, thus we can connect each $y'_n$ and $y''_n$ in $V^\text{con,x}_j$ by a curve and find a sequence $\tilde{y}_n \to x$ for which $\mathcal{P}^a(\text{resp. } b)(\tilde{y}_n) = \epsilon^*$. Thus $x \in V^p_j \cap T^p_j$. Consider a countable dense set $\{\epsilon_p\}_{p \in \mathbb{Z}_+}$ in $[0, k+1]$ of regular values of $\mathcal{P}^a(\text{resp. } b)$ so that for any two $\epsilon' \neq \epsilon''$, there is a separating $\epsilon_p \in (\epsilon', \epsilon'')$. Define $V^p_j = (\mathcal{P}^a(\text{resp. } b))^{-1}(\epsilon_p)$. Thus any $b$-irregular point $x$ is in the closure of the union $\bigcup_{p \in \mathbb{Z}_+} V^p_j$. After consideration of case (2), we will prove that $V^p_j$ is $b$-irregular over $V_i$ at those $x$.

Consider case (2). By Lemma 1 in this case if $x \in \text{Sing}_a(\text{resp. } b)(V^p_j, V_i)$, then $x$ belongs to $V^p_i - \text{Un}_a(V^p_j, V_i)$. Therefore, there are two sequences $\{y''_n\}, \{y''_n\}$ in a local connected component $V^\text{con,x}_j$ tending to $x$ such that $T^\text{reg}_V V^p_j \to \tau_0$, $T^\text{reg}_V V^p_j \to \tau_1$, and $\tau_0 \neq \tau_1$. Choose an orientation of $T^\text{reg}_V V^p_j$. By connecting $y''_n$ locally with all other points $\{y''_n\}$ one can induce an orientation on all other $T^\text{reg}_V V^p_j$ so that the orientations of $\tau_0$ and $\tau_1$ coincide with the orientations of the limits.

There is an orientable $(m-k)$-plane $l_j$ separating $\tau_0$ and $\tau_1$ and transversal to $V^p_j$ (by the transversality theorem). Consider the orthogonal projection $\pi_j$ along $l_j$ onto its orthogonal complement $l_j^\perp$. Denote by $p_{l_j,j}$ its restriction to $V^p_j$, $p_{l_j,j} = \pi_j|_{V^p_j}$; $V^p_j \to l_j^\perp$. Denote by $\text{Crit}(l_j, V^p_j)$ the set of critical points of $p_{l_j,j}$ in $V^p_j$ where the rank of $p_{l_j,j}$ is not maximal. Then $\text{Crit}(l_j, V^p_j)$ is a semivariety in $V^p_j$ and $\dim \text{Crit}(l_j, V^p_j) < \dim V^p_j$. Connect two points $y''_n$ and $y''_n$ by a curve in $V^p_j$, then $T^\text{reg}_V V^p_j$ deforms continuously to $T^\text{reg}_V V^p_j$. Then by Rolle’s lemma there is a critical point of $p_{l_j,j}$ in $V^\text{con,x}_j$ arbitrarily close to $x$. Thus $x \in \text{Crit}(l_j, V^p_j)$.

By the transversality theorem there is a countable dense set of orientable $(m-k)$-planes $\{l_j^r\}_{r \in \mathbb{Z}_+}$ transversal to $V^p_j$ such that they separate any two distinct orientable

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3 One can show that this case is impossible.
By Lemma 1 we know that \( V_i \setminus \bigcup_n \{ \text{Crit}(l_j^r, V_j) \setminus \text{Crit}(l_j^r, V_i) \} \). Therefore, we have

\[
V_i \setminus \bigcup_n \{ \text{Crit}(l_j^r, V_j) \setminus \text{Crit}(l_j^r, V_i) \} \subset \bigcup_{r \in \mathbb{Z}^+} \{ \bar{V}_p \cap V_i \}.
\]

Notice that if \( \bar{V}_p \cap V_i \) has dimension less than \( \dim V_i \), then there is “nothing to do”, so, the only case to study is when the former dimension is larger.

The definitions of \( P^n \) (resp. \( b \)) and \( \pi_{j,s} \) explicitly imply that (7) is satisfied, because \( P^n \) (resp. \( b \))(\( y_n \)) has a positive limit point for any \( \{ y_n \} \subset V_p^p \). If one projects along a smaller plane \( (T_{y_n} V_p^p \subset T_{y_n} V_j) \), then the size of the projection is larger. Thus for the Kuo map \( P^n \) (resp. \( b \)): \( V_j \to \mathbb{R} \), defined in (5), the sequence \( P^n \) (resp. \( b \)) also has a positive limit point. Now to separate interior and boundary points of the closures \( \bar{V}_p^p \) in \( V_i \) define the set \( V_p = (V_i \cap \bar{V}_p^p) \setminus \text{Int}_V(V_i \cap \bar{V}_p^p) \). This completes the proof of the lemma and Theorem 2.

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References


