MATHER THEORY, WEAK KAM, AND VISCOSITY SOLUTIONS OF HAMILTON-JACOBI PDE'S

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1. Motivation

Consider a C^2 smooth Hamiltonian $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$, where $\mathbb{T}^n$ is the standard n-dimensional torus and $\mathbb{R}^n$ is the n-dimensional Euclidean space, $r \geq 3$. Denote coordinates $x = (x_1, \dots, x_n) \in \mathbb{T}^n$ and $p = (p_1, \dots, p_n) \in \mathbb{R}^n$. Call x – *position*. We shall investigate dynamics of the Hamiltonian equations:

$$\begin{cases} \dot{x} = \partial_p H(x,p), \\ \dot{p} = -\partial_x H(x,p). \end{cases} \tag{1}$$

We call the following three assumptions *standard assumptions*
• H is *convex* in p, i.e. for all $x \in \mathbb{T}^n$ we have that the Hessian matrix $\partial^2_{p_i p_j} H(x,p)$ is positive definite for all $p \in \mathbb{R}^n$.
• H is *superlinear* in p, i.e. for all $x \in \mathbb{T}^n$ we have that $\lim H(x,p)/|p| \to +\infty$ as $|p| \to +\infty$
• The flow defined by (1) is *complete*, i.e. for each initial condition $(x_0, p_0) \in \mathbb{T}^n \times \mathbb{R}^n$ solutions of (1) exists for all time.

Actually weak KAM and Mather theories we describe below work for a Hamiltonian defined on any smooth compact manifold. We consider the case of a Hamiltonian on a torus for exposition purposes only.

The Basic Example is the standard mechanical system: $H(x,p) = \frac{\langle p,p \rangle}{2} + V(x)$ with $\langle \cdot, \cdot \rangle$ — Euclidean scalar product and $V(x)$ — C^2 smooth potential on $\mathbb{T}^n$.

Let $\{(x(t), p(t))\}_{t \in R}$ be a trajectory of (1) with an initial conditions $(x(0), p(0)) = (x_0, p_0)$. Denote by $\hat{x}(t) \in \mathbb{R}^n$ a lift of $x(t) \in \mathbb{T}^n$ to the universal cover $\mathbb{R}^n$ over $\mathbb{T}^n = \mathbb{R}^n / \mathbb{T}^n$.

Definition 1.1. (Position) *rotation vector* of a trajectory $\{(x(t), p(t))\}_{t \in R}$ (if exists) equals average asymptotic velocity of x, i.e.

$$\lim_{|t| \to +\infty} \frac{\hat{x}(t) - \hat{x}(0)}{t} = h(x_0, p_0). \tag{2}$$

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It is easy to see that rotation vector does not depend on a lift and it is invariant along trajectories of (1). Pose a

Question: *Given rotation number $h \in \mathbb{R}^n$. Is there a trajectory of (1) of rotation vector h or, similarly, is there an invariant set $\mathcal{M}_h \subset \mathbb{T}^n \times \mathbb{R}^n$ of the flow (1) with this property?*

We first state an incorrect statement and then correct it later:

Pseudotheorem: *For any C^2 smooth Hamiltonian $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying standard assumptions and any vector $h \in \mathbb{R}^n$ a nonempty (action-minimizing) invariant set $\mathcal{M}_h \subset \mathbb{T}^n \times \mathbb{R}^n$ exists, i.e. each trajectory of (1) from $\mathcal{M}_h$ has rotation vector h.*

We shall define "action-minimizing" later. This Theorem is easy for $n = 1$. In the case $n = 2$ it is true "locally", which is a fundamental result of Aubry-Mather theory [AD, Ma1]. In general for $n > 2$ without restriction for a set $\mathcal{M}_h$ to be action-minimizing the answer is not known. For action-minimizing sets there are various counterexamples including the classical one by Hedlund (see e.g. [L]). Below we shall present a weaker form of this statement which holds true. Moreover, the family of sets $\{\mathcal{M}_h\}_h$ is intimately related to viscosity solutions of Hamilton-Jacobi equations.

2. KAM example

Recall that a *rigid rotation* of $\mathbb{T}^n$ is a flow given by $\dot{x} = p$ or, equivalently, $\hat{x}(t) = \hat{x}(0) + pt$. Notice that each point in $\mathbb{T}^n$ has the rotation vector p and the rotation vector does not depend on a choice of a coordinate system on $\mathbb{T}^n$. Now recall a basic result of KAM theory. Suppose $H_0(p)$ is C^∞ smooth (resp. analytic) and $H_\varepsilon(x,p) = H_0(p) + \epsilon H_1(x,p)$ is a C^∞ (resp. analytic) small perturbation of a Hamiltonian $H_0(p)$.

Kolmogorov-Arnold-Moser Theorem (see e.g. [AKN]) *If the Hessian matrix $\partial^2_{p_i p_j} H_0(p)$ is nondegenerate for all p in the unit ball $B^n \subset \mathbb{R}^n$, then for small enough ϵ*

- *most of trajectories in $\mathbb{T}^n \times B^n$ of the Hamiltonian flow (1) of $H = H_\varepsilon$ belong to invariant (KAM) n-dimensional tori;*
- *dynamics on each of those tori is smoothly (resp. analytically) conjugate to a rigid rotation;*
- *for most of rotation vectors $h \in B^n$ there is a (KAM) invariant torus $\mathbb{T}^n_{h,\varepsilon}$ with a flow on it smoothly (resp. analytically) conjugate to the rigid rotation $\dot{x} = h$;*
- *each such a (KAM) torus $\mathbb{T}^n_{h,\varepsilon}$ is C^∞ small perturbation of the torus $\mathbb{T}^n_h = \{x \in \mathbb{T}^n,\ p = h\}$.*

Remarks

1. Conjugacy to a rigid rotation implies that each trajectory on such an invariant torus $\mathbb{T}^n_{h,\varepsilon}$ has rotation vector h.

2. Vectors $h \in B^n$ for which invariant tori exist can be described by a diophantine condition on h.
3. Most of trajectories and most of rotation vectors mean of Lebesgue measure $O(\sqrt{\varepsilon})$-close to full in $\mathbb{T}^n \times B^n$ and in B^n respectively.
4. In notations of previous section these tori $\mathbb{T}^n_{h,\varepsilon}$'s is a class of examples of $\mathcal{M}_h$'s. Most of these tori carry an invariant measure which is a smooth deformation of n-dimensional measure $Leb_{\mathbb{T}^n} \times \delta_h$ supported on $\mathbb{T}^n \times \mathbb{R}^n$ for the unperturbed system.

The question: is there an invariant set $\mathcal{M}_h$ of (1) with trajectories having *any* ahead given (e.g. nondiophantine) rotation number? remains valid.

3. From Hamiltonian to Euler-Lagrange equations via Legendge trasformation

To study trajectories of a Hamiltonian flow (1) we make a smooth change of coordinates so that the Hamiltonian flow becomes an Euler-Lagrange flow (see (4) below). The latter flow has a large class of solutions, so-called *action-minimizing or minimizers*, obtained by minimization of action. In what follows we shall study those.

Define a *Legendre transform* of a C^r smooth function $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ with $r \geq 2$

$$L(x, v) = \inf_{p \in \mathbb{R}^n} \{\langle v, p\rangle - H(x, p)\}, \tag{3}$$

where $< \cdot, \cdot >$ is the standard scalar product in $\mathbb{R}^n$. Denote it $L(x, v) = (\mathcal{L} \circ H)(x, v)$.

Lemma 3.1. *If $H(x, p)$ is C^r smooth in (x, p) and convex and superlinear in $p \in \mathbb{R}^n$, then so is $L(x, v)$ in $v \in \mathbb{R}^n$ and $\mathcal{L}_H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ is a C^{r-1} diffeomorphism given by $\mathcal{L}_H : (x, p) \to (x, v)$, $v = \partial_p H(x, p)$. Moreover, the Hamiltonian flow (1) under this diffeomorphism becomes Euler-Lagrange flow*

$$\begin{cases} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \\ \frac{dx}{dt} = \dot{x}. \end{cases} \tag{4}$$

If Hamilton flow (1) is complete the so is the Euler-Lagrange one.

By analogy with Hamiltonian we call convexity, superlinearity of $L(x, v)$ in v and completeness of Euler-Lagrange flow (4) *the standard assumptions* on L. From now on we shall investigate Euler-Lagrange flow (4).

Let $\gamma : [a, b] \to \mathbb{R}^n$ be an absolutely continuous curve lifted from $\mathbb{T}^n$ to the universal cover $\mathbb{R}^n$ and denote by $d\gamma(t) = (\gamma(t, \dot{\gamma}(t))$ the 1-jet of γ. $d\gamma(t)$ is defined for a.e. t. Then

$$\mathcal{A}(\gamma) = \int_a^b L(d\gamma(t))dt \tag{5}$$

is well defined and called *action* along γ.

4. c-minimizers are solutions of E-L equation

First we give an example to motivate the concept of c-minimizer and then define it. This concept, introduced by Mather [Ma2], is a tool to choose among *all possible* trajectories of the Hamiltonian flow (1) *some* which might have certain rotation vector and are minimal.

An absolutely continuous curve $\gamma : [a, b] \to \mathbb{R}^n$ is called *minimal or action-minimizing* subject to fixed end points if for any other absolutely continuous curve $\tilde{\gamma} : [a, b] \to \mathbb{R}^n$ with $\gamma(a) = \tilde{\gamma}(a)$ and $\tilde{\gamma}(b) = \gamma(b)$ we have

$$\mathcal{A}(\gamma) \le \mathcal{A}(\tilde{\gamma}). \tag{6}$$

Similarly, $\gamma : [a, b] \to \mathbb{R}^n$ is minimal subject to one fixed endpoint if for any other absolutely continuous curve $\tilde{\gamma} : [a, b] \to \mathbb{R}^n$ with $\gamma(a) = \tilde{\gamma}(a)$ we have (6). Similarly, one can say $\gamma : [a, +\infty) \to \mathbb{R}^n$ minimal subject to one fixed end point if it is minimal for every $[a, b]$ subject to end points fixed with $a < b < +\infty$.

Model Example: Suppose $H(x, p) = \frac{\langle p,p\rangle}{2}$, then $L(x, v) = \frac{\langle v,v\rangle}{2}$. Let's return to the starting question of finding trajectories with a rotation vector. Fix a vector $h \in \mathbb{R}^n$. If we minimize L, then minimizers with one fixed end point don't have a unique rotation vector. Consider minimization of so called *c-action*

$$\mathcal{A}_c(\gamma) = \int_a^b \{L(d\gamma(t)) - \langle c, \dot{\gamma}(t)\rangle\}\ dt. \tag{7}$$

Notice $\frac{\langle\dot{\gamma}(t),\dot{\gamma}(t)\rangle}{2} - \langle c, \dot{\gamma}(t)\rangle = \frac{\langle\dot{\gamma}(t)-c,\dot{\gamma}(t)-c\rangle-\langle c,c\rangle}{2}$. Easy calculation show that one fixed endpoint minimizer is a straight line whose rotation vector equals c. So c can be thought as a "control" parameter of a direction of motion. It motivates the following definition due to Mather:

Definition 4.1. An absolutely continuous curve $\gamma : [a, b] \to \mathbb{R}^n$ is *c-minimal* subject to one fixed end point if $\gamma(a)$ is fixed and $\gamma(b)$ varies and we have that $\gamma : [a, b] \to \mathbb{R}^n$ minimizes c-action.

Theorem 4.2. *For any C^2 smooth $L : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying standard assumptions and any $c \in \mathbb{R}^n$*
1. (Tonelli) c-minimizer always exists in the class of absolutely continuous curves.
2. Each c-minimizer satisfies Euler-Lagrange equation (4).

3. Completeness of Euler-Lagrange flow (4) implies that such a minimizer is always C^1 smooth.

Remark 4.3. Since a c-minimizer satisfies Euler-Lagrange equation after Legendre change of coordinates it satisfies Hamiltonian equation (1).

Unfortunately, as we pointed out at the end of the first section even adjusting c it is not always possible to direct a c-minimal trajectory into a certain direction. However, an *averaged* statement, which we are about to state, holds true.

5. Mather c-minimal measures

By analogy with c-minimal trajectories, which minimize c-action, we define invariant probability measures which also minimize c-action. Let $\mathcal{P}_L$ be the space of probability measures on $\mathbb{T}^n \times \mathbb{R}^n$ invariant with respect to the Euler-Lagrange flow (4).

Definition 5.1. A measure $\mu \in \mathcal{P}_L$ is called *c-minimal* if c-action of μ, given by

$$\mathcal{A}_c(\mu) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \{L(x,v) - \langle c, v\rangle\} \, d\mu(x,v) \tag{8}$$

realizes minimum of $A_c(\nu)$ over all measures ν in $\mathcal{P}_L$.

Theorem 5.2. [Ma2, Ma3] *For any $c \in \mathbb{R}^n$ there is a c-minimal measure μ invariant with respect to Euler-Lagrange flow (4).*

Question: What do c-minimal measures have to do with trajectories having rotation vector?

6. Rotation vector of a Probability Measure and relation to Mather sets

Rotation vector of a probability measure μ is rouhgly an average of rotation vectors of trajectories in the support of μ. In contrast to rotation vector of a trajectory it always exists. The definition is as follows: $\rho(\mu) \in \mathbb{R}^n$ is a *rotation vector of* μ if

$$\rho(\mu) = \int_{\mathbb{T}^n \times \mathbb{R}^n} v d\mu(x,v). \tag{9}$$

Rotation vector of a convex linear combination of probability measures is the same convex combination of the corresponding rotation vectors.

Examples:

1. If $\mu = Leb_{\mathbb{T}^n} \times \delta_h$ is the product of Lebesgue probability measure on $\mathbb{T}^n$ and δ-measure concentrated at $h \in \mathbb{R}^n$, then $\rho(\mu) = h$.
2. As KAM Example (sect. 2) suggests if $H(x,p)$ is a C^∞ small perturbation

of a positive definite $H_0(p)$ or, equivalently, $L(x, v)$ is a C^∞ small perturbation of a positive definite $L_0(v) = (\mathcal{L} \circ H_0)(v)$, then in $\mathbb{T}^n \times B^n$ most points belong to invariant tori with dynamics conjugated to a rigid one. By KAM Theorem each of these tori carries an n-dimensional measure which is small deformation of $\mu = Leb_{\mathbb{T}^n} \times \delta_h$ for h being the rotation vector. So such a measure is *singular* as a measure in $2n$-dimensional space, but *smooth* as a measure on a graph over $\mathbb{T}^n$.

Now we are ready to state a correction to the Pseudo Theorem.

Theorem 6.1. *(Mather [Ma3]) For any rotation vector $h \in \mathbb{R}^n$ there is a probability measure μ invariant with respect to Euler-Lagrange flow (4) whose rotation vector $\rho(\mu) = h$.*

A measure with rotation vector $\rho(\mu) = h$ is called h-minimal if $\mathcal{A}(\mu) = \inf_{\rho(\nu)=h} \mathcal{A}(\nu)$. Denote by

$$\mathcal{M}_h = \overline{\cup_{h-\text{minimal } \mu} \text{ supp } \mu}. \tag{10}$$

This is also called *Mather set.*

Relation Theorem: *The following two sets coincide*

$$\cup_{h\in\mathbb{R}^n} \mathcal{M}_h = \cup_{c\in\mathbb{R}^n} \mathcal{M}^c. \tag{11}$$

A good starting point to understand definitions above is Aubry-Mather theory. This is a theory about minimizers of large class of convex time periodic Lagrangians $L(x, v, t)$, where $x, t \in \mathbb{T}$, $v \in \mathbb{R}$. In other words Aubry-Mather theory corresponds to the case of 1.5 degrees of freedom. See [Ba, MaF] for an introduction.

7. Zoo of minimizers and Graph Theorems

There are the following major types of minimizers:

$$\begin{array}{lcll}
\mathcal{M}^c & = & \overline{\cup_{c-\text{minimal } \mu} \text{ supp } \mu} & \text{— Mather set} \\
\cap & & & \\
\mathcal{A}^c & = & \overline{\cup \text{ regular } c-\text{minimizers}} & \text{— Aubry set} \\
\cap & & & \\
\mathcal{N}^c & = & \overline{\cup\, c-\text{minimizers}} & \text{— Mane set} \\
\cap & & & \\
E & = & \text{Energy Surface.} &
\end{array}$$

We refer to [Ma3] for definition of regular c-minimizer. This diagram combines results of various people see e.g. [CI].

Graph Theorem: [Ma3] *For any $c \in \mathbb{R}^n$ the corresponding Aubry (resp. Mather) set $\mathcal{M}^c \subseteq \mathcal{A}^c \subset \mathbb{T}^n \times \mathbb{R}^n$ is a (Lipschitz) graph over $\mathbb{T}^n$. More exactly, if $\pi : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n$ is the natural projection, then $\pi^{-1}|_{\pi\mathcal{A}^c} : \pi\mathcal{A}^c \to \mathcal{A}^c$ is Lipschitz.*

8. Shift of c

Let $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be a C^2 smooth Hamiltonian satisfying standard assumptions and $L : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ its Legendre transform. For each $c \in \mathbb{R}^n$ define shifted Hamiltonian and Lagrangian as follows: $L_c(x, v) = L(x, v) - \langle c, v\rangle$ and $H_c(x, p)$ is Legendre transform of L_c. Notice that both H_c and L_c satisfy standard assumptions. Moreover, for each $c' \in \mathbb{R}^n$ we have that c'-minimizer of L_c is $(c' + c)$-minimizer of L. Therefore, it suffices to consider only the case of 0-minimizers or Aubry (resp. Mather) set $\mathcal{M}^c \subseteq \mathcal{A}^c$ of L is Aubry (resp. Mather) set $\mathcal{M}^0 \subseteq \mathcal{A}^0$ of L_c.

Integrable example: Let $H(x, p) = \frac{\langle p,p\rangle}{2}$ or $L(x, v) = \frac{\langle v,v\rangle}{2}$. For each $c \in \mathbb{R}^n$ denote $L_c(x, v) = \frac{\langle v,v\rangle}{2} - \langle c, v\rangle$ and by $H_c(x, p) = \frac{\langle p-c,p-c\rangle}{2}$. Notice that Legendre transform of H_c is L_c. The Euler-Lagrange equation of L_c is $\dot{x} = v - c,\ \dot{v} = 0$. Each torus $\mathbb{T}^n \times \{v = h\} \subset \mathbb{T}^n \times \mathbb{R}^n$ is invariant. Let $u : \mathbb{T}^n \to \mathbb{R}$ be a constant function. Then

$$\mathcal{G}raph\ (\nabla u) = \{(x, \nabla u(x)) :\ x \in \mathbb{T}^n\} \supseteq \mathbb{T}^n_c = \mathbb{T}^n \times \{v = 0\}. \tag{12}$$

Question: *Is each Aubry (resp. Mather) set $\mathcal{M}^c \subseteq A^c$ embedable into $\mathcal{G}raph\ (\nabla u)$ for some $u : \mathbb{T}^n \to \mathbb{R}$ with interesting properties?*

9. Weak KAM theorem

Define semigroup of non-linear operators $(T^-_t)_{t\ge 0}$ from the space of continuous functions $C^0(\mathbb{T}^n)$ on $\mathbb{T}^n$ into itself associated with the Euler-Lagrange flow (4). This semigroup is well known in PDE and in Calculus of Variations and is called *Lax-Oleinik semigroup.* To define it fix $u \in C^0(\mathbb{T}^n)$ and $t > 0$. For $x \in \mathbb{T}^n$ we set

$$T^-_t\ u(x) = \inf_{\gamma}\{u(\gamma(0)) + \int_0^t L(d\gamma(s))ds\}, \tag{13}$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0, t] \to \mathbb{T}^n$ such that $\gamma(t) = x$.

T^-_t defines a semigroup (see e.g. [F2], Cor. 4.2.4). Define *weak KAM homological equation*

$$(14) \qquad T_t^- \ u(x) = u(x) - kt \quad \text{for all } t \geq 0.$$

Solve it means find a number k and a reasonably regular function $u : \mathbb{T}^n \to \mathbb{R}$ which satisfy this equation.

Geometrically this equation means that we are looking for functions $u : \mathbb{T}^n \to \mathbb{R}$ such that $\mathcal{G}raph\ (\nabla u)$ is invariant under backward Euler-Lagrange flow (4). Indeed, one can check that of $(x_0, v_0) = (x_0, \nabla_x u|_{x=x_0})$ is an initial condition, where $\nabla_x u(x)$ is defined, then $(x_t, v_t) = (x_t, \nabla_x u|_{x=x_t})$, because $u(x_t) = u(x_0) + \int_0^t L(d\gamma(s))ds$ along a trajectory $\gamma : [0, t] \to \mathbb{T}^n$. More exactly, if Φ_t is time t map of (4), then

$$(15) \qquad \Phi_t(\mathcal{G}raph\ (\nabla u)) \subseteq \mathcal{G}raph\ (\nabla u) \quad \text{for all } t \leq 0.$$

Back to intergrable Example: When $L(x, v) = \frac{\langle v,v\rangle}{2} - \langle c, v\rangle + k$ for some $c \in \mathbb{R}^n$ and $k \in \mathbb{R}$ we have that for $u(x) = const$ and any position $\hat{x}(t) = \hat{x} + ht$ KAM homological equation (14) holds.

Weak KAM Theorem: (Fathi [F2], Thm.0.1.2) *For some $k \in R$ the backward invariant graph equation $T_t^- \ u = u - kt$ has a Lipschitz solution $u : \mathbb{T}^n \to \mathbb{R}$. Moreover, Aubry (resp. Mather) set belong to the invariant graph:*

$$(16) \qquad \mathcal{M}^0 \subset \mathcal{A}^0 \subset \mathcal{G}raph\ (\nabla u)$$

Remark 9.1. Gradient of a Lipschitz function is defined almost everywhere by Rademacher's theorem, therefore, $\mathcal{G}raph\ (\nabla u)$ exists for a.e. $x \in \mathbb{T}^n$. The second part of the Theorem can be deduced from [F2], Thm. 5.2.8. First, reversing the time in (13) one could consider the analog of KAM homological equation like (14). Solutions to this equation also exists. Thus solutions to forward and backward invariant graph equations do exist and they coincide on the Aubry set $\mathcal{A}^0$, i.e. defined forward and backward in time. See [F2] for more details.

From the point of view of dynamical systems $\mathcal{G}raph\ (\nabla u)$ solving (13) (resp. its analog forward in time) is an unstable (resp. stable) set of the Aubry set $\mathcal{A}^0$.

10. Weak KAM and viscosity solutions of Hamilton-Jacobi PDE's

It turns out that solutions to the invariant graph equation has a very different interpretation. Recall $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is a C^2 smooth Hamiltonian. The equation

$$(17) \qquad H(x, \nabla u(x)) = k \quad \text{for some } k \in \mathbb{R}$$

is called a (stationary) *Hamilton-Jacobi equation.* It is a fundamental equation for (see Evans [E] or Crandall & P.-L. Lions [CL]).

Definition 10.1. (Very Weak Solution) A Lipschitz function $u : \mathbb{R}^n \to \mathbb{R}$ is a very weak viscosity solution of Hamilton-Jacobi equation if it satisfies (17) at differentiability points of u.

The definition makes sense by Rademacher's Theorem.

Definition 10.2. (Viscosity Solution) A function $u : \mathbb{T}^n \to \mathbb{R}$ is a *subsolution* of $H(x, \nabla_x u(x)) = k$ if for every C^1 function $\varphi : \mathbb{T}^n \to \mathbb{R}$ and every point $x_0 \in \mathbb{T}^n$ such that $u - \varphi$ has a *maximum* at x_0, we have $H(x_0, \varphi(x_0)) \le k$.

A function $u : \mathbb{T}^n \to \mathbb{R}$ is a *supersolution* of $H(x, \nabla_x u(x)) = k$ if for every C^1 function $\varphi : \mathbb{T}^n \to \mathbb{R}$ and every point $x_0 \in \mathbb{T}^n$ such that $u - \varphi$ has a *minimum* at x_0, we have $H(x_0, \varphi(x_0)) \ge k$.

A function $u : \mathbb{T}^n \to \mathbb{R}$ is a *solution* of $H(x, \nabla_x u(x) + \text{const}) = c$ on $\mathbb{T}^n$ if it is *both* a subsolution and supersolution.

One can prove that a C^1 function $u : \mathbb{T}^n \to \mathbb{R}$ is a viscosity solution of $H(x, \nabla_x u(x)) = k$ if and only if it is a classical solution. Moreover, the function $u : \mathbb{T}^n \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of $H(x, \nabla_x u(x)) = k$ if and only if $H(x, \nabla_x u(x)) \le k$ (resp. $H(x, \nabla_x u(x)) \ge k$ for all $x \in \mathbb{T}^n$ (see [F2] sect. 7.2).

Theorem 10.3. *(Fathi* [F2] *Thm. 7.6.2) Let H be C^2 smooth and satisfy the standard assumptions and L be its Legendre transform. A continuous function $u : \mathbb{T}^n \to \mathbb{R}$ is a viscosity solution of Hamilton-Jacobi equation (17) if and only if it is Lipschitz and also satisfies equation $T_t^-\ u = u - kt$ for some $k \in \mathbb{R}$ and each $t \ge 0$.*

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