A C^r UNIMODAL MAP WITH AN ARBITRARY FAST GROWTH OF THE NUMBER OF PERIODIC POINTS

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Abstract. In this paper we present a surprising example of a C^r unimodal map of an interval $f : I \to I$ whose number of periodic points $P_n(f) = \{x \in I : f^n x = x\}$ grows faster than any ahead given sequence along a subsequence $n_k = 3^k$. This example also shows that "non-flatness" of critical points is necessary for Martens-de Melo-van Strien Theorem [MMS] to hold.

In this paper we investigate growth of the number of periodic points of C^r maps $f : I \to I$ of the interval $I = [-1, 1]$. Denote by $C^r(I, I)$ the space of such maps with the uniform C^r-topology.

Definition 1. A map $f : I \to I$ is called Artin-Mazur (A-M) if for some $C > 0$ we have

$$P_n(f) \leq \exp(Cn) \quad \forall n \in \mathbb{Z}_+ \tag{1}$$

Artin-Mazur [AM] proved that for any $0 \leq r \leq \infty$ in $C^r(I, I)$ the set of A-M maps is C^r-dense (see also [K1]). It turns out that for 1-dimensional maps much more can be said about A-M maps.

Definition 2. A C^r map $f : I \to I$ has non-flat critical points if for any critical point $c \in I$ one of its higher derivatives does not vanish at c. If the map f is only C^3, then we say that f has a non-flat critical point if there is a local C^3 diffeomorphism ϕ with $\phi(c) = 0$ such that $f(x) = |\phi(x)|^a + f(c)$ for some $a \geq 2$.

Theorem 1. [MMS] *Let $r \geq 2$ and $f : I \to I$ be a C^r map with no flat critical points. Then for some $n_0 \in \mathbb{Z}_+$ and $\gamma > 0$ for any periodic point $x = f^n x$ of period $n > n_0$ we have $|(f^n)'x| > 1 + \gamma$.*

Corollary 1. *In the setting above we have $P_n(f) \leq |I|(2\gamma)^{-1}\|f\|^n_{C^2}$ and, therefore, such an f satisfies A-M property (1) and maps with this property form an open dense set in $C^r(I, I)$ for any $r \geq 2$.*

Our main result is a counterpart of Martens-de Melo-van Strien theorem.

Theorem 2. *For any sequence $a = \{a_n\}_{n \in \mathbb{Z}_+}$ and any $\infty > r \geq 0$ there exists a C^r unimodal map $f : I \to I$ such that for any $k \in \mathbb{Z}_+$ we have $P_{3^k}(f) > a_{3^k}$. The map is C^∞ everywhere except the critical point.*

Remark 1. Certainly the above map f has a flat critical point in the sense of definition 2. Otherwise, it contradicts theorem 1.

It turns out that superexponential growth of the number of periodic point for higher dimensional maps not only exists, but is (Baire) generic in certain open sets in the space of C^r smooth maps of manifolds [K2], which is turn based on [GST].

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0.1. **Degenerate periodic points.** Let $f : I \to I$ be a C^r map. We say that a periodic point $f^n x = x$ is *neutral of order* $k \le r$ if $(f^n)'x = 1,\ (f^n)^{(s)} = 0$ for $s = 1, \dots, k-1$ and $(f^n)^{(k)} \neq 0$[1].

Remark 2. If f has a neutral periodic point $x = f^n x$ of order $k > r$, then by a C^r-perturbation creates arbitrary many periodic points of period n close to x. Thus, if one can create a C^r unimodal map with an infinite number of neutral periodic point $\{p_k\}_{k\in\mathbb{Z}_+}$ of periods 3^k whose orbits are isolated, then this proves theorem 2.

0.2. **Fixed point of a renormalization operator with a degenerate critical point.** A *unimodal map* is an endomorphism of the interval I. It is of the form $f = \phi \circ q_t$, where $\phi \in \mathrm{Diff}^2_+(I)$ is an orientation preserving C^2 diffeomorphism of I and $q_t : I \to I,\ t \in [0,1]$ defined by $q_t(x) = -2t|x|^\alpha + 2t - 1$. The exponent $\alpha > 1$ is called the *critical exponent* of f. The map q_t is called the *canonical folding map* with *peak-value* $t \in [0,1]$. The peak-value determines the maximum $q_t(0) = 2t - 1$. The above form for the canonical folding map is not just a choice for convenience it naturally arises [Ma]. The diffeomorphism ϕ is called the *diffeomorphic part* of f. Notice that $f(-1) = f(1) = -1$. The collection of unimodal maps with chosen critical exponent $\alpha > 1$ is denoted by $\mathcal{U}_\alpha$.

Let $\mathcal{U}_\alpha$ be the collection of unimodal maps whose peak-value is high enough such that the unimodal map has a fixed point $p \in (0,1)$. For every $f \in \mathcal{U}_\alpha$ we can consider the first return map to the interval $[-p,p]$. If the peak-value is not too high the first return map will be just $f^2|_{[-p,p]}$, the unimodal map f is called *renormalizable*. The unimodal map obtained by rescaling this first return-map to $[-p,p]$ is called the *renormalization* of f. The operator defined this way is called the *renormalization operator*. Lanford [L] and, later, Sullivan [S], proved that there is a fixed point for the renormalization operator.

More generally, a unimodal map $f \in \mathcal{U}_\alpha$ is called *renormalizable* if and only if there exists an expanding periodic point $p \in (-1,1)$ such that the first return map to the central interval $C = [-p,p]$ is a of the form $f^q : C \to C$ with $f^q(p) = p$ and $q \ge 2$. The first return map to C will be, up to rescaling, a unimodal map. This unimodal map is a *renormalization* of f. Notice that a renormalization is completely determined by the periodic point p. In particular, when $q = 3$ and the renormalization operator is well-defined. By theorem of Epstein [E] such an operator has a fixed point $f_\alpha \in \mathcal{U}_\alpha$[2]. Moreover, f_α is real analytic.

Let $\alpha = [2r+2]$, $f_\alpha \in \mathcal{U}_\alpha$ denote a fixed point of the renormalization operator for $q = 3$, and 0 be the critical point of f_α. Then it has critical point of order $\ge 2r$ and satisfies the following properties.

- f_α has a fixed point, denoted by p;
- p has a neighborhood $[p^-, p^+]$ free from the forward orbit of 0;
- for each $k \in \mathbb{Z}_+$ there is a periodic point p_k of period 3^k and $p_k = \lambda^k p$ for some $\lambda \in (-1,0)$, where $[-\lambda,\lambda]$ is the interval where the renormalization $f^3_\alpha : [-\lambda,\lambda] \to [-\lambda,\lambda]$ of f_α is defined and equals f_α after rescaling;
- each of intervals $I_k = [\lambda^k p^-, \lambda^k p^+],\ k \in \mathbb{Z}_+$ is free from the forward orbit of 0;

[1] for m-dimensional maps with $m \ge 2$ a periodic point x is called *saddle-node of order* k if absolute values of eigenvalues of its linearization are all but one different from 1 and along the exceptional eigendirection the first derivative is 1 and others of order up to $k-1$ are zero

[2] See Martens [Ma] for more detailed analysis of such fixed points

• derivatives along periodic orbits p_k's for each integer s satisfy $\left(f_\alpha^{3^k}\right)^{(s)}(p_k) = \lambda^{k(1-s)} f_\alpha^{(s)}(p)$.

The Main Idea: We shall prove that an arbitrary small C^r-perturbation $\tilde f$ of f_α for a sufficiently large k_0 and all $k > k_0$ has a neutral periodic point of order $r+1$ at p_k. Moreover, orbits $O_k(f_\alpha) = \{f_\alpha^s(p_k)\}_{s=0}^{3^k-1}$ of p_k's are isolated from each other. Therefore, a C^r small perturbation of $\tilde f$ around of p_k's, for each $k > k_0$ can create explosions of the number of periodic points from p_k's of periods 3^k.

0.3. **Calculations.** In this section we show that, indeed, for a large enough k by a small C^r perturbation inside of I_k. we can obtain a neutral periodic point p_k of order $r+1$. The crucial point of the proof is that *for any set of constants $D_1, \dots, D_{r+1}$ a small C^r perturbations $\tilde f$ of f_α in a small neighbourhood of critical point, say at p_k, $k \gg 1$, allows to change all its first $r+1$ derivatives by a constant, i.e. $\tilde f^{(s)}(p_k) = D_s f_\alpha^{(s)}(p_k)$, $1 \le s \le r+1$.*

Suppose f_α and some $\tilde f$ coincide outside of the union of intervals $\cup_{k \ge k_0} I_k$. Then, since for each $k \in \mathbb{Z}_+$ the trajectory of p_k, excluding p_k, does not visit I_k, we have $f_\alpha^{(s)}(f^l p_k) = \tilde f^{(s)}(f^l p_k)$ for all $s, k \in \mathbb{Z}_+$, and $1 \le l < 3^k$. Then it is proven by direct calculation that there is a change of first $r+1$ derivatives of f_α to $\{\tilde f^{(s)}(p_k) = D_s^k f_\alpha^{(s)}(p_k)\}_{s=1}^{k-1}$ so that p_k becomes neutral periodic of order $r+1$.

To make calculation in the case of general r easier to comprehend we discuss the case $r=2$ first. We fix a large k and consider the trajectory $O_k(f_\alpha)$ of p_k. All the points of $O_k(f_\alpha)$ except p_k are away from I_k. It suffices to C^2 perturb f_α to $\tilde f$ only inside I_k keeping the trajectory $O_k(f_\alpha)$ of p_k fixed so that p_k becomes neutral of order 3 and C^2-norm of the perturbation decays to 0 as k tends to infinity.

Put $\tilde f = f_\alpha + \delta_k$, where δ_k is a function supported in I_k such that

- $\delta_k(\lambda^k p) = \delta(\lambda^k p^-) = \delta(\lambda^k p^+) = 0, \quad \lambda^k p = p_k;$
- $\delta_k^{(s)}(\lambda^k p^-) = \delta_k^{(s)}(\lambda^k p^+) = 0$ for $s = 0, 1, 2;$
- $\delta_k^{(s)}(\lambda^k p) \sim \lambda^{(6-s)k}$ for $s = 0, 1, 2;$
- $\|\delta_k\|_{C^2} \to 0$ as $k \to 0$.

The last item is clearly satisfied because the support of δ_k is of length $\sim \lambda^k$ and C^2-smallness requires δ_k be of order $o(\lambda^{3k})$.

Straghtforward calculation gives

$$
(2) \qquad \begin{aligned}
(f_\alpha^{3^k})'(p_k) &= f'_\alpha(f^{3^k-1} p_k) \dots f'_\alpha(p_k) = f'_\alpha(p) \\
(f_\alpha^{3^k})''(p_k) &= (f_\alpha^{3^k})'(p_k) \left(\sum_{l=0}^{3^k-1} \frac{f''_\alpha(f_\alpha^l p_k)}{f'_\alpha(f_\alpha^l p_k)} \right) = \lambda^{-k} f''_\alpha(p).
\end{aligned}
$$

The left-hand side equalities is the 5-th remark about properties of renormalization. It is easy to see that change of $f'_\alpha(p_k)$ and $f''_\alpha(p_k)$ by a constant suffices to satisfy equations $(\tilde f^{3^k})'(p_k) = 1$ and $(\tilde f^{3^k})''(p_k) = 0$. Now we prove similar statement in the general case $r \ge 2$.

Proposition 1. *For each $k \in \mathbb{Z}_+$ there is a uniformly bounded in k collection of constants $\{D_s^k\}_{s=1}^r$ such that if $f^{(s)}(p_k) = D_s^k f^{(s)}(p_k)$, then $(f^{3^k})^{(s)}(p_k)$ equals 1 for $s = 1$ and 0 for $s = 2, \dots, r$.*

Proof: The proof is in two steps.
Step 1: Let $f, g : I \to I$ be two C^∞ functions and f has zero of order $a \in \mathbb{Z}_+$ at zero, denoted by $\mathrm{ord}_0 f = a$.

Lemma 1. *For any $s < a$ there is a collection of polynomials with integer coefficients $\{P_{sl}(x_1, \dots, x_{s-l+1})\}_{1 \le l \le s \le a}$ such that for each*

$$(3)\quad \begin{aligned} (g \circ f)^{(s)}(x) &= \sum_{l=1}^{s} g^{(l)}(fx) P_{sl}\left(f'(x), \dots, f^{(s-l+1)}(x))\right), \text{ where} \\ P_{sl}\left(f'(x), \dots, f^{(s-l+1)}(x))\right) &= \sum_{i_1+\dots+i_r = s-l+1,\ i_j > 0} a_{i_1 \dots i_r} f^{(i_1)}(x) \dots f^{(i_r)}(x). \end{aligned}$$

with $P_{k0}(x) \equiv 0$ for all k, $P_{11}(x) = f'(x)$, and

$$(4)\quad \begin{aligned} &P_{(s+1)l}\left(f'(x), \dots, f^{(s-l+2)}(x)\right) = \\ &\frac{d}{dx} P_{sl}\left(f'(x), \dots, f^{(s-l+1)}(x))\right) + f'(x) P_{s(l-1)}\left(f'(x), \dots, f^{(s-l+2)}(x))\right) \end{aligned}$$

and if function $Q_{sl}(x) = P_{sl}\left(f'(x), \dots, f^{(s-l+1)}(x))\right)$, then $\mathrm{ord}_0 Q_{sl} = la - s$.

Proof is by induction in s: For $s = 1$ we have $(g \circ f)'(x) = g'(f(x))f'(x)$. So $P_{11}(x) = f'(x)$ and $\mathrm{ord}_0 f'(x) = a - 1$.

Suppose lemma is proven for $s = n$. Prove it for $s = n + 1$.

$$(5)\quad \begin{aligned} (g \circ f)^{n+1}(x) &= \left(\sum_{l=1}^{n} g^{(l)}(f(x)) P_{nl}\left(f'(x), \dots, f^{(s-l+1)}(x)\right)\right)' = \\ &\sum_{l=1}^{n+1} g^{(l)}(f(x)) \left\{ \frac{d}{dx} P_{nl}\left(f'(x), \dots, f^{(s-l+1)}(x)\right) \right. \\ &\quad \left. + f'(x) P_{n(l-1)}\left(f'(x), \dots, f^{(s-l+2)}(x)\right) \right\}, \end{aligned}$$

where the polynomials $P_{(n+1)l}$ as defined in the lemma. By assumption $\mathrm{ord}_0 Q_{nl} = la - n$ so $\mathrm{ord}_0 \frac{d}{dx} Q_{nl} = la - (n+1)$ and $\mathrm{ord}_0 f' \cdot Q_{n(l-1)} = (l-1)a - n + a - 1 = la - (n+1)$. This completes the proof. Q.E.D.

Step 2: For all $k \in \mathbb{Z}_+$ by renormalization property $\left(f^{3^k}\right)^{(s)}(p_k) = \lambda^{k(1-s)} f^{(s)}(p)$. Fix $k \in \mathbb{Z}_+$ and $1 \le s \le r$. If we can prove that for some set of constants $\{D_s\}_{s=1}^r$ independent of k we have $g(x) = f^{3^k-1}(x)$ and all terms in (3) are bounded by $D_s|\lambda^{k(1-s)}|$, then changing $f^{(s)}(p_k)$ to $f^{(s)}(p_k)$ by an explicitly computable constant we can achieve $((f^{3^k-1})^{(s)}(x) = 0$ (or $((f^{3^k-1})'(x) = 1$ if $s = 1$).

Lemma 2. *There is a constant C independent of a, s, l such that for all $1 \le l \le s < a$ in notations of the above lemma we have*

$$(6)\quad \left|(f^{3^k-1})^{(l)}(fp_k) P_{sl}(f'(p_k), \dots, f^{(s-l+1)}(p_k))\right| \le C\lambda^{k(1-s)}.$$

Proof: It suffices to prove the estimate for $l = s$ and use induction is s. Once one differentiate expression (3) for $g(x) \equiv f^{3^k-1}(x)$ and $s = n$ we see that zero of each term in the sum obeys estimate $\mathrm{ord}_0 Q_{sl} = al - n$ and differentiation decrease order of zero by 1 or increase the value by λ^{-k} (up to constant). Thus the first term for

$(n+1)$-st derivative is bounded by the sum of absolute values of the others. This completes the proof of the lemma. Q.E.D.

In a view of arguments before the last lemma. This also completes the proof of the Proposition. Q.E.D.

Now we construct the perturbation in the form $f = \tilde{f} + \sum_{k>k_0} \delta_k$, where δ_k is a function supported in $[\lambda^k p^-, \lambda^k p^+]$. Each δ_k has the following properties

- $\delta_k(\lambda^k p) = \delta(\lambda^k p^-) = \delta(\lambda^k p^+) = 0, \quad \lambda^k p = p_k$;
- $\delta_k^{(s)}(\lambda^k p^-) = \delta_k^{(s)}(\lambda^k p^+) = 0$ for each $s = 0, \dots, r$;
- $\delta_k^{(s)}(\lambda^k p) = (D_s - 1) f^{(s)}(\lambda^k p) \sim \lambda^{(2r+2-s)k}$ for each $s = 0, \dots, r$;
- $\|\delta_k\|_{C^r} \to 0$ as $k \to 0$.

The last item can be satisfied because the support of δ_k is of length $\sim \lambda^k$ and C^r-smallness requires δ_k be of order $o(\lambda^{(r+1)k})$.

By iterative application of calculation of lemma 1 one can check that all periodic points p_k's for k sufficiently large are neutral of order $r+1$. This completes the proof of the Main theorem. Q.E.D.

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