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The Hilbert–Arnold Problem and an Estimate of the Cyclicity of Polycycles on the Plane and in Space

V. Yu. Kaloshin

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1. Polycycles on the plane. The following definition is motivated by the Hilbert–Arnold problem about the uniform boundedness of the number of limit cycles for generic families of vector fields on the sphere S^2 (e.g., see [1, 6]).

Let B be a domain in Euclidean space.

Definition 1. A *polycycle* γ of a vector field $\dot{x} = v(x, \varepsilon^*)$, $x \in S^2$, $\varepsilon^* \in B$, is a cyclically ordered finite sequence of equilibria $p_1, \dots, p_k$ (not necessarily distinct) and a sequence of distinct phase curves $\gamma_1, \dots, \gamma_k$ such that γ_j connects p_j with $p_{j+1 \mod k}$ for each j.

The *cyclicity* of γ is the least number $\mu(\gamma)$ with the following property: there exists a neighborhood $U \subseteq S^2$ of γ and a neighborhood $V \subseteq B$ of ε^* such that each vector field $v(\,\cdot\,, \varepsilon)$, $\varepsilon \in V$, has at most $\mu(\gamma)$ coexisting cycles in U and the Hausdorff distance between each of these cycles and γ tends to zero as $\varepsilon \to \varepsilon^*$.

An equilibrium point is said to be *elementary* if the matrix of the linearized field at this point has at most one zero eigenvalue. A polycycle having only elementary equilibria is said to be *elementary.*

The *elementary cyclicity* $E(n)$ is the maximum possible cyclicity of a polycycle occurring in a generic n-parameter family of C^∞ vector fields on S^2.

Theorem 1. *The estimate $E(n) \leqslant 2^{25n^2}$ holds for each positive integer n.*

In [6] it was shown that $E(n) < \infty$.

2. Bifurcations of polycycles in space. Definition 1 of a polycycle on the plane can be extended to the multidimensional setting. Arguments similar to those in [6, p. 3] show that an infinite family of limit cycles of bounded length lying in a compact set always has a limit polycycle. Let us give a classical example of a polycycle.

2.1. The Shilnikov polycycle. The Shilnikov polycycle is a polycycle that occurs in a generic one-parameter family of vector fields in $\mathbb{R}^3$. It consists of a saddle-focus p with eigenvalues $\lambda > 0$ and $\mu \pm i\omega$, $\mu > 0$, $\lambda + \mu > 0$, and a homoclinic trajectory $\gamma = W^u(p) \cap W^s(p)$.

Consider a Poincaré section T transversal to the phase curve $\gamma \setminus \{p\}$. For an open subset $U \subset T$, the Poincaré map $\Delta \colon U \to T$ is well defined. In contrast with the two-dimensional case, Δ may have not only fixed points, but also points of period greater than 1. An analysis of the topology of Δ shows that it has countably many coexisting Smale horseshoes [4, 9]. One can show that Δ has infinitely many periodic points of period k for each $k \in \mathbb{Z}_+$. This is caused in particular by the presence of complex conjugate eigenvalues.

2.2. An estimate of the cyclicity of a multidimensional polycycle. Let $\gamma \subset \mathbb{R}^N$ be a polycycle of a vector field $\dot{x} = v(x, \varepsilon^*)$, $x \in \mathbb{R}^N$, $\varepsilon^* \in B$. Then γ can be represented as the union of some equilibria $\{p_j\}_{j \in J}$ and phase curves $\{\gamma_j\}_{j \in J}$ connecting these equilibria. A tube neighborhood T_γ of γ is the union of neighborhoods $\{U_j\}_{j \in J}$ of the equilibria $\{p_j\}_{j \in J}$ and tube neighborhoods $\{T_j\}_{j \in J}$ of the phase curves $\{\gamma_j\}_{j \in J}$.

Definition 2. The *k-cyclicity* of a polycycle γ is the least number $\mu(\gamma, k)$ with the following property: there exists a tube neighborhood $T_\gamma \subset \mathbb{R}^N$ of γ, a neighborhood $V \subseteq B$ of ε^*, and Poincaré sections (hypersurfaces) T_j transversally intersecting γ_j at a single point for each $j \in J$ such that for each $\varepsilon \in V$ the vector field $v(\,\cdot\,, \varepsilon)$ has at most $\mu(\gamma, k)$ coexisting cycles in T_γ and

Moscow State University, Department of Mechanics and Mathematics; e-mail: kaloshin@math.princeton.edu.
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each of these cycles intersects each T_j at exactly k distinct points. Each of these cycles is called a *k-winding cycle* or a *k-cycle*.

An equilibrium is said to be *quasielementary* if the matrix of the linearized field has only real eigenvalues and either they are nonresonant or singly resonant, or one of then is zero with Lojasiewicz exponent 1 and the other are nonresonant and distinct.* A polycycle having only quasielementary equilibria is said to be quasielementary.

The *quasielementary k-cyclicity* $\mathrm{QE}(N, n, k)$ is the maximum possible k-cyclicity of a quasielementary polycycle occurring in a generic C^∞ family of vector fields in $\mathbb{R}^N$.

Theorem 2. *For any positive integers N, n, and k, the quasielementary k-cyclicity admits the estimate* $\mathrm{QE}(N, n, k) \leqslant 2^{T^2}$, *where* $T = 6Nnk$.

2.3. Multiplicities of germs of generic maps. An *n-generic C^m germ* is a germ occurring in a generic global C^m map $f\colon \mathbb{R}^n \to \mathbb{R}^N$. For example, for $n = N = 2$, by Whitney's theorem about generic maps of two-dimensional manifolds, there are three types of 2-generic germs: 1-1, folds, and cusps [2].

Definition 3. Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a C^m germ. Its *geometric multiplicity* $G_0(f)$ is the maximum number of preimages in a neighborhood of zero:

$$G_0(f) = \sup_{r\to 0+} \sup_{y\in\mathbb{R}^n} \#\{|x| < r : x \in \phi^{-1}(y)\},$$

where ϕ is some representative of f ($G_0(f)$ is independent of the choice of a representative).

Theorem 3. *The estimate $G_0(f) \leqslant 2^{n(n-1)/2} n^n$ holds for an n-generic C^{n+1} germ $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$.*

One can also estimate the geometric multiplicity of the composition $P \circ j^m f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, where $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^N, 0)$ is a generic C^{n+1} germ and $P\colon J^m(\mathbb{R}^n, \mathbb{R}^N) \to (\mathbb{R}^n, 0)$ is a polynomial of given degree d with regular points. Namely, the estimate $G_0(P \circ j^m f) \leqslant 2^{n(n-1)/2}(dn)^n$ holds, which is independent of the dimension of the intermediate space. This estimate is used in the proof of Theorems 1 and 2.

The proof of Theorems 1–3 uses the normal form theory [5, 6], Khovanskii's method [6, 8], and some new facts [7] from the theory of stratified sets. The proof of Theorem 2 also relies on a special multijet transversality theorem [3].

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*The definition of nonresonant and singly resonant eigenvalues can be found, e.g., in [5, 6].