GROWTH RATE OF THE NUMBER OF PERIODIC POINTS

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February, 2003

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Chapter 1

A Problem of the Growth of the Number of Periodic Points and Decay of Hyperbolicity for Generic Diffeomorphisms.

1.1 Introduction

Let $\text{Diff}^r(M)$ be the space of C^r diffeomorphisms of a finite-dimensional smooth compact manifold M with the uniform C^r-topology, where $\dim M \geq 2$, and let $f \in \text{Diff}^r(M)$. Consider the number of periodic points of period n

$$P_n(f) = \#\{\text{isolated } x : \ x = f^n(x)\}. \tag{1.1}$$

The main question of this paper is:

Question 1.1.1. *How fast can $P_n(f)$ grow with n for a C^r "generic" diffeomorphism f?*

We put the word "generic" in brackets because as the reader will see the answer depends on notion of genericity. We call a diffeomorphism $f \in \text{Diff}^r(M)$ an *Artin-Mazur diffeomorphism* (or simply *A-M diffeomorphism*) if the number of isolated periodic orbits of f grows at most exponentially fast, *i.e.* for some number $C > 0$,

$$P_n(f) \leq \exp(Cn) \quad \text{for all } n \in \mathbb{Z}_+. \tag{1.2}$$

Artin & Mazur [AM] proved that *for any* $0 \le r \le \infty$*, A-M diffeomorphisms are dense in Diff*$^r(M)$ *with the uniform* C^r*-topology.*

We say that a point $x \in M$ of period n for f is hyperbolic if $Df^n(x)$, the derivative of f^n at x, has no eigenvalues with modulus 1. (Notice that a hyperbolic solution to $f^n(x) = x$ must also be isolated.) We call $f \in \mathrm{Diff}^r(M)$ a strongly Artin-Mazur diffeomorphism if for some number $C > 0$,

$$P_n(f) \le \exp(Cn) \quad \text{for all } n \in \mathbb{Z}_+, \tag{1.3}$$

and all periodic points of f are hyperbolic. In [K1] an elementary proof of the following extension of the Artin-Mazur result is given.

Theorem 1.1.2. *For any* $0 \le r < \infty$*, strongly A-M diffeomorphisms are dense in* $\mathrm{Diff}^r(M)$ *with the uniform* C^r*-topology.*

According to the standard terminology a set in $\mathrm{Diff}^r(M)$ is called *residual* if it contains a countable intersection of open dense sets and a property is called *(Baire) generic* if diffeomorphisms with that property form a residual set. In what follows we say

Definition 1.1.3. *An open set* $\mathcal{N} \subset \mathrm{Diff}^r(M)$ *has a supergrowth or a supergrowth is generic in* $\mathcal{N}$ *if for any given sequence* $a = \{a_n\}_{n\in\mathbb{Z}_+}$ *there is a generic set* $\mathbb{R}_a$ *in* $\mathcal{N}$ *depending on the sequence* a_n *with the property if* $f \in \mathbb{R}_a$*, then for infinitely many* $n_k \in \mathbb{Z}_+$ *we have* $Pn_k(f) > a_{n_k}$*.*

It turns out the A-M property is *not generic*, as is shown in [K2]. Moreover it is proven

Theorem 1.1.4. *For any* $2 \le r < \infty$ *there is an open set* $\mathcal{N} \subset \mathrm{Diff}^r(M)$ *where supergrowth is generic.*

Notice that the Theorem implies the first statement, since A-M diffeomorphisms have empty intersection with $\mathbb{R}_a$ for any superexponentially fast growing sequence $a = \{a_n\}_{n\in\mathbb{Z}}$ and any two generic sets must have non-empty intersection. The proof of this result is based on a result of Gonchenko-Shilnikov-Turaev [GST1]. Two slightly different detailed proofs of their result are given in [K2] and [GST2]. The proof in [K2] relies on a strategy outlined in [GST1]. An example of a C^∞ smooth diffeomorphism on a torus with arbitrary fast growth of the number of periodic points has been constructed earlier by Rozales-Gonsales [RG].

In relation to the problem of growth of the number of periodic points Smale [Sm] posed the following question (Problem 4.5, p.765): *Is dynamical* ζ_f*-function generically rational (i.e. is* ζ_f *rational for a residual set of* $f \in \mathrm{Diff}^r(M)$*)?*[1]

Bowen asked the following question in his book [Bo]: *Let* $h(f)$ *denote the topological entropy of* f*. Is the property that* $h(f) = \limsup_{n\to\infty} \log P_n(f)/n$ *generic with respect to the* C^r *topology?*

Theorem 1.1.4 implies negative answer to both question, since both to have a positive answer require A-M to be generic. The following problems natural arise from the above results:

Problem 1.1.5. *Prove that in the space of* C^r *smooth vector fields on a compact manifold* M *A-M vector fields are* C^r*-dense for any* $1 \le r$*. Are they* C^0 *or* C^1 *dense?*

Problem 1.1.6. *For any* $1 \le r$ *prove that in the space of* C^r *smooth volume-preserving or symplectic diffeomorphisms A-M diffeomorphisms are* C^r*-dense.*

Problem 1.1.7. *Construct an analytic diffeomorphism of a compact manifold with a superexponential growth of the number of isolated periodic points*[2]*.*

1.2 Surface diffeomorphisms, Newhouse Phenomenon, and Palis' Conjecture

Actually Theorem 1.1.4 for diffeomorphisms of a 2-dimensional compact manifold M^2 can be extended to the following Theorem.

Theorem 1.2.1. *For any* $2 \le r < \infty$ *in* $\mathrm{Diff}^r(M^2)$ *outside of an open set of hyperbolic diffeomorphisms there is a* C^1*-dense* C^r*-open set satisfying Theorem 1.1.4. In other words, outside of hyperbolic diffeomorphisms supergrowth is* C^1*-dense* C^r*-generic*[3]*.*

Definition 1.2.2. *We say that a* C^r *smooth diffeomorphism* $f : M \to M$ *has a homoclinic tangency (HT in brief) if for some saddle periodic point* p *its stable and unstable manifolds* $W^s(p)$ *and* $W^u(p)$ *respectively have a point of tangency (see Fig. 2 Chapter 2).*

[1] In [Si] it is shown that for the 3-dimensional torus there is no generic set on which the dynamical ζ_f-function is rational.

[2] Yoccoz informed the author that without restriction on isolatedness of periodic points he can produce examples of analytic germs with an arbitrary fast growth of the number of periodic points

[3] C^1-dense C^r-generic contains a countable intersection of C^1-dense C^r-open nested sets.

Notice that diffeomorphisms with HT can occur in any dimension and this definition is dimension independent.

Theorem 1.2.3. *[Ne] For any $2 \leq r$ the C^r-closure of surface diffeomorphisms with HT has a nonempty interior in* $\mathrm{Diff}^r(M^2)$.

Definition 1.2.4. *We say that an open set $\mathcal{N} \subset \mathrm{Diff}^r(M^2)$ is a Newhouse domain if diffeomorphisms with HT are dense in $\mathcal{N}$.*

Theorem 1.2.5. *[K2] For any open set $\mathcal{N} \subset \mathrm{Diff}^r(M^2)$ which is a Newhouse domain Theorem 1.1.4 holds.*

Definition 1.2.6. *We say that a diffeomorphism $f \in \mathrm{Diff}^r(M)$ is hyperbolic if almost every point is attracted to a uniformly hyperbolic attractor and there is only a finite number of those.*

Conjecture 1.2.7. (Palis) *For any $1 \leq r$ the C^r-closure of hyperbolic surface diffeomorphisms and diffeomorphisms with HT coincides with* $\mathrm{Diff}^r(M^2)$.

Theorem 1.2.8. *[PS] For any $1 \leq r$ the C^1-closure of hyperbolic surface diffeomorphisms and diffeomorphisms with HT coincides with* $\mathrm{Diff}^r(M^2)$.

Theorems 1.2.3 and 1.2.8 along with Remark 1.2.5 imply Theorem 1.2.1.

1.3 3-dimensional and higher dimensional diffeomorphisms

For 3-dimensional diffeomorphisms hyperbolic and diffeomorphisms with HT do not form a dense set. There is at least another open set which we shall define.

Definition 1.3.1. *We say that a C^r smooth diffeomorphism $f : M^3 \to M^3$ has a heteroclinic cycle (HC in brief) if for two saddle periodic points p and q with different indices such that $W^u(p)$ and $W^s(q)$ (resp. $W^s(p)$ and $W^u(q)$) are 2-dimensional and have 1-dimensional transversal intersection. Moreover, $W^s(p)$ and $W^u(q)$ (resp. $W^u(p)$ and $W^s(q)$) also intersect.*

Theorem 1.3.2. *[D, BD] For any $1 \leq r$ the C^r-closure of diffeomorphisms with HC has a nonempty interior in* $\mathrm{Diff}^r(M^3)$.

Definition 1.3.3. *We say that an open set $\mathcal{B} \subset \mathrm{Diff}^r(M^2)$ has a blender if diffeomorphisms with HC are dense in $\mathcal{B}$.*

Theorem 1.3.4. *[KP] For any open set $\mathcal{B} \subset \mathrm{Diff}^r(M^3)$ which has a blender Theorem 1.1.4 holds.*

This Theorem exhibits also a different scenario for superexponential growth different from the one in Theorem 1.1.4 produced by HT-phenomenon. It is believed that the following conjecture holds true

Conjecture 1.3.5. *[P] For any $1 \leq r$ the C^r-closure of hyperbolic diffeomorphisms and diffeomorphisms with HT and HC coincides with $\mathrm{Diff}^r(M^3)$.*

Similarly as Theorems 1.2.3 and 1.2.8 along with Remark 1.2.5 imply Theorem 1.1.4 we have that Theorem 1.3.4, an analog of Theorem 1.1.4 for 3-dimensional domains (to be proven), and Conjecture 1.3.5 imply the following

Theorem 1.3.6. *For any $2 \leq r < \infty$ in $\mathrm{Diff}^r(M^3)$ outside of an open set of hyperbolic diffeomorphisms there is an open C^1-dense set satisfying Theorem 1.1.4. In other words, outside of hyperbolic diffeomorphisms supergrowth is C^1-dense C^r-generic set.*

It is natural to pose the following

Conjecture 1.3.7. *For any $\dim M \geq 2$ and any $1 \leq r$ in $\mathrm{Diff}^r(M)$ outside of hyperbolic diffeomorphisms supergrowth is C^r-generic.*

Theorems 1.2.1 and 1.3.6 are justification that Conjecture 1.3.7 holds true. It would be great if one can find a proof of this conjecture without proving Palis' conjecture or Palis-type conjecture. So far proposed way is through these conjectures.

1.4 Superexponential growth in 1-dimension

In this section we state another surprising result which says that even for 1-dimensional maps with one critical point superexponential of the number of periodic points is possible.

Theorem 1.4.1. *[KK] For any sequence $a = \{a_n\}_{n\in\mathbb{Z}_+}$ there is a C^r smooth unimodal map $f : I \to I$ of an interval such that for some $k_0 \in \mathbb{Z}_+$ and any $k > k_0$ we have $P_{3^k}(f) > a_{3^k}$. Moreover, the map is C^∞ everywhere except the critical point.*

Definition 1.4.2. *A C^r smooth $f : I \to I$ has a non-flat critical point at x_0 if after C^r coordinates change $\phi : U_{x_0} \to U_{x_0}$ in a neighborhood of x_0 we have that $(\phi \circ f)^{(s)}(x_0) \neq 0$ for some $1 < s \leq r$.*

Theorem 1.4.3. *[MMS] Let $f : I \to I$ be a C^r smooth map with no flat critical points and $r \geq 2$. Then for some $n_0 \in \mathbb{Z}_+$ and $\gamma > 0$ for each periodic point x of period $n > n_0$ we have that $|(f^n)'(x)| > 1 + \gamma$.*

Corollary 1.4.4. *With f as above and each $n > n_0$ we have $P_n(f) \leq |I|(2\gamma)^{-1}\|f\|^n_{C^2}$.*

This Corollary follows from Proposition 1.5.2. Theorem 1.4.1 also says that requirement of non-flatness of critical points is not only sufficient, but also necessary.

1.5 Growth of the number of periodic points for prevalent diffeomorphisms and Arnold's problem

1.5.1 Arnold's problem

However, it seems unnatural that if you pick a diffeomorphism at random then it may have an arbitrarily fast growth of number of periodic points. Moreover, Baire generic sets in Euclidean spaces can have zero Lebesgue measure. Phenomena that are Baire generic, but have a small probability are well-known in dynamical systems, KAM theory, number theory, etc. (see [O], [HSY], [K3] for various examples). This partially motivates the problem posed by Arnold [A]:

Problem 1.5.1. *Prove that "with probability one" $f \in \mathrm{Diff}^r(M)$ is an A-M diffeomorphism.*

Arnold suggested the following interpretation of "with probability one": *for a (Baire) generic finite parameter family of diffeomorphisms $\{f_\varepsilon\}$, for Lebesgue almost every ε we have that f_ε is A-M.* As Theorem 1.3 shows, a result on the genericity of the set of A-M diffeomorphisms based on (Baire) topology is likely to be extremely subtle, if possible at all. We use instead a notion of "probability one" based on prevalence [HSY, K3], which is independent of Baire genericity. We also are able to state the result in the form Arnold suggested for generic families using this measure-theoretic notion of genericity. For more detailed discussion of prevalence we refer to

[HSY, KH]. Our first main prevalent result is a partial solution to Arnold's problem. It says that *for a prevalent diffeomorphism* $f \in \mathrm{Diff}^r(M)$*, with* $1 < r < \infty$*, and all* $\delta > 0$ *there exists* $C = C(\delta) > 0$ *such that for all* $n \in \mathbb{Z}_+$,

$$P_n(f) \leq \exp(Cn^{1+\delta}). \tag{1.4}$$

1.5.2 Quantitative hyperbolicity and growth of the number of periodic points

The Kupka-Smale theorem (see e.g.[PM]) states that for a generic diffeomorphism all periodic points are hyperbolic and all associated stable and unstable manifolds intersect one another transversally. [K3] shows that the Kupka-Smale theorem also holds on a prevalent set. We shall say *how hyperbolic are the periodic points, as function of their period, for a Baire generic (resp. prevalent) diffeomorphism* f*?*

Recall that a linear operator $L : \mathbb{R}^N \to \mathbb{R}^N$ is *hyperbolic* if it has no eigenvalues on the unit circle $\{|z| = 1\} \subset \mathbb{C}$. Denote by $|\cdot|$ the Euclidean norm in $\mathbb{C}^N$. Then we define the *hyperbolicity* of a linear operator L by

$$\gamma(L) = \inf_{\phi\in[0,1)} \inf_{|v|=1} |Lv - \exp(2\pi i\phi)v|. \tag{1.5}$$

We also say that L is γ-hyperbolic if $\gamma(L) \geq \gamma$. In particular, if L is γ-hyperbolic, then its eigenvalues $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ are at least γ-distant from the unit circle, i.e. $\min_j ||\lambda_j| - 1| \geq \gamma$. The *hyperbolicity* of a periodic point $x = f^n(x)$ of period n, denoted by $\gamma_n(x, f)$, equals the hyperbolicity of the derivative $Df^n(x)$ of f^n at points x, *i.e.* $\gamma_n(x, f) = \gamma(Df^n(x))$. Similarly to the number of periodic points $P_n(f)$ of period n, define

$$\gamma_n(f) = \min_{\{x:\ x=f^n(x)\}} \gamma_n(x, f). \tag{1.6}$$

The idea of Gromov and Yomdin [Y] of measuring hyperbolicity is that a γ-hyperbolic point of period n of a C^2 diffeomorphism f has an $M_2^{-2n}\gamma$-neighborhood (where $M_2 = \|f\|_{C^2}$) free from periodic points of the same period[4]. In [KH] we prove the following result.

[4]In [Y] hyperbolicity is introduced as the minimal distance of eigenvalues to the unit circle. This way of defining hyperbolicity does not guarantee the existence of a $M_2^{-2n}\gamma$-neighborhood free from periodic points of the same period; see [KH]

Proposition 1.5.2. *Let M be a compact manifold of dimension N, let $f : M \to M$ be a C^2 diffeomorphism that has only hyperbolic periodic points, and let $M_2 = \max(\|f\|_{C^2}, 2)$. Then there is a constant $C = C(M) > 0$ such that for each $n \in \mathbb{Z}_+$ we have*

$$P_n(f) \leq C \ (M_2)^{2nN} \gamma_n(f)^{-N}. \tag{1.7}$$

Similar statement can be proven for $C^{1+\rho}$ diffeomorphisms with $0 < \rho \leq 1$. Proposition 1.5.2 implies that a lower estimate on a decay of hyperbolicity $\gamma_n(f)$ gives an upper estimate on growth of number of periodic points $P_n(f)$. Therefore, a natural question is:

Question 1.5.3. *How fast can $\gamma_n(f)$ decay with n for a C^r "generic" diffeomorphism f?*

For Baire generic $f \in \mathrm{Diff}^r(M)$, the existence of lower bound on a rate of decay of $\gamma_n(f)$ would imply the existence of an upper bound on a rate of growth of the number of periodic points $P_n(f)$, whereas no such bound exists by Theorem 1.1.4. Thus again we consider genericity in the measure-theoretic sense of prevalence. Our second main prevalent result, which in view of Proposition 1.5.2 implies the first main prevalent result, is that *for a prevalent diffeomorphism $f \in \mathrm{Diff}^r(M)$, with $1 < r < \infty$, and for any $\delta > 0$ there exists $C = C(\delta) > 0$ such that*

$$\gamma_n(f) \geq \exp(-Cn^{1+\delta}). \tag{1.8}$$

Now we shall discuss in more detail our definition of prevalence ("probability one") in the space of diffeomorphisms $\mathrm{Diff}^r(M)$.

1.5.3 Prevalence in the Space of Diffeomorphisms $\mathrm{Diff}^r(M)$

The space of C^r diffeomorphisms $\mathrm{Diff}^r(M)$ of a compact manifold M is a Banach manifold. Locally we can identify it with a Banach space, which gives it a local linear structure in the sense that we can perturb a diffeomorphism by "adding" small elements of the Banach space. As we described in the previous section, the notion of prevalence requires us to make additive perturbations with respect to a probability measure that is independent of the place that we make the perturbation. Thus although there is not a unique way to put a linear structure on $\mathrm{Diff}^r(M)$, it is important to make a choice that is consistent throughout the Banach manifold.

The way we make perturbations on $\mathrm{Diff}^r(M)$ by small elements of a Banach space is as in [K1]. First we embed M into the interior of the closed unit ball

$B^N \subset \mathbb{R}^N$, which we can do for N sufficiently large by the Whitney Embedding Theorem [W]. Choose any embedding of M into $\mathbb{R}^N$. Then we consider a closed neighborhood $U \subset B^N$ of M and Banach space $C^r(U)$ of C^r functions from U to $\mathbb{R}^N$. Next we extend every element $f \in \mathrm{Diff}^r(M)$ to an element $\tilde{f} \in C^r(U)$ that is strongly contracting in the directions transverse to M. It is done in two steps:
First we consider a manifold $\tilde{M} = M \times [0,1]/\sim$, where the equivalence relation is defined by $(x,0) \sim (f(x),1)$ for all $x \in M$. $\tilde{M}$ is as smooth as f is and carries a naturally defined vector field X_f whose time one map, restricted to $M \times \{0\}$ coincides with f. Such a construction is usually called *suspension*.
Now embed $\tilde{M}$ into the interior of the closed unit ball $B^N \subset \mathbb{R}^N$. One can extend the vector field X_f to a tube neighborhood of $\tilde{M}$ so it contracts strongly in the directions transverse to $\tilde{M}$. A corresponding Poincare return map along the extended vector field is $\tilde{f}$ we need[5].

By Sacker [Sac] and Fenichel [F] persistence of invariant manifolds, since $\tilde{f}$ has M as an invariant manifold, if we add to $\tilde{f}$ a small perturbation in $g \in C^r(U)$, the perturbed map $\tilde{f}+g$ has an invariant manifold in U that is close to M. Then $\tilde{f}+g$ restricted to its invariant manifold corresponds in a natural way to an element of $\mathrm{Diff}^r(M)$, which we consider to be the perturbation of $f \in \mathrm{Diff}^r(M)$ by $g \in C^r(U)$. In this way we reduce the problem to the study of maps in $\mathrm{Diff}^r(U)$, the open subset of $C^r(U)$ consisting of those elements that are diffeomorphisms from U to some subset of its interior. The details of this construction are described in [KH].

The perturbation measure μ that we use here is supported within the analytic functions in $C^r(B^N)$. In this sense we foliate $\mathrm{Diff}^r(B^N)$ by analytic leaves that are compact and overlapping. The main prevalent result then says that *for every analytic leaf $L \subset \mathrm{Diff}^r(B^N)$ and every $\delta > 0$, for almost every diffeomorphism $f \in L$ in the leaf L both (1.4) and (1.8) are satisfied.* Now we define an analytic leaf as a "Hilbert brick" in the space of analytic functions and a natural Lebesgue product probability measure μ on it.

1.5.4 Formulation of Main Prevalent Results

Fix a coordinate system $x = (x_1, \dots, x_N) \in \mathbb{R}^N \supset B^N$ and the scalar product $\langle x, y\rangle = \sum_i x_i y_i$. Let $\alpha = (\alpha_1, \dots, \alpha_N)$ be a multiindex from $\mathbb{Z}^N_+$, and let $|\alpha| = \sum_i \alpha_i$. For a point $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ we write $x^\alpha = \prod_{i=1}^N x_i^{\alpha_i}$. Associate to a real

[5]Thank you C.Carminati for pointing out that two steps process clarifies the way extension $\tilde{f}$ is constructed

analytic function $\phi : B^N \to \mathbb{R}^N$ the set of coefficients of its expansion:

$$\phi_{\vec{\varepsilon}}(x) = \sum_{\alpha \in \mathbb{Z}_+^N} \vec{\varepsilon}_\alpha x^\alpha. \tag{1.9}$$

Denote by $W_{k,N}$ the space of N-component homogeneous vector-polynomials of degree k in N variables and by $\nu(k, N) = \dim W_{k,N}$ the dimension of $W_{k,N}$. According to the notation of the expansion (1.9), denote coordinates in $W_{k,N}$ by $\vec{\varepsilon}_k = \big(\{\vec{\varepsilon}_\alpha\}_{|\alpha|=k}\big) \in W_{k,N}$.

In $W_{k,N}$ we use a scalar product that is invariant with respect to orthogonal transformation of $\mathbb{R}^N \supset B^N$ (see [KH]), defined as follows:

$$\langle \vec{\varepsilon}_k, \vec{\nu}_k \rangle_k = \sum_{|\alpha|=k} \binom{k}{\alpha}^{-1} \langle \vec{\varepsilon}_\alpha, \vec{\nu}_\alpha \rangle, \quad \|\vec{\varepsilon}_k\|_k = \big(\langle \vec{\varepsilon}_k, \vec{\varepsilon}_k \rangle_k\big)^{1/2}. \tag{1.10}$$

Denote by $B_k^N(r) = \{\vec{\varepsilon}_k \in W_{k,N} : \ \|\vec{\varepsilon}_k\|_k \le r\}$ the closed r-ball in $W_{k,N}$ centered at the origin. Let $Leb_{k,N}$ be Lebesgue measure on $W_{k,N}$ induced by the scalar product (1.10) and normalized by a constant so that the volume of the unit ball is one: $Leb_{k,N}(B_k^N(1)) = 1$.

Fix a nonincreasing sequence of positive numbers $\vec{\mathbf{r}} = (\{r_k\}_{k=0}^\infty)$ such that $r_k \to 0$ as $k \to \infty$ and define a Hilbert brick of size $\vec{\mathbf{r}}$

$$\begin{aligned} HB^N(\vec{\mathbf{r}}) =& \{\vec{\varepsilon} = \{\varepsilon_\alpha\}_{\alpha \in \mathbb{Z}_+^N} : \text{ for all } k \in \mathbb{Z}_+, \|\varepsilon_k\|_k \le r_k\} \\ =& B_0^N(r_0) \times B_1^N(r_1) \times \cdots \times B_k^N(r_k) \times \cdots \subset W_{0,N} \times W_{1,N} \times \cdots \times W_{k,N} \ldots . \end{aligned} \tag{1.11}$$

Define *a product Lebesgue probability measure* $\mu_{\vec{\mathbf{r}}}^N$ *associated to the Hilbert brick* $HB^N(\vec{\mathbf{r}})$ *of size* $\vec{\mathbf{r}}$ by normalizing for each $k \in \mathbb{Z}_+$ the corresponding Lebesgue measure $Leb_{k,N}$ on $W_{k,N}$ to the Lebesgue probability measure on the r_k-ball $B_k^N(r_k)$:

$$\mu_{k,r}^N = r^{-\nu(k,N)}\ Leb_{k,N} \quad \text{and} \quad \mu_{\vec{\mathbf{r}}}^N = \times_{k=0}^\infty \mu_{k,r_k}^N. \tag{1.12}$$

Definition 1.5.4. *Let $f \in \text{Diff}^r(B^N)$ be a C^r diffeomorphism of B^N into its interior. We call $HB^N(\vec{\mathbf{r}})$ a Hilbert brick of an admissible size $\vec{\mathbf{r}} = (\{r_k\}_{k=0}^\infty)$ with respect to f if*
A) for each $\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}})$, the corresponding function $\phi_{\vec{\varepsilon}}(x) = \sum_{\alpha \in \mathbb{Z}_+^N} \vec{\varepsilon}_\alpha x^\alpha$ is analytic on B^N;
B) for each $\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}})$, the corresponding map $f_{\vec{\varepsilon}}(x) = f(x) + \phi_{\vec{\varepsilon}}(x)$ is a diffeomorphism from B^N into its interior, i.e. $\{f_{\vec{\varepsilon}}\}_{\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}})} \subset \text{Diff}^r(B^N)$;
C) for all $\delta > 0$ and all $C > 0$, the sequence $r_k \exp(Ck^{1+\delta}) \to \infty$ as $k \to \infty$.

Remark 1.5.5. *The first and second conditions ensure that the family* $\{f_{\vec{\varepsilon}}\}_{\vec{\varepsilon}\in HB^N(\vec{\mathbf{r}})}$ *lie in an analytic leaf within the class of diffeomorphisms* $\mathrm{Diff}^r(B^N)$*. The third condition provides us enough freedom to perturb. It is important for our method to have infinitely many parameters to perturb. If* r_k*'s were decaying too fast to zero it would make our family of perturbations essentially finite-dimensional.*

An example of an admissible sequence $\vec{\mathbf{r}} = (\{r_k\}_{k=0}^\infty)$ is $r_k = \tau/k!$, where τ depends on f and is chosen sufficiently small to ensure that condition (B) holds. Notice that the diameter of $HB^N(\vec{\mathbf{r}})$ is then proportional to τ, so that τ can be chosen a s some multiple of the distance from f to the boundary of $\mathrm{Diff}^r(B^N)$.

Main Prevalent Theorem. *For any* $0 < \rho \le \infty$ *and any* $C^{1+\rho}$ *diffeomorphism* $f \in \mathrm{Diff}^{1+\rho}(B^N)$*, consider a Hilbert brick* $HB^N(\vec{\mathbf{r}})$ *of an admissible size* $\vec{\mathbf{r}}$ *with respect to* f *and the family of analytic perturbations of* f

$$\{f_{\vec{\varepsilon}}(x) = f(x) + \phi_{\vec{\varepsilon}}(x)\}_{\vec{\varepsilon}\in HB^N(\vec{\mathbf{r}})} \tag{1.13}$$

with the Lebesgue product probability measure $\mu_{\vec{\mathbf{r}}}^N$ *associated to* $HB^N(\vec{\mathbf{r}})$*. Then for every* $\delta > 0$ *and for* $\mu_{\vec{\mathbf{r}}}^N$*-a.e.* $\vec{\varepsilon}$ *there is* $C = C(\vec{\varepsilon}, \delta) > 0$ *such that for all* $n \in \mathbb{Z}_+$

$$\gamma_n(f_{\vec{\varepsilon}}) > \exp(-Cn^{1+\delta}) \quad P_n(f_{\vec{\varepsilon}}) < \exp(Cn^{1+\delta}). \tag{1.14}$$

Since a countable intersection of prevalent subsets of a Banach space is prevalent [HSY], the Main Prevalent Theorem implies the results stated in terms of prevalence in the introduction. Indeed, we can decompose $\mathrm{Diff}^r(B^N)$ into a nested countable union of sets $\mathcal{S}_j$ that are each a positive distance from the boundary of $\mathrm{Diff}^r(B^N)$ and for each $j \in \mathbb{Z}^+$ choose an admissible sequence $\vec{\mathbf{r}}_j$ that is valid for all $f \in \mathcal{S}_j$.

Similar results can be proved for $C^{1+\rho}$ smooth 1-dimensional maps. Moreover, 1-dimensional case is a good starting point to learn the method [KH].

In [KH] we deduce from the Main Prevalent Theorem the following more general result. Let $\gamma \ge 0$ and $f \in \mathrm{Diff}^{1+\rho}(B^N)$ be a $C^{1+\rho}$ diffeomorphism for some $\rho > 0$. A point $x \in B^N$ is called (n,γ)*-periodic* if $\|f^n(x) - x\| \le \gamma$ and (n,γ)*-hyperbolic* if $\gamma_n(x,f) = \gamma(Df^n(x)) \ge \gamma$. Notice that a point can be (n,γ)-hyperbolic regardless of its periodicity, but this property is of interest primarily for (n,γ)-periodic points. For positive C and δ let $\gamma_n(C,\delta) = \exp(-Cn^{1+\delta})$.

Theorem 1.5.6. *Given the hypotheses of the Main Prevalent Theorem, for every* $\delta > 0$ *and for* $\mu_{\vec{\mathbf{r}}}^N$*-a.e.* $\vec{\varepsilon}$ *there is* $C = C(\vec{\varepsilon}, \delta) > 0$ *such that for all* $n \in \mathbb{Z}_+$*, every* $(n, \gamma_n^{1/\rho}(C,\delta))$*-periodic point* $x \in B^N$ *is* $(n, \gamma_n(C,\delta))$*-hyperbolic. (Here we assume* $0 < \rho \le 1$*; in a space* $\mathrm{Diff}^{1+\rho}(B^N)$ *with* $\rho > 1$*, the statement holds with* ρ *replaced by* 1*.)*

This result together with Proposition 1.5.2 implies the Main Theorem, because any periodic point of period n is (n, γ)-periodic for any $\gamma > 0$. Also in the statement of the Main Prevalent Theorem and Theorem 1.5.6 the unit ball B^N can be replaced by an bounded open set $U \subset \mathbb{R}^N$. After scaling U can be considered as a subset of the unit ball B^N.

One can define a distance on a compact manifold M and almost periodic points of diffeomorphisms of M. Then one can cover $M = \cup_i U_i$ by coordinate charts and define hyperbolicity for almost periodic points using these charts $\{U_i\}_i$ (see [Y] for details). This gives a precise meaning to the following result.

Theorem 1.5.7. *Let $\{f_\varepsilon\}_{\varepsilon \in B^m} \subset \mathrm{Diff}^{1+\rho}(M)$ be a generic m-parameter family of diffeomorphisms of a compact manifold M for some $\rho > 0$. Then for every $\delta > 0$ and almost every $\varepsilon \in B^m$ there is a constant $C = C(\varepsilon, \delta)$ such that every $(n, \gamma_n^{1/\rho}(C, \delta))$-periodic point x in B^N is $(n, \gamma_n(C, \delta))$-hyperbolic. (Here again we assume $0 < \rho \leq 1$, replacing ρ with 1 in the conclusion if $\rho > 1$.)*

In [KH] we also give a precise meaning to the term *generic*.

Chapter 2

Superexponential Growth of the Number of Periodic Points in Newhouse domains.

In this section we discuss somewhat improved version of Theorem 1.2.5, which says that in a Newhouse domain $\mathcal{N} \subset \text{Diff}^r(M^2)$ supergrowth is C^r-generic with any $1 \leq r < \infty$. We outline the proof of it from [K2] with the main emphasis to point out the original ideas from [GST1]. This represents the *first mechanism* of arbitrary fast growth of the number of periodic points we are going to present. The mechanism for arbitrary fast growth for 1-dimensional maps and 3-dimensional diffeomorphisms with a heteroclinic cycle is presented in the next chapter. Another extremely important result of this chapter, due to Gonchenko-Shilnikov-Turaev [GST1], is that *it is impossible to describe bifurcations of a surface diffeomorphism exhibiting a homoclinic tangency with any finite-parameter family.*

Point out also that Theorem 1.2.5 also implies Theorem 1.1.4 (see [K1] sect. 4.2). The main result of this chapter is the following

Theorem 2.0.8. *Let $2 \leq r < \infty$ and M be a compact 2-dimensional manifold. Let $\mathcal{N} \subset \text{Diff}^r(M)$ be a Newhouse domain. Then for an arbitrary sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ there exists a residual set $\mathcal{R}_a \subset \mathcal{N}$, depending on the sequence $\{a_n\}_{n=1}^{\infty}$, with the property that $f \in \mathcal{R}_a$ implies that*

$$\limsup_{n\to\infty} P_n(f)/a_n = \infty.$$

Moreover, there is a dense set $\mathcal{D}$ in $\mathcal{N}$ such that any diffeomorphism $f \in \mathcal{D}$ has a curve of periodic points.

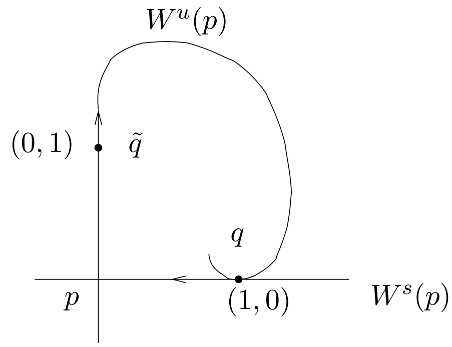

Figure 2.1: Homoclinic tangency

The key to the proof of this Theorem is the following result of Gonchenko-Shilnikov-Turaev [GST1, GST2]

2.1 Gonchenko-Shilnikov-Turaev Theorem [GST1] and degenerate periodic orbits in Newhouse domains

Assume that a C^r diffeomorphism f exhibits a homoclinic tangency. By the Newhouse theorem [Ne], in each C^r neighborhood of a diffeomorphism f exhibiting a homoclinic tangency there exists a Newhouse domain.

Let us define a degenerate periodic point of order k or a k-degenerate periodic point. Sometimes, it is also called a *saddlenode periodic orbit of multiplicity* $k+1$.

Definition 2.1.1. *Let f be a C^s diffeomorphism of a smooth manifold having a periodic orbit p of period m. A periodic point p is called k-degenerate, where $k < s$, if the linear part of f^m at point p has a multiplier $\nu = 1$ while the other multipliers are different in absolute value from the unit and a restriction of f to the central manifold in some coordinate system can be written in the form*

$$x \mapsto x + l_{k+1}x^{k+1} + o(x^{k+1}). \tag{2.1}$$

Let $s > r$. Then C^s diffeomorphisms are dense in the space $\mathrm{Diff}^r(M)$ and, therefore, in any Newhouse domain $\mathcal{N} \subset \mathrm{Diff}^r(M)$ (see e.g. [PM]).

Theorem 2.1.2. *(Theorem 4, [GST1]) For any positive integers $s > k \geq r$ the set of C^s diffeomorphisms having a k-degenerate periodic orbit is dense in a Newhouse domain $\mathcal{N} \subset \mathrm{Diff}^r(M)$.*

This theorem and Theorem 1.2.3 of Newhouse imply the following extremely important result:

Corollary 2.1.3. *[GST1] Let $f \in \mathrm{Diff}^r(M)$ be a diffeomorphism exhibiting a homoclinic tangency. There is no finite number s such that a generic s-parameter family $\{f_\varepsilon\}$ unfolding a diffeomorphism $f_0 = f$ is a versal family of f_0 meaning that the family $\{f_\varepsilon\}$ describes all possible bifurcations occurring next to f. Indeed, to describe all possible bifurcations of a k-degenerate periodic orbit one needs at least $k+1$ parameters and k can be arbitrary large.*

Once Theorem 2.1.2 is proved the proof of Theorem 2.0.8 can be completed by inductive application of the following idea. Let f be a C^s diffeomorphism from a Newhouse domain $\mathcal{N} \subset \mathrm{Diff}^r(M)$ with a k-degenerate periodic orbit p of period, say n, of f for $s > k \geq r$, then p is *flat periodic point* along the central manifold with respect to the C^r topology, namely, by a C^r-perturbation one can make the restriction to the central manifold be *the identical map.* It allows us either to create a *curve of periodic orbits* or *split* p into *any ahead given number of* hyperbolic periodic orbits of the same period (or double the period of p) by a small C^r-perturbation. Since, created periodic orbits are hyperbolic they persist under perturbations. Moreover, after a perturbation we are still in a Newhouse domain one can iterate this procedure of creating a k-degenerate periodic orbits and splitting them without destroying what was done in previous stages (see section 2.7 [K2]).

In what follows we need a few notions related to a saddle periodic point. These definitions will be needed in the proof of Theorem 2.1.2.

Definition 2.1.4. *Let f be a C^s diffeomorphism of a 2-dimensional manifold M and let p be a saddle periodic point of period m, namely, $f^m(p) = p$ with eigenvalues λ and μ, $\lambda < 1 < \mu$. The saddle exponent of p is the number $\rho(p, f) = \frac{-\log\lambda}{\log\mu}$. We call p a ρ-shrinking saddle, where $\rho = \rho(p, f)$. If ρ is greater than some r, then p is also called at least r-shrinking.*

A saddle p is called nonresonant if for any pair of positive integers n and m such that the number $\lambda^n\mu^m$ is different from 1.

2.2 A Scheme of a Proof of Theorem 2.1.2

Theorem 2.1.2 is stated in ([GST1], Thm.4). A proof of this theorem is outlined there. We present a slightly different outline based on [K2]. [K2] in turn essentially uses ideas given in [GST1]. See also [GST2] for a detail proof. In what follows a C^r-perturbation means a small C^r-perturbation. The proof of Theorem 2.1.2 consists of four steps.

The first step. From the existence of a homoclinic tangency of a dissipative saddle, we deduce the existence (after a C^r-perturbation) of a homoclinic tangency of an at least k-shrinking saddle, $k > r$[1].

The second step. From the existence of a homoclinic tangency of an at least k-shrinking saddle, we create a k-floor tower (defined in section 2.3) after a C^r-perturbation (see Fig. 2.3 for $k = 3$).

The third step. From the existence of a k-floor tower, we show that a C^r-perturbation can make a k-th order homoclinic tangency.

The fourth step. From the existence of a k-th order homoclinic tangency we construct by a C^r-perturbation a k-th order degenerate periodic orbit of an arbitrarily high period.

Notice that the way we construct a k-tower is slightly different from the one in [GST1, GST2].

Steps one and four above are done in [K2] by direct calculations (see sect. 2.2 and 2.3 there) so we shall skip description of these steps. Steps two and three are highly original are require a beautiful idea of Gonchenko-Shilnikov-Turaev [GST1] of a *separatrix tower*. In the next section we define a separatrix tower or simply a tower and in the following section explain how towers can produce homoclinic tangencies of arbitrary large orders.

2.3 The second step: Construction of a k-floor tower

Consider a C^∞ diffeomorphism f with a nonresonant saddle periodic point p exhibiting a homoclinic tangency at a point q. First, we give a definition of *a k-floor tower.* Recall that U denotes a neighborhood of the homoclinic tangency q. Let $\tilde{p}$ be a saddle periodic orbit of f, $\tilde{p} \in U$. Then denote by $W^s_{loc}(\tilde{p})$ (resp. $W^u_{loc}(\tilde{p})$) the first connected component of the intersection of stable (resp. unstable) manifold $W^s(\tilde{p})$ (resp. $W^u(\tilde{p})$) with U.

Definition 2.3.1. *A k-floor tower is a collection of k saddle periodic points $p_1, \dots, p_k$*

[1]Theorem 2.1.2 can be done without k-shrinking saddles (see [GST2]), but it is easy to find such a saddle (Corollary 4 [K2])

(of different periods) such that $W^u_{loc}(p_i)$ is tangent to $W^s_{loc}(p_{i+1})$ for $i = 1, \ldots, k-1$, and $W^u_{loc}(p_k)$ intersects $W^s_{loc}(p_1)$ transversally (see Fig. 2.3 for $k = 3$).

For determines we shall consider towers with saddles posed in turn to the left and to the right, i.e. saddles $p_1, p_3, \ldots$ are to the right of a point of homoclinic tangency and saddles $p_2, p_4, \ldots$ are to the left of homoclinic tangency.

Construction of a k-floor tower is an intermediate step in the proof of Theorem 2.1.2. Notice that Fig. 2.3 might be misleading, because in order to get a natural shape of a 3-floor tower one should shrink the figure along the horizontal direction by a sufficiently large factor. If one draw a 3-floor tower with in a natural size such a figure it would become unreadable.

In this section we prove the following

Lemma 2.3.2. *([K2], lemma 2) For any positive integer k a C^r diffeomorphism f exhibiting a homoclinic tangency for an at least r-shrinking saddle periodic orbit p admits a C^r-perturbation $\tilde{f}$ such that $\tilde{f}$ has a k-floor tower. If q is a point of homoclinic tangency of f, then the aforementioned tower of $\tilde{f}$ is located in a neighborhood U of q.*

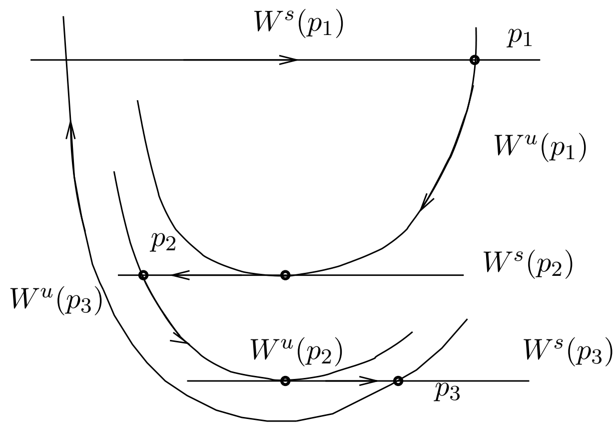

Figure 2.2: A 3-floor tower.

Proof: We prove this Lemma using localized perturbation technic. As usual consider normal coordinates for a nonresonant saddle p. Induce coordinates in U by normal coordinates for the point p and the diffeomorphism f. Simple calculations (see e.g. prop. 1 [K2]) give existence of the contour described on Fig. 2.3 in the case

$k = 3$. Indeed, consider an increasing sequence of numbers $n_1, \ldots, n_k$ such that for each $i = 1, \ldots, k$ the following two properties hold:

1) T_{n_i} intersects $f^{n_i+N}(T_{n_i})$ and they form a horseshoe;

2) n_{i+1} is the largest number such that $T_{n_{i+1}}$ and $f^{n_i+N}(T_{n_i})$ intersect in a horseshoe-like way, i.e., that they bound an open set.

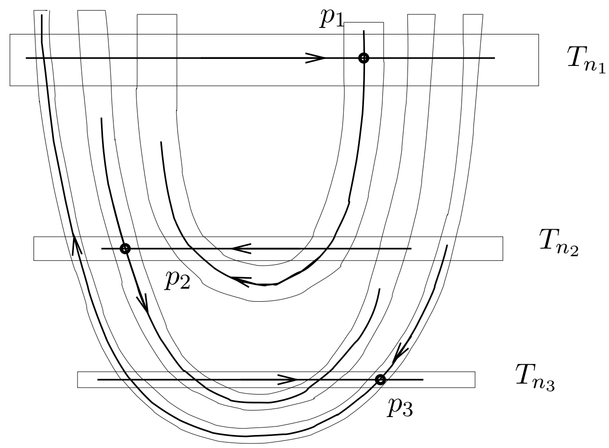

Figure 2.3: An uncomplete 3-floor tower.

For each $i = 1, \ldots, k$ condition 1) implies existence of a saddle periodic point $p_i \in T_{n_i} \cap f^{n_i+N}(T_{n_i})$ of period $n_i + N$ and condition 2) that $W^s_{loc}(p_{i+1})$ and $W^u_{loc}(p_i)$ intersect.

Let U be equipped with normal coordinates. Define the maximal distance in the vertical direction between $W^s_{loc}(p_i)$ and $W^u_{loc}(p_i)$, denoted by s_i, as the maximum of distance between any two points $x \in W^s_{loc}(p_i)$ and $y \in W^u_{loc}(p_i)$, which is below $W^s_{loc}(p_i)$ such that x and y have the same $\bar{x}$-coordinate. Denote the vertical distance between centers of T_{n_i} and $T_{n_{i+1}}$ by t_i (see Fig. 2.3). So, $s_i - t_i$ is the distance by which one should lift $W^u_{loc}(p_i)$ to create heteroclinic tangency with $W^s_{loc}(p_{i+1})$. By direct calculation from section 2.2 of [K2] we get for $t_i = \mu^{-n_i} - \mu^{-n_{i+1}}$.

Lemma 2.3.3. *([K2], Prop.2) If the saddle p having a homoclinic tangency is at least r-shrinking, then the ratio $\frac{s_i - t_i}{t_i^r}$ is arbitrarily small for each $i = 1, \ldots, k-1$.*

If such a ratio is small it is easy to see that by a C^r-perturbation one can create a heteroclinic tangency (see Fig. 2.3)

In order to construct a k-floor tower one needs to create a heteroclinic tangency of $W^s_{loc}(p_{n_{i+1}})$ and $W^u_{loc}(p_{n_i})$ by a C^r-perturbation. We construct it by "bending" $W^u_{loc}(p_{n_i})$.

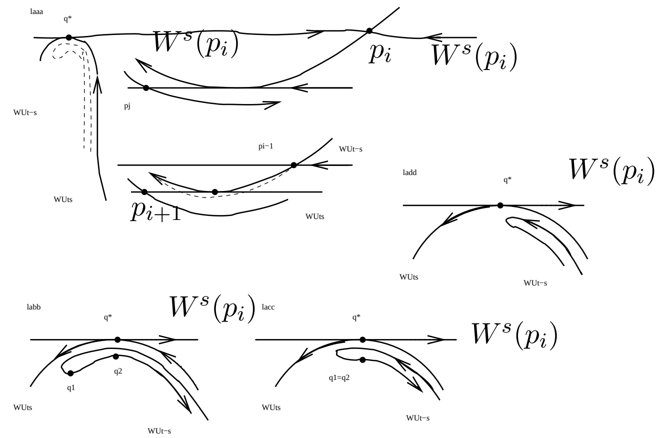

Figure 2.4: A localized perturbation for a floor of a tower.

Another way to construct a k-floor tower, used in [GST1, GST2], is by fixing the eigenvalue $\mu > 1$ and varying the other eigenvalue $\lambda < 1$ of the saddle p exhibiting homoclinic tangency. Simple calculations (see e.g. prop. 1 [K2]) shows that the rectangle T_n is centered at $(1, \mu^{-1})$ and the curvilinear rectangle $f^{n+N}(T_n)$ is $C\delta_n^\rho = C\lambda^n$ distant away from $W^s(p)$, therefore, by changing λ one can vary the position of $f^{n+N}(T_n)$ without changing the position of T_n. But, in this case one needs some additional geometric argument to construct all heteroclinic tangencies of a k-tower simultaneously.

2.4 The third step: Construction of a $k - th$ order tangency

We shall prove that by a perturbation of a $(k+1)$-floor tower one can create a k-th order homoclinic tangency. Let us start with a 2-nd order tangency for a 3-floor tower and then use induction in k then.

Proposition 2.4.1. *[GST1], [K2] A perturbation of a 3-floor tower can create a 2-nd order homoclinic tangency.*

Proof: Eventhough this lemma is proven in [GST1, GST2], [K2] we present a proof of it, which is simple and elegant. It also gives a pretty good idea of how a k-floor tower might produce a k-th order homoclinic tangency by a C^r-perturbaton.

Step 1. Let us create a 1-st order tangency of $W^u_{loc}(p_2)$ and $W^s_{loc}(p_1)$. Start with a 3-tower on Fig. 2.3. "Push" $W^u_{loc}(p_2)$ down $W^s_{loc}(p_3)$. Denote by γ the tongue (the part) of $W^u_{loc}(p_2)$ underneath $W^s_{loc}(p_3)$. The tongue γ is in the sector of the saddle hyperbolic point p_3, therefore, under iteration of f γ will be stretched along $W^u_{loc}(p_3)$ and for some s $f^s(\gamma) \cap W^s(p_1) \neq \emptyset$. Varying the size of the tongue γ we can create a heteroclinic tangency (see Fig. 2.4.a) with $i = 0$). Denote a point of tangency by q^*. Only two parts of $W^u(p_2)$ are depicted on Fig. 2.4 a): first part — starting part of $W^u(p_2)$ at p_2 and second — image of γ after a number of iterations under f (in above notations $f^s(\gamma)$).

Assume that saddle p_1 is nonresonant. Then there is normal coordinates around p_1 linearizing f. Induce by f normal coordinates in a neighborhood of U^* of q^*. In what follows we shall use these coordinate systems in U^*.

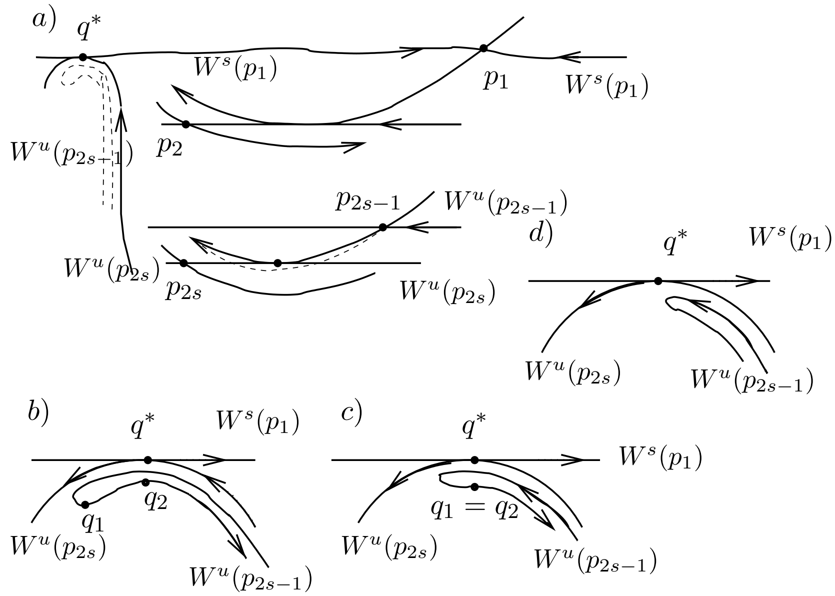

Figure 2.5: A 2-nd (even) order tangency.

Step 2. Let us create a 2-nd order homoclinic tangency of $W^u_{loc}(p_1)$ and $W^s_{loc}(p_1)$. Start with a contour on Fig. 2.4 a). "Push" $W^u_{loc}(p_1)$ down $W^s_{loc}(p_2)$. Denote by γ^1 the tongue (the part) of $W^u(p_1)$ underneath $W^s(p_2)$. Some iterate of the tongue γ^1 of $W^u_{loc}(p_1)$ come into U^*. U^* has normal coordinates and naturally defined the horizontal and the vertical directions. Now our goal is varying the size of γ^1 construct a tangency of some iterate of γ^1 to the horizontal direction in U^*.

Fix some coordinates in a neighborhood of $\tilde{U}$ of a tangency $\tilde{q}$ of $W^u_{loc}(p_1)$ and

$W^s_{loc}(p_2)$. Consider a 1-parameter family of diffeomorphisms $\{f_\varepsilon\}$, where ε is the maximal distance of $W^u_{loc}(p_1) \cap \tilde{U}$ and $W^s_{loc}(p_2) \cap \tilde{U}$ in the vertical direction.

Let $\gamma_\varepsilon = W^u_{loc}(p_1) \cap \{y \leq 0\}$. Fix $\varepsilon > 0$ and $s = s(\varepsilon)$ such that $\gamma^s_\varepsilon = f^s(\gamma^1_\varepsilon) \cap U^* \neq \emptyset$ (see Fig. 2.4 b)). The curve γ^s_ε has two points q_1 and q_2 of tangency to the horizontal direction. As ε decreases q_1 and q_2 approach one to the other and for some critical value ε^* they collide and $q_1 = q_2$ (see Fig. 2.4 c)).

At the point $q_1 = q_2$ the local unstable manifolds $W^u_{loc}(p_1)$ has a 2-nd order tangency to the horizontal direction. Let this point have coordinates $(\varepsilon_1, \varepsilon_2)$ in U^*. Lifting $W^u_{loc}(p_2)$ by ε_2 we can create a 2-nd order tangency at point $(\varepsilon_1, 0)$. This completes the proof of the Proposition. Q.E.D.

For construction of higher order tangencies we refer to [GST2], [K2] and point out that the idea is the same as for 3-rd order, but requires a more delicate calculations (see e.g. [K2], prop 5). This exposition shows the third step of the proof of Theorem 2.0.8 is done. As we pointed out above steps one and four can be done by direct calculations and their description is omitted here (see [GST2, K2] for details).

Chapter 3

Superexponential Growth of the Number of Periodic Points for 1-dimensional maps and 3-dimensional heteroclinic cycles

In this chapter we prove Theorem 1.4.1 in the case $r = 2$ and outline the proof of Theorem 1.3.4 from [KP]. The main emphasis will be to point out the ideas for the proofs and refer to the papers [KK, KP] for more involved details. These two proofs represent the *second and third mechanisms* of arbitrary fast growth of the number of periodic points we are going to present. These results are indications of the fact that there are many scenarios superexponential growth of the number of periodic point can occur outside of hyperbolic diffeomorphisms. Theorems 1.2.5 and 1.3.4 support Conjecture 1.3.7, provided Conjecture 1.3.5 is true.

3.1 Superexponential growth of periodic points for 1-dimensional maps

This section is devoted to 1-dimensional example in the case $r = 2$. In the case $r \geq 2$ the proof is almost the same, just calculations are more involved [KK]. Denote by $C^r(I, I)$ the space of such maps with the uniform C^r-topology. As we pointed out in the introduction there is a C^r-dense set of Artin-Mazur maps. Moreover, by Martens-de Melo-van Strien Theorem (see Theorem 1.4.3 and its Corollary) those maps which are not Artin-Mazur form a set of codimension r. The main result of the first part of this chapter is a counterpart of Martens-de Melo-van Strien theorem. Namely, Theorem 1.4.1. Below we describe the basic idea of the proof of this Theorem.

Let $f : I \to I$ be a C^r map. We say that a periodic point $f^n x = x$ is *neutral of order* $k \leq r$ if $(f^n)'x = 1,\ (f^n)^{(s)} = 0$ for $s = 1, \dots, k-1$ and $(f^n)^{(k)} \neq 0$[1].

Remark 3.1.1. *If f has a neutral periodic point $x = f^n x$ of order $k > r$, then by a C^r-perturbation creates arbitrary many periodic points of period n close to x. Thus, if one can create a C^r unimodal map with an infinite number of neutral periodic point $\{p_k\}_{k \in \mathbb{Z}_+}$ of periods 3^k whose orbits are isolated, then this proves Theorem 1.4.1.*

3.1.1 Fixed point of a renormalization operator with a degenerate critical point

A *unimodal map* is an endomorphism of the interval I. It is of the form $f = \phi \circ q_t$, where $\phi \in \mathrm{Diff}^2_+(I)$ is an orientation preserving C^2 diffeomorphism of I and $q_t : I \to I,\ t \in [0,1]$ defined by $q_t(x) = -2t|x|^\alpha + 2t - 1$. The exponent $\alpha > 1$ is called the *critical exponent* of f. The map q_t is called the *canonical folding map* with *peak-value* $t \in [0,1]$. The peak-value determines the maximum $q_t(0) = 2t - 1$. The above form for the canonical folding map is not just a choice for convenience it naturally arises [Ma]. The diffeomorphism ϕ is called the *diffeomorphic part* of f. Notice that $f(-1) = f(1) = -1$. The collection of unimodal maps with chosen critical exponent $\alpha > 1$ is denoted by $\mathcal{U}_\alpha$.

Let $\mathcal{U}_\alpha$ be the collection of unimodal maps whose peak-value is high enough such that the unimodal map has a fixed point $p \in (0,1)$. For every $f \in \mathcal{U}_\alpha$ we can consider the first return map to the interval $[-p,p]$. If the peak-value is not too high the first return map will be just $f^2|_{[-p,p]}$, the unimodal map f is called *renormalizable.* The unimodal map obtained by rescaling this first return-map to $[-p,p]$ is called the *renormalization* of f. The operator defined this way is called the *renormalization operator.* Lanford [L] and, later, Sullivan [S], proved that there is a fixed point for the renormalization operator.

More generally, a unimodal map $f \in \mathcal{U}_\alpha$ is called *renormalizable* if and only if there exists an expanding periodic point $p \in (-1,1)$ such that the first return map to the central interval $C = [-p,p]$ is a of the form $f^q : C \to C$ with $f^q(p) = p$ and

[1] for m-dimensional maps with $m \geq 2$ a periodic point x is called *saddle-node of order k* if absolute values of eigenvalues of its linearization are all but one different from 1 and along the exceptional eigendirection the first derivative is 1 and others of order up to $k-1$ are zero

$q \geq 2$. The first return map to C will be, up to rescaling, a unimodal map. This unimodal map is a *renormalization* of f. Notice that a renormalization is completely determined by the periodic point p. In particular, when $q = 3$ and the renormalization operator is well-defined. By theorem of Epstein [E] such an operator has a fixed point $f_\alpha \in \mathcal{U}_\alpha$[2]. Moreover, f_α is real analytic.

Let $\alpha = [2r + 2]$, $f_\alpha \in \mathcal{U}_\alpha$ denote a fixed point of the renormalization operator for $q = 3$, and 0 be the critical point of f_α. Then it has critical point of order $\geq 2r$ and satisfies the following properties.

- f_α has a fixed point, denoted by p;
- p has a neighborhood $[p^-, p^+]$ free from the forward orbit of 0;
- for each $k \in \mathbb{Z}_+$ there is a periodic point p_k of period 3^k and $p_k = \lambda^k p$ for some $\lambda \in (-1, 0)$, where $[-\lambda, \lambda]$ is the interval where the renormalization $f_\alpha^3 : [-\lambda, \lambda] \to [-\lambda, \lambda]$ of f_α is defined and equals f_α after rescaling;
- each of intervals $I_k = [\lambda^k p^-, \lambda^k p^+],\ k \in \mathbb{Z}_+$ is free from the forward orbit of 0;
- derivatives along periodic orbits p_k's for each integer s satisfy $\left(f_\alpha^{3^k}\right)^{(s)}(p_k) = \lambda^{k(1-s)} f_\alpha^{(s)}(p)$.

The Main Idea: We shall prove that an arbitrary small C^r-perturbation $\tilde{f}$ of f_α for a sufficiently large k_0 and all $k > k_0$ has a neutral periodic point of order $r + 1$ at p_k. Moreover, orbits $O_k(f_\alpha) = \{f_\alpha^s(p_k)\}_{s=0}^{3^k-1}$ of p_k's are isolated from each other. Therefore, a C^r small perturbation of $\tilde{f}$ around of p_k's, for each $k > k_0$ can create explosions of the number of periodic points from p_k's of periods 3^k.

3.1.2 Calculations

In this section we show that, indeed, for a large enough k by a small C^r perturbation inside of I_k. we can obtain a neutral periodic point p_k of order $r+1$. The crucial point of the proof is that *for any set of constants $D_1, \ldots, D_{r+1}$ a small C^r perturbations $\tilde{f}$ of f_α in a small neighborhood of a critical point, say at $p_k,\ k \gg 1$, allows to change all its first $r + 1$ derivatives by a constant, i.e. $\tilde{f}^{(s)}(p_k) = D_s f_\alpha^{(s)}(p_k),\ 1 \leq s \leq r + 1$.*

Suppose f_α and some $\tilde{f}$ coincide outside of the union of intervals $\cup_{k \geq k_0} I_k$. Then, since for each $k \in \mathbb{Z}_+$ the trajectory of p_k, excluding p_k, does not visit I_k, we have $f_\alpha^{(s)}(f^l p_k) = \tilde{f}^{(s)}(f^l p_k)$ for all $s, k \in \mathbb{Z}_+$, and $1 \leq l < 3^k$. Then it is proven by direct calculation that there is a change of first $r + 1$ derivatives of f_α to $\{\tilde{f}^{(s)}(p_k) = D_s^k f_\alpha^{(s)}(p_k)\}_{s=1}^{k-1}$ so that p_k becomes neutral periodic of order $r + 1$.

[2] See Martens [Ma] for more detailed analysis of such fixed points

To make calculation in the case of general r easier to comprehend we discuss the case $r = 2$ first. We fix a large k and consider the trajectory $O_k(f_\alpha)$ of p_k. All the points of $O_k(f_\alpha)$ except p_k are away from I_k. It suffices to C^2 perturb f_α to $\tilde{f}$ only inside I_k keeping the trajectory $O_k(f_\alpha)$ of p_k fixed so that p_k becomes neutral of order 3 and C^2-norm of the perturbation decays to 0 as k tends to infinity.

Put $\tilde{f} = f_\alpha + \delta_k$, where δ_k is a function supported in I_k such that

- $\delta_k(\lambda^k p) = \delta(\lambda^k p^-) = \delta(\lambda^k p^+) = 0, \quad \lambda^k p = p_k;$
- $\delta_k^{(s)}(\lambda^k p^-) = \delta_k^{(s)}(\lambda^k p^+) = 0$ for $s = 0, 1, 2;$
- $\delta_k^{(s)}(\lambda^k p) \sim \lambda^{(6-s)k}$ for $s = 0, 1, 2;$
- $\|\delta_k\|_{C^2} \to 0$ as $k \to 0$.

The last item is clearly satisfied because the support of δ_k is of length $\sim \lambda^k$ and C^2-smallness requires δ_k be of order $o(\lambda^{3k})$.

Straghtforward calculation gives

$$\begin{aligned} (f_\alpha^{3^k})'(p_k) &= f'_\alpha(f^{3^k-1}p_k)\dots f'_\alpha(p_k) = f'_\alpha(p) \\ (f_\alpha^{3^k})''(p_k) &= (f_\alpha^{3^k})'(p_k)\left(\sum_{l=0}^{3^k-1}\frac{f''_\alpha(f_\alpha^l p_k)}{f'_\alpha(f_\alpha^l p_k)}\right) = \lambda^{-k} f''_\alpha(p). \end{aligned} \tag{3.1}$$

The left-hand side equalities is the 5-th remark about properties of renormalization. It is easy to see that change of $f'_\alpha(p_k)$ and $f''_\alpha(p_k)$ by a constant suffices to satisfy equations $(\tilde{f}^{3^k})'(p_k) = 1$ and $(\tilde{f}^{3^k})''(p_k) = 0$. To prove similar statement in the general case $r \geq 2$ more involved calculations are required (see [KK]).

3.2 Superexponential Growth of periodic points for 3-dimensional heteroclinic cycles

In this section we outline the proof of Theorem 1.3.4 from [KP]. This Theorem represents another mechanism for creation of superexponential growth of the number of periodic points outside of the set of hyperbolic diffeomorphisms. It is also another indication toward Conjecture 1.3.7.

The initial idea of the proof of Theorem 1.3.4 is the same as for Theorem 1.2.5. We shall prove that diffeomorphisms having a degenerate periodic orbits of all orders are dense.

Theorem 3.2.1. *Let $1 \leq r < \infty$ and M be a compact 3-dimensional manifold. Let $\mathcal{B} \subset \mathrm{Diff}^r(M)$ be a domain with a blender. Then for any positive integers $s > k \geq r$ the set of C^s diffeomorphisms having a k-degenerate periodic orbit is dense in $\mathcal{B} \subset \mathrm{Diff}^r(M)$.*

Then arguments from Section 2.1 independently of dimension of M show that Theorem 3.2.1 implies Theorem 1.3.4. Similarly to Section 2.2 we outline the proof of Theorem 3.2.1 in the next Section. Below we formulate several auxiliary definitions.

Definition 3.2.2. *In definition 1.2.2 of an HC (heteroclinic cycle) we call periodic points p, q vertices of HC.*

We say that such an HC is k-degenerate if its vertices p, q are k-degenerate periodic points for some $k > 0$.

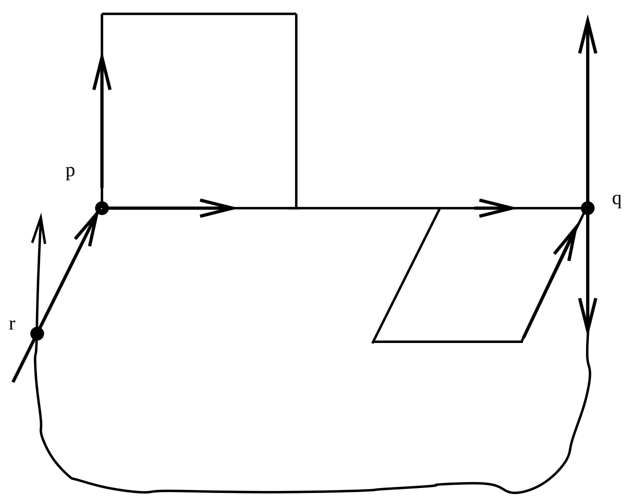

Figure 3.1: A Heteroclinic Cycle

3.2.1 A Scheme of a Proof of Theorem 3.2.1

In couple of passages below we say *create a dynamic object* meaning that there is a small C^r-perturbation of a starting diffeomorphism (before a creation) so that a perturbed diffeomorphism has a "created" property. For example, Theorem 2.1.2 implies that from a diffeomorphism in a Newhouse domain one can *create* a homoclinic tangency, i.e. there is a C^r close diffeomorphism with a homoclinic tangency.

The proof of Theorem 3.2.1 consists of k stages steps. Each stage, except the last one consists of three steps. Start with a diffeomorphism $f : M^3 \to M^3$ having an HC.

Stage 1: *Construction of a* 1*-degenerate HC*

The first step. We approximate dynamics near HC by dynamics of a 1-paramater family of 1-dimensional maps (see Fig. 3.2.2).
The second step. Within this family of 1-dimensional maps we can find a 1-degenerate periodic point (or simply a saddlenode). Then using a reduction from Step 1 we deduce existence of a 1-degenerate periodic point p_1 for a C^r-perturbation of f, denoted by f_1.
The third step. Without destroying p_1, we create another 1-degenerate periodic point q_1 for f_2.
The forth step. Now we create a 1-degenerate HC for f_3 with p_1 and q_1 as vertices.

Stage 2: *Construction of a* 2*-degenerate HC*

In this Step we basically repeat Stage 1 for higher order degenerate HCs. Namely:
The first step. We approximate dynamics near HC by dynamics of a 1-paramater family of 1-dimensional maps (see Fig. 3.2.2).
The second step. Using such a reduction deduce existence of a 2-degenerate periodic point p_2 for a C^r-perturbation of f_4, denoted by f_4.
The third step. Similarly, without destroying p_2, we create another 2-degenerate periodic point q_2 for f_5.
The forth step. Now we create for f_6 a 2-degenerate HC with p_2 and q_2 as vertices.

Following this algorithm after the (k-1)-st stage we have an $(k-1)$-degenerate HC with verices $(k-1)$-degenerate periodic points p_{k-1} and q_{k-1} as vertices. The first stage of the k-th step gives existence of a k-denenerate periodic point p_k. This is the required statement which completes the proof of Theorem 3.2.1.

3.2.2 Model 1-dimensional maps

In this subsection we present a family of 1-dimensional maps which arises in the first steps of each stage and show how such a family can produce a more degenerate periodic orbit than those in the family.

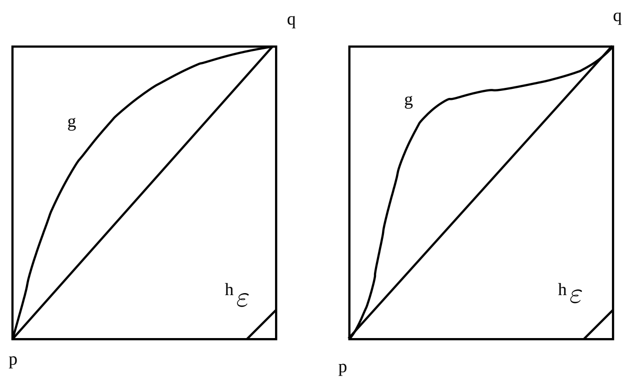

Figure 3.2: Model 1-dimensional maps

Denote the unit interval by I. Fix a small $\varepsilon > 0$. Consider a pair of C^r-smooth maps g and h_ε, $r \geq 1$ of the following form:

$$\begin{gathered} g : I \to I,\ g(0) = 0,\ g'(0) > 1,\ g(1) = 1,\ g'(1) < 1, \\ g(x) > x\ textupforevery\ x \in I,\ g \text{ is monotonically increasing} \\ h_\varepsilon : [1-\varepsilon, 1] \to [0, \varepsilon],\ h(x) = 1 - x. \end{gathered} \tag{3.2}$$

The way dynamics of this pair is defined is as follows: if $x \in [1-\varepsilon]$ we always apply g and if $x \in [1-\varepsilon, 1]$, then we can apply either g or h_ε.

Heuristically connection between an HC on Fig. 3.2 and this family is the following. Suppose a diffeomorphism $f : M^3 \to M^3$ has an HC. Consider a 1-parameter family f_ε perturbing $f = f_0$, where ε measures the oriented minimal distance between continuation of $W^s(p)$, denoted $W^s(p_\varepsilon)$, and continuation of $W^u(q)$, denoted $W^u(q_\varepsilon)$ in a small neighborhood of r (see Fig. 3.2). For every small ε continuations $W^s(q_\varepsilon) \cap W^u(p_\varepsilon)$ form a smooth segment, say J. Moreover, dynamics induced on J is smoothly conjugate to a smooth map $\tilde{g}$ of the unit interval I and $\tilde{g}$ satisfies conditions (3.2) of g. Indeed, one end of J is repelling, the other is attracting, and no fixed points in between.

To see heuristically the analog of the map h_ε on Fig. 3.2 first notice that for $\varepsilon = 0$ we have a sequence of point on $W^u(q)$ converging to q which under forward iterations approach p or, equivalently, belong to $W^s(p)$. By analogy $h_0(1) = 0$. Now pick $\varepsilon > 0$, then the intersection of local parts of $W^s(p_\varepsilon)$ and $W^u(q_\varepsilon)$ becomes empty. So trajectories on $W^u(q)$ can start arbitrarily close to q, but don't have to approach p. In terms of h_ε we have $h_\varepsilon(1) = \varepsilon \neq 0$. Below we formulate the statement which is necessary for the proof.

In a view of the discussion above one might guess that if we consider a k-degenerate HC with vertices p and q, then the corresponding family has the same h_ε, but a different C^r smooth g, $r \geq k$, where g has the following properties:

$$\begin{aligned} &g : I \to I,\ g(0) = 0,\ g'(0) = 1,\ g(1) = 1,\ g'(1) = 1, \\ &g^{(s)}(0) = g^{(s)}(1) = 0 \text{ for } s = 2, \dots, k, \\ &g(x) > x,\ x \in I \text{ and } g \text{ is monotonically increasing} \\ &h_\varepsilon : [1-\varepsilon, 1] \to [0, \varepsilon],\ h(x) = 1 - x. \end{aligned} \tag{3.3}$$

Pairs of maps with conditions (3.2) and (3.3) satisfy the following

Lemma 3.2.3. *[KP] For a pair of a map g and a family $\{h_\varepsilon\}_{\varepsilon>0}$ satisfying (3.2) there is a sequence ε_n decreasing to 0 such that for each ε_n there is an integer $p_n \in \mathbb{Z}_+$ and a point $x \in I$ which is 1-degenerate periodic of period p_n. More exactly, $h_\varepsilon \circ g^{p_n - 1}(x) = x$ and $(h_\varepsilon \circ g^{p_n-1})'(x) = 1$.*

Similarly, for any $k \in \mathbb{Z}_+$ and any pair of a map g and a family $\{h_\varepsilon\}_{\varepsilon>0}$ satisfying (3.3) there is a sequence ε_n decreasing to 0 and a sequence of C^r perturbations g_n which C^r converges to g such that for each ε_n there is an integer $p_n \in \mathbb{Z}_+$ and a point $x \in I$ which is 1-degenerate periodic point of period p_n. More exactly, $h_\varepsilon \circ g_n^{p_n-1}(x) = x, (h_\varepsilon \circ g_n^{p_n-1})'(x) = 1, (h_\varepsilon \circ g_n^{p_n-1})^{(s)}(x) = 0$ for every $s = 2, \dots, k+1$.

Proof of the first part: We present a proof only of the first part and of the second part for $k = 2$ in order not to overload with technical details. Once the second part of the lemma is stated the proof of it is direct calculations. In [KP] we consider $2k$-parameter family. The first k parameters unfold the fixed point 0 and the second k unfold 1. One can show that for a sufficiently large period p within the family the following equations are solvable:

· 1-derivative along a trajectory of length p is 1,
· 2-nd derivative along the trajectory is zero, and so on
· $(k+1)$-st derivative is zero.

Moreover, values of $2k$-parameters solving these $(k+1)$ equations tend to zero as k tends to infinity.

The proof for the first part is a trivial consequence of the following pair of remarks. Call a point $x \in I$ *(g, h_ε)-periodic of period p* if $h_\varepsilon \circ g_n^{p-1}(x) = x$

Suppose for some $\varepsilon > 0$ there is an (g, h_ε)-periodic point x of period p, then p depends on ε and as ε tends to zero p tends to infinity. This follows from the fact that x has to belong to the interval $[0, \varepsilon]$ and as ε tends to zero the number of iterates by g to get from $[0, \varepsilon]$ to $[1-\varepsilon, 1]$ tends to infinity.

Therefore, for each large enough p there is a segment $[\varepsilon_p^-, \varepsilon_p^+]$ in the ε-parameter space for which (g, h_ε)-periodic point of period p exists. In order to disapear any periodic point has to pass through a bifurcation point. At that point (the end point of $[\varepsilon_p^-, \varepsilon_p^+]$) the periodic point has to be 1-degenerate. This completes the proof of the first part of the lemma.

To prove the second part of the lemma for $k = 2$ we start with a pair of a C^r map g and a family $\{h_\varepsilon\}_{\varepsilon>0}$ satisfying (3.3) for $k = 2$. Suppose $g''(0) = b_1$ and $g''(1) = b_2$ are nonzero, otherwise, make an arbitrary small perturbation to have it. Let $a = (a_1, a_2) \in \mathbb{R}^2$. Consider the 2-parameter family $g_a(x) = g(x)+a_1x(x-1)^2-a_2x^2(x-1)$. Conditions (3.3) are fulfilled for all small $a_1 > 0,\ a_2 > 0$. State several remarks:
1. Since $h'_\varepsilon(x) \equiv 1$ where it is defined, $(h_\varepsilon \circ g_n^{p-1})'(x) = 1$ if and only if $(g^{p-1})'(x) = 1$.
2. x is (g, h_ε)-periodic point of period p if $x \in [0, \varepsilon]$ and $g_n^{p-1}(x) \in [1 - \varepsilon, 1]$.
3. For any $a_1,\ a_2 > 0,\ a = (a_1, a_2)$ application of the first part of this lemma show that for any sufficiently large $p \in \mathbb{Z}_+$ there exists $\varepsilon = \varepsilon(a, p) > 0$ and a 1-degenerate (g_a, h_ε)-periodic point x of period p.
4. Fix $\varepsilon_0 > 0$. Define $p_-(x) = \#\{s \geq 0 :\ g^s(x) \in [0, \varepsilon]\}$ and $p_+(x) = \#\{0 \leq s \leq p-1 :\ g^s(x) \in [1-\varepsilon, 1]\}$. As ε tends to zero and x is a 1-degenerate (g_a, h_ε)-periodic, then $p_\pm(x)$ tends to infinity and $p_+(x)/p_-(x)$ tends to a_2/a_1.
5. Notice that

$$(h_\varepsilon \circ g_a^{p-1})''(x) = (h_\varepsilon \circ g_a^{p_n-1})'(x) \left(\sum_{j=0}^{p-1} \frac{g_a''(g_a^j(x))}{g_a'(g_a^j(x))} \right). \tag{3.4}$$

Notice also that $g_a''(x)\ b_1 + a_1 + a_2$ for x near zero and $g_a''(x)\ b_2 + a_1 + a_2$ for x near one. Therefore, with notations of part 3) above as ε tends to zero and a_1, a_2 are small positive we have

$$\frac{(h_\varepsilon \circ g_a^{p-1})''(x)}{p} \to \frac{p_-(x)}{p} \frac{b_2 + a_1 + a_2}{1 + a_1} + \frac{p_+(x)}{p} \frac{b_1 + a_1 + a_2}{1 + a_2} + o(1). \tag{3.5}$$

By part 4) $p_+(x)/p_-(x) \to a_2/a_1$. Thus, monotonically changing the ratio a_2/a_1 and keeping them small we can solve the equation $(h_\varepsilon \circ g_a^{p-1})''(x) = 0$ for a 1-degenerate (g_a, h_ε)-periodic point x of period p. This proves that x constructed in this way is actually a 2-degenerate (g_a, h_ε)-periodic point x of period p. Parts 4) and 5) require some calculations (see [KP] for details). The proof of the lemma for $k = 2$ is complete. Q.E.D.

Chapter 4

Newton Interpolation Polynomials and their application for perturbation of trajectories

The proof of Main Prevalent Theorem from Section 1.5.4 is extremely long and overloaded with technical details. The main purpose of this chapter is to present application of Newton Interpolation Polynomials to study dynamics of recurrent trajectories. This is one of crucial ideas for the proof of Main Prevalent Theorem from [KH, K4]. This chapter is based on announcement of Main Prevalent Theorem [HK] part II.

4.1 Perturbation of recurrent trajectories by Newton Interpolation Polynomials

In order to keep the notations and formulas simple as we formalize this approach, we consider the case of 1-dimensional maps, but the reader should always have in mind that our approach is designed for multidimensional diffeomorphisms. Let $f : I \to I$ be a C^1 map on the interval $I = [-1, 1]$. Recall that a trajectory $\{x_k\}_{k\in\mathbb{Z}}$ of f is called *recurrent* if it returns arbitrarily close to its initial position — that is, for all $\gamma > 0$ we have $|x_0 - x_n| < \gamma$ for some $n > 0$. A very basic question is how much one should perturb f to make x_0 periodic. Here is an elementary Closing Lemma that gives a simple partial answer to this question.

Closing Lemma. *Let $\{x_k = f^k(x_0)\}_{k=0}^n$ be a trajectory of length $n + 1$ of a map $f : I \to I$. Let $u = (x_0 - x_n)/\prod_{k=0}^{n-2}(x_{n-1} - x_k)$. Then x_0 is a periodic point of period* *n of the map*

$$f_u(x) = f(x) + u \prod_{k=0}^{n-2} (x - x_k). \tag{4.1}$$

Of course f_u is close to f if and only if u is sufficiently small, meaning that $|x_0 - x_n|$ should be small compared to $\prod_{k=0}^{n-2} |x_{n-1} - x_k|$. However, this product is likely to contain small factors for recurrent trajectories. In general, it is difficult to control the effect of perturbations for recurrent trajectories. The simple reason why is because *one can not perturb f at two nearby points independently*.

The Closing Lemma above also gives an idea of how much we must change the parameter u to make a point x_0 that is (n, γ)-periodic not be (n, γ)-periodic for a given $\gamma > 0$, which as we described above is one way to make a map that is "bad" for the initial condition x_0 become "good". To make use of our other alternative we must determine how much we need to perturb a map f to make a given x_0 be (n, γ)-hyperbolic for some $\gamma > 0$.

Perturbation of hyperbolicity. *Let $\{x_k = f^k(x_0)\}_{k=0}^{n-1}$ be a trajectory of length n of a C^1 map $f : I \to I$. Then for the map*

$$f_v(x) = f(x) + v(x - x_{n-1}) \prod_{k=0}^{n-2} (x - x_k)^2 \tag{4.2}$$

such that $v \in \mathbb{R}$ and

$$\Big| |(f_v^n)'(x_0)| - 1 \Big| = \left| \left| \prod_{k=0}^{n-1} f'(x_k) + v \prod_{k=0}^{n-2} (x_{n-1} - x_k)^2 \prod_{k=0}^{n-2} f'(x_k) \right| - 1 \right| > \gamma \tag{4.3}$$

we have that x_0 is an (n, γ)-hyperbolic point of f_v.

One more time we can see the product of distances $\prod_{k=0}^{n-2} |x_{n-1} - x_k|$ along the trajectory is important quantitative characteristic of how much freedom we have to perturb.

The perturbations (4.1) and (4.2) are reminiscent of Newton interpolation polynomials. Let us put these formulas into a general setting using singularity theory.

Given $n > 0$ and a C^1 function $f : I \to \mathbb{R}$ we define an associated function $j^{1,n} f : I^n \to I^n \times \mathbb{R}^{2n}$ by

$$j^{1,n} f(x_0, \ldots, x_{n-1}) = (x_0, \ldots, x_{n-1}, f(x_0), \ldots, f(x_{n-1}), f'(x_0), \ldots, f'(x_{n-1})) \tag{4.4}$$

In singularity theory this function is called the *n-tuple* 1*-jet* of f. The ordinary 1-jet of f, usually denoted by $j^1 f(x) = (x, f(x), f'(x))$, maps I to the 1*-jet space*

$\mathcal{J}^1(I,\mathbb{R}) \simeq I \times \mathbb{R}^2$. The product of n copies of $\mathcal{J}^1(I,\mathbb{R})$, called the *multijet space*, is denoted by

$$\mathcal{J}^{1,n}(I,\mathbb{R}) = \underbrace{\mathcal{J}^1(I,\mathbb{R}) \times \cdots \times \mathcal{J}^1(I,\mathbb{R})}_{n \text{ times}}, \tag{4.5}$$

and is equivalent to $I^n \times \mathbb{R}^{2n}$ after rearranging coordinates. The n-tuple 1-jet of f associates with each n-tuple of points in I^n all the information necessary to determine how close the n-tuple is to being a periodic orbit, and if so, how close it is to being nonhyperbolic.

The set

$$\Delta_n(I) = \left\{ \{x_0, \ldots, x_{n-1}\} \times \mathbb{R}^{2n} \subset \mathcal{J}^{1,n}(I,\mathbb{R}) : \exists\, i \neq j \text{ such that } x_i = x_j \right\} \tag{4.6}$$

is called the *diagonal* (or sometimes the *generalized diagonal*) in the space of multijets. In singularity theory the space of multijets is defined outside of the diagonal $\Delta_n(I)$ and is usually denoted by $\mathcal{J}^1_n(I,\mathbb{R}) = \mathcal{J}^{1,n}(I,\mathbb{R}) \setminus \Delta_n(I)$ (see [GG]). It is easy to see that *a recurrent trajectory $\{x_k\}_{k\in\mathbb{Z}_+}$ is located in a neighborhood of the diagonal $\Delta_n(I)$ in the space of multijets for a sufficiently large n.* If $\{x_k\}_{k=0}^{n-1}$ is a part of a recurrent trajectory of length n, then the product of distances along the trajectory

$$\prod_{k=0}^{n-2} |x_{n-1} - x_k| \tag{4.7}$$

measures how close $\{x_k\}_{k=0}^{n-1}$ is to the diagonal $\Delta_n(I)$, or how independently one can perturb points of a trajectory. One can also say that (4.7) is a quantitative characteristic of how recurrent a trajectory of length n is. Introduction of this *product of distances along a trajectory* into the analysis of recurrent trajectories is a new point of our paper.

4.2 Newton interpolation and blow-up along the diagonal in multijet space

Now we present a construction due to Grigoriev and Yakovenko [GY] which puts the "Closing Lemma" and "Perturbation of Hyperbolicity" statements above into a general framework. It is an interpretation of Newton interpolation polynomials as an algebraic blow-up along the diagonal in the multijet space. In order to keep the notations and formulas simple we continue in this section to consider only the 1-dimensional case.

Consider the $2n$-parameter family of perturbations of a C^1 map $f : I \to I$ by polynomials of degree $2n-1$

$$f_\varepsilon(x) = f(x) + \phi_\varepsilon(x), \qquad \phi_\varepsilon(x) = \sum_{k=0}^{2n-1} \varepsilon_k x^k, \tag{4.8}$$

where $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{2n-1}) \in \mathbb{R}^{2n}$. The perturbation vector ε consists of coordinates from the Hilbert brick $HB^1(\mathbf{r})$ of analytic perturbations defined in Section 3 of the previous article. Our goal now is to describe how such perturbations affect the n-tuple 1-jet of f, and since the operator $j^{1,n}$ is linear in f, for the time being we consider only the perturbations ϕ_ε and their n-tuple 1-jets. For each n-tuple $\{x_k\}_{k=0}^{n-1}$ there is a natural transformation $\mathcal{J}^{1,n} : I^n \times \mathbb{R}^{2n} \to \mathcal{J}^{1,n}(I, \mathbb{R})$ from ε-coordinates to jet-coordinates, given by

$$\mathcal{J}^{1,n}(x_0, \dots, x_{n-1}, \varepsilon) = j^{1,n}\phi_\varepsilon(x_0, \dots, x_{n-1}). \tag{4.9}$$

Instead of working directly with the transformation $\mathcal{J}^{1,n}$, we introduce intermediate u-coordinates based on Newton interpolation polynomials. The relation between ε-coordinates and u-coordinates is given implicitly by

$$\phi_\varepsilon(x) = \sum_{k=0}^{2n-1} \varepsilon_k x^k = \sum_{k=0}^{2n-1} u_k \prod_{j=0}^{k-1} (x - x_{j (\mathrm{mod}\ n)}). \tag{4.10}$$

Based on this identity, we will define functions $\mathcal{D}^{1,n} : I^n \times \mathbb{R}^{2n} \to I^n \times \mathbb{R}^{2n}$ and $\pi^{1,n} : I^n \times \mathbb{R}^{2n} \to \mathcal{J}^{1,n}(I, \mathbb{R})$ so that $\mathcal{J}^{1,n} = \pi^{1,n} \circ \mathcal{D}^{1,n}$, or in other words the diagram in Figure 4.2 commutes. We will show later that $\mathcal{D}^{1,n}$ is invertible, while $\pi^{1,n}$ is invertible away from the diagonal $\Delta_n(I)$ and defines a blow-up along it in the space of multijets $\mathcal{J}^{1,n}(I, \mathbb{R})$.

The intermediate space, which we denote by $\mathcal{DD}^{1,n}(I, \mathbb{R})$, is called *the space of divided differences* and consists of n-tuples of points $\{x_k\}_{k=0}^{n-1}$ and $2n$ real coefficients $\{u_k\}_{k=0}^{2n-1}$. Here are explicit coordinate-by-coordinate formulas defining $\pi^{1,n}$:

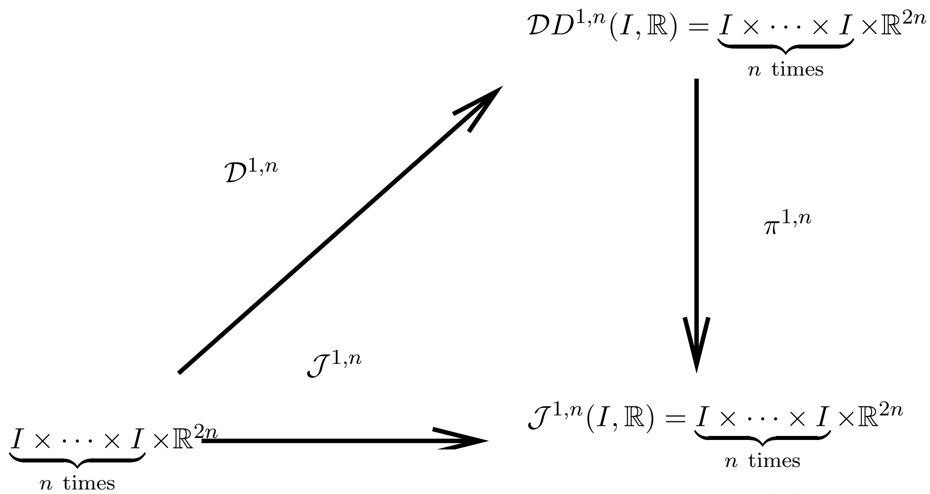

Figure 4.1: An Algebraic Blow-up along the Diagonal $\Delta_n(I)$

$\mathcal{DD}^{1,n}(I,\mathbb{R}) \to \mathcal{J}^{1,n}(I,\mathbb{R})$.

$$\begin{aligned}
\phi_\varepsilon(x_0) &= u_0,\\
\phi_\varepsilon(x_1) &= u_0 + u_1(x_1 - x_0),\\
\phi_\varepsilon(x_2) &= u_0 + u_1(x_2 - x_0) + u_2(x_2 - x_0)(x_2 - x_1),\\
&\vdots\\
\phi_\varepsilon(x_{n-1}) &= u_0 + u_1(x_{n-1} - x_0) + \cdots + u_{n-1}(x_{n-1} - x_0)\ldots(x_{n-1} - x_{n-2}),\\
\phi'_\varepsilon(x_0) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{2n-1} u_k \prod_{j=0}^{k-1}(x - x_{j(\text{mod } n)})\right)\Big|_{x=x_0},\\
&\vdots\\
\phi'_\varepsilon(x_{n-1}) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{2n-1} u_k \prod_{j=0}^{k-1}(x - x_{j(\text{mod } n)})\right)\Big|_{x=x_{n-1}},
\end{aligned} \tag{4.11}$$

These formulas are very useful for dynamics. For a given base map f and initial point x_0, the image $f_\varepsilon(x_0) = f(x_0) + \phi_\varepsilon(x_0)$ of x_0 depends only on u_0. Furthermore the image can be set to any desired point by choosing u_0 appropriately — we say then that it depends nontrivially on u_0. If x_0, x_1, and u_0 are fixed, the image $f_\varepsilon(x_1)$ of x_1 depends only on u_1, and as long as $x_0 \neq x_1$ it depends nontrivially on u_1. More generally for $0 \le k \le n-1$, if pairwise distinct points $\{x_j\}_{j=0}^{k}$ and coefficients $\{u_j\}_{j=0}^{k-1}$ are fixed, then the image $f_\varepsilon(x_k)$ of x_k depends only and nontrivially on u_k.

Suppose now that an n-tuple of points $\{x_j\}_{j=0}^{n}$ not on the diagonal $\Delta_n(I)$ and Newton coefficients $\{u_j\}_{j=0}^{n-1}$ are fixed. Then derivative $f'_\varepsilon(x_0)$ at x_0 depends only and nontrivially on u_n. Likewise for $0 \leq k \leq n-1$, if distinct points $\{x_j\}_{j=0}^{n}$ and Newton coefficients $\{u_j\}_{j=0}^{n+k-1}$ are fixed, then the derivative $f'_\varepsilon(x_k)$ at x_k depends only and nontrivially on u_{n+k}.

As Figure 4.2 illustrates, these considerations show that for any map f and any desired trajectory of distinct points with any given derivatives along it, one can choose Newton coefficients $\{u_k\}_{k=0}^{2n-1}$ and explicitly construct a map $f_\varepsilon = f + \phi_\varepsilon$ with such a trajectory. Thus we have shown that $\pi^{1,n}$ is invertible away from the diagonal $\Delta_n(I)$ and defines a blow-up along it in the space of multijets $\mathcal{J}^{1,n}(I,\mathbb{R})$.

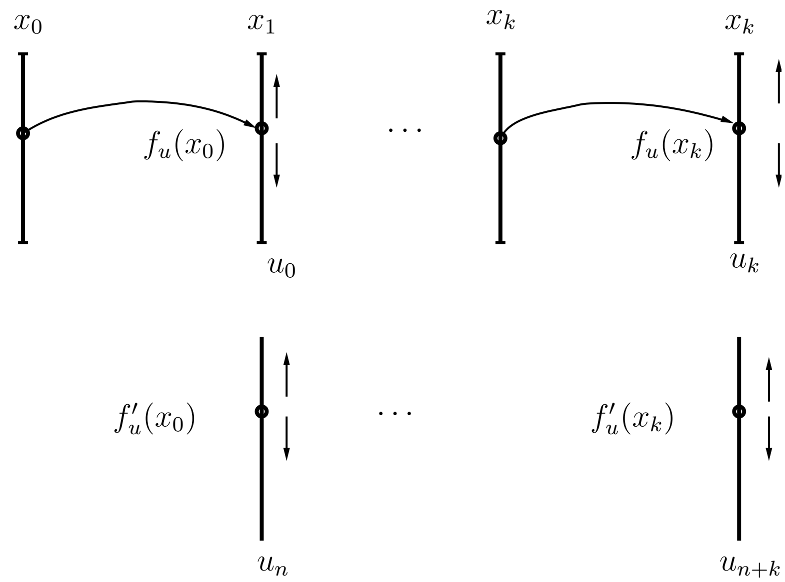

Figure 4.2: Newton coefficients and their action

Next we define the function $\mathcal{D}^{1,n} : I^n \times \mathbb{R}^{2n} \to \mathcal{DD}^{1,n}(I,\mathbb{R})$ explicitly using so-called divided differences. Let $g : \mathbb{R} \to \mathbb{R}$ be a C^r function of one real variable.

Definition 4.2.1. *The* first order divided difference *of g is defined as*

$$\Delta g(x_0, x_1) = \frac{g(x_1) - g(x_0)}{x_1 - x_0} \tag{4.12}$$

for $x_1 \neq x_0$ and extended by its limit value as $g'(x_0)$ for $x_1 = x_0$. Iterating this construction we define divided differences of the m-th order for $2 \leq m \leq r$,

$$\Delta^m g(x_0, \ldots, x_m) = \frac{\Delta^{m-1} g(x_0, \ldots, x_{m-2}, x_m) - \Delta^{m-1} g(x_0, \ldots, x_{m-2}, x_{m-1})}{x_m - x_{m-1}} \tag{4.13}$$

for $x_{m-1} \neq x_m$ *and extended by its limit value for* $x_{m-1} = x_m$.

A function loses at most one derivative of smoothness with each application of Δ, so $\Delta^m g$ is at least C^{r-m} if g is C^r. Notice that Δ^m is linear as a function of g, and one can show that it is a symmetric function of $x_0, \ldots, x_m$; in fact, by induction it follows that

$$\Delta^m g(x_0, \ldots, x_m) = \sum_{i=0}^{m} \frac{g(x_i)}{\prod_{j \neq i}(x_i - x_j)} \tag{4.14}$$

Another identity that is proved by induction will be more important for us, namely

$$\Delta^m \ x^k(x_0, \ldots, x_m) = p_{k,m}(x_0, \ldots, x_m), \tag{4.15}$$

where $p_{k,m}(x_0, \ldots, x_m)$ is 0 for $m > k$ and for $m \leq k$ is the sum of all degree $k - m$ monomials in $x_0, \ldots, x_m$ with unit coefficients,

$$p_{k,m}(x_0, \ldots, x_m) = \sum_{r_0 + \cdots + r_m = k-m} \ \prod_{j=0}^{m} x_j^{r_j}. \tag{4.16}$$

The divided differences form coefficients for the Newton interpolation formula. For all C^∞ functions $g : \mathbb{R} \to \mathbb{R}$ we have

$$\begin{aligned} g(x) = &\Delta^0 g(x_0) + \Delta^1 g(x_0, x_1)(x - x_0) + \ldots \\ &+ \Delta^{n-1} g(x_0, \ldots, x_{n-1})(x - x_0) \ldots (x - x_{n-2}) \\ &+ \Delta^n g(x_0, \ldots, x_{n-1}, x)(x - x_0) \ldots (x - x_{n-1}) \end{aligned} \tag{4.17}$$

identically for all values of $x, x_0, \ldots, x_{n-1}$. All terms of this representation are polynomial in x except for the last one which we view as a remainder term. The sum of the polynomial terms is the degree $(n-1)$ *Newton interpolation polynomial* for g at $\{x_k\}_{k=0}^{n-1}$. To obtain a degree $2n-1$ interpolation polynomial for g and its derivative at $\{x_k\}_{k=0}^{n-1}$, we simply use (4.17) with n replaced by $2n$ and the $2n$-tuple of points $\{x_{k(\text{mod } n)}\}_{k=0}^{2n-1}$.

Recall that $\mathcal{D}^{1,n}$ was defined implicitly by (4.10). We have described how to use divided differences to construct a degree $2n-1$ interpolating polynomial of the form on the right-hand side of (4.10) for an arbitrary C^∞ function g. Our interest then is in the case $g = \phi_\varepsilon$, which as a degree $2n-1$ polynomial itself will have no remainder term and coincide exactly with the interpolating polynomial. Thus $\mathcal{D}^{1,n}$ is given coordinate-by-coordinate by

$$
\begin{aligned}
u_m &= \Delta^m \left(\sum_{k=0}^{2n-1} \varepsilon_k x^k \right) (x_0, \dots, x_{m \ (mod \ n)}) \\
&= \varepsilon_m + \sum_{k=m+1}^{2n-1} \varepsilon_k p_{k,m}(x_0, \dots, x_{m \ (mod \ n)})
\end{aligned}
\tag{4.18}
$$

for $m = 0, \dots, 2n-1$. We call the transformation given by (4.18) the *Newton map*. Notice that for fixed $\{x_k\}_{k=0}^{2n-1}$, the Newton map is linear and given by an upper triangular matrix with units on the diagonal. Hence it is Lebesgue volume-preserving and invertible, whether or not $\{x_k\}_{k=0}^{2n-1}$ lies on the diagonal $\Delta_n(I)$.

We call the basis of monomials

$$
\prod_{j=0}^{k} (x - x_{j(\text{mod } n)}) \quad \text{for} \quad k = 0, \dots, 2n-1 \tag{4.19}
$$

in the space of polynomials of degree $2n-1$ the *Newton basis* defined by the n-tuple $\{x_k\}_{k=0}^{n-1}$. The Newton map and the Newton basis, and their analogue in dimension N, are useful tools for perturbing trajectories and estimating the measure of "bad" parameter values $\vec{\varepsilon} \in HB^N(\mathbf{r})$.

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