GEOMETRIC PROOFS OF MATHER'S CONNECTING AND ACCELERATING THEOREMS

V. KALOSHIN

Abstract. In this paper we present simplified proofs of two important theorems of J.Mather. The first (connecting) theorem [Ma2] is about wandering trajectories of exact area-preserving twist maps naturally arising for Hamiltonian systems with 2 degrees of freedom. The second (accelerating) theorem is about dynamics of generic time-periodic Hamiltonian systems on two-torus (2.5-degrees of freedom). Mather [Ma6] proves that for a generic time-dependent mechanical Hamiltonian there are trajectories whose speed goes to infinity as time goes to infinity, in contrast to time-independent case, where there is a conservation of energy.

The results of this paper are not new and the main purpose is to present simplified geometric proofs of two important theorems of J. Mather [Ma2, Ma6]. Both theorems are particular examples of instabilities in Hamiltonian systems or what is sometimes called *Arnold's diffusion.* Recently Mather [Ma7] annonced a proof of existence of Arnold's diffusion for a generic nearly integrable Hamiltonian systems with 2.5 and 3-degrees of freedom using his variational approach developed in [Ma2]–[Ma6].

The first (connecting) Mather's theorem says that inside of a Birkhoff region of instability there are trajectories connecting any two Aubry-Mather sets, i.e. given any two Aubry-Mather sets Σ_ω and $\Sigma_{\omega'}$ inside of a Birkhoff region of instability there is a trajectory α-asymptotic to Σ_ω and ω-asymptotic to $\Sigma_{\omega'}$. Recently J. Xia [X] gave a simplified proof of the first result using the same variational approach as Mather. The second (accelerating) theorem says that a "generic" Hamiltonian time periodic system on the 2-torus $\mathbb{T}^2$ has trajectories whose speed goes to infinity as time goes to infinity. Different from [Ma6] proofs of this result are given by Bolotin-Treschev [BT] and Delshams-de la Llave-Seara [DLS1]. Our proof of the second theorem combines a geometric approach and Mather's variational approach. The second theorem is proved using ideas from the proof of the first theorem. Let's give the rigorous statement of both results.

First show that exact area-preserving twist maps naturally arise for Hamiltonian systems with two degrees of freedom. Indeed, let $H : \mathbb{R}^4 \to \mathbb{R}$ be a C^2-smooth function and consider the corresponding Hamiltonian system

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$$(1) \qquad \begin{cases} \dot{x} = H_y(x,y) \\ \dot{y} = -H_x(x,y) \end{cases} \qquad x, y \in \mathbb{R}^2.$$

Denote by $\varphi^t_H(x_0, y_0) = (x_t, y_t)$ the time t map along the trajectories of (1). Each trajectory starting at some point $(x_0, y_0) \in \mathbb{R}^4$ belongs to its energy surface $E^3 = E^3_{(x_0,y_0)} \subset \{H(x,y) = H(x_0,y_0) = \text{const}\} \subset \mathbb{R}^4$. Take a *Poincare section* $S^2 \subset E^3$ transverse to the vector field (1) and a trajectory $\{\varphi^t_H(p)\}_{t \geq 0}$ of a point $p \in S^2$. Let $\varphi^{\tau(p)}_H(p) \in S^2$ be the first return to S^2. For a nearby point $q \in S^2$ also there is a point $\varphi^{\tau(q)}_H(q) \in S^2$ of the first intersection with S^2 for $\tau(q)$ close to $\tau(p)$. This defines the *Poincare return map* which sends a point $q \in U \subset S^2$ into the point $\mathcal{P}(q) = \varphi^{\tau(q)}_H(q)$ of the first return of its forward trajectory to S^2 with U being the set where such a return exists. Given by the Hamiltonian flow φ^t_H the map $\mathcal{P} : U \to S^2$ preserves an area form on S^2. Such an area form is the restriction of the standard symplectic Darboux form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ on $\mathbb{R}^4$ onto S and is non-degenerate, because (1) is transverse to S^2. The Poincare map $\mathcal{P}$ preserves this area form. Moreover, the regularity properties of $\mathcal{P}$ are the same as those of H, i.e. if H is C^r (with $R \geq 1$), then $\mathcal{P}$ is also C^r in the region where it is defined. To formulate the other two important properties of $\mathcal{P}$ we bring the domain of definition of $\mathcal{P}$ to the standard form.

Let $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R} = \{(\theta, r) \in S^1 \times \mathbb{R}\}$ be the cylinder with the standard Lebesgue area form $d\theta \wedge dr$ and let $C \subset \mathbb{A}$ be an open which intersects every vertical line $\{\theta\} \times \mathbb{R}$ in an open interval. A non-contructable Jordan curve γ on $\mathbb{A}$ homeomorphic to a circle is called *rotational.* Consider a C^1-smooth orientation and area preserving map $\mathcal{P} : C \to \mathbb{A}$. $\mathcal{P}$ is called

- *exact* (or with no up/down drift) if $\mathcal{P}$ has zero flux, i.e. for any rotational curve $\gamma \subset C$ area of the regions above γ and below $\mathcal{P}(\gamma)$ equals area below γ and above $\mathcal{P}(\gamma)$[1];
- *monotone twist* (or simply twist) if for any vertical curve $l = \{\theta\} \times \mathbb{R}$ in $\mathbb{A}$ its image $\mathcal{P}(l)$ intersects every vertical line $\{\tilde{\theta}\} \times \mathbb{R}$ with a nonzero angle.

Assume also that $\mathcal{P}$ is homotopic to the inclusion map. We shall call a map with the above properties an *EAPT* (exact orientation and area preserving twist).

Important examples of EAPTs of a cylinder are *billiards* in convex bounded regions, *the plane restricted three-body* problem, the *standard map* of the 2-torus or Frenkel-Kontorova model, and etc (see e.g. [MF] and [Mo] II.4).

0.1. **Mather's connecting theorem.** A compact region $C \subset \mathbb{A}$ is called *a Birkhoff region of instability (BRI)* if C is a compact $\mathcal{P}$-invariant set whose frontier consists of two components denoted by C_- and C_+ both rotational curves and no

[1] Area preservation implies that this property is independent of the choice of γ and is necessary to have invariant curves at all

other rotational invariant curves in between. For convenience we call the upper frontier C_+— the top and the lower frontier C_- —the bottom. Let C be a BRI. Since both frontiers are invariant under $\mathcal{P}$ it induces two homeomorphisms of the circles. Therefore, there are two well-defined rotation numbers ω_- and ω_+ for $\mathcal{P}|_{C_-}$ and $\mathcal{P}|_{C_+}$ respectively. It follows from the twist condition that $\omega_- < \omega_+$.

Theorem 1. (Aubry [AL]-Mather [Ma1]) *For any rotation number $\omega \in [\omega_-, \omega_+]$ there is an invariant set Σ_ω, called Aubry-Mather set, such that every orbit in Σ_ω has a rotation number of winding around the cylinder equal ω.*[2]

If a rotation number is rational $\omega = p/q \in \mathbb{Q}$, then generically Σ_ω is a finite union of period orbits of period q. In a highly degenerate case Σ_ω might be a rotational curve. If a rotation number is irrational, then Σ_ω is either a Denjoy-Cantor set or a rotational curve.

Mather's Connecting Theorem. *For any two rotation numbers $\omega, \omega' \in [\omega_-, \omega_+]$ inside of rotation interval of Birkhoff region of instability there is a point $p \in C$ such that its $\omega(\alpha)$-limit set of p is contained in Σ_ω (in $\Sigma_{\omega'}$) respectively.*

In this paper first we shall prove a weaker version of this theorem (see just below) and then for generic EAPTs extend it to a strong version.

Mather's Weak Connecting Theorem. *In the setting of the above theorem for any positive ε there is a point $p \in C$ such that for some positive n_+ and negative n_- we have $\mathcal{P}^{n_\pm}(p)$ belongs to the ε-neighborhood of Σ_ω and $\Sigma_{\omega'}$ respectively.*

The original proof of Mather [Ma2] using variational method is quite complicated and involved. Recently it was significantly simplified by J. Xia [X]. It might give some insight in how to estimate on diffusion time. Topological arguments presented here are sufficiently simple and based on Birkhoff's invariant set theorem. These arguments are qualitative and seem to give no insight *on diffusion time.* In the case $\Sigma_\omega = C_-$ and $\Sigma_{\omega'} = C_+$ Mather's Weak Connecting Theorem is the Theorem of Birkhoff [B1] and Mather's Connecting Theorem was also proved by Le Calvez [L1] using clever topological arguments.

0.2. **Mather's accelerating theorem.** Let $\mathbb{T}^2$ be the two-torus with a C^2-smooth Riemannian metric ρ, defined on the tangent bundle $\mathbf{T}\mathbb{T}^2$ of $\mathbb{T}^2$ and by duality can be also defined on the cotangent bundle $\mathbf{T}^*\mathbb{T}^2$. Denote by $S\mathbb{T}^2$ ($S^*\mathbb{T}^2$) denotes the unit (co)tangent bundle of $\mathbb{T}^2$ and by $T_q(p)$ the associated kinetic energy to the metric ρ, i.e.

$$T_q(p) = \rho_q(p,p)/2, \quad p \in \mathbf{T}^*_q\mathbb{T}^2, \tag{2}$$

where $\mathbf{T}^*_q\mathbb{T}^2$ denotes the cotangent bundle of $\mathbb{T}^2$ at q. Let $U : \mathbb{T}^2 \times \mathbb{T} \to \mathbb{R}$ be a C^2-smooth time periodic function on $\mathbb{T}^2$ which is the potential energy. We

[2] We shall give a more precise statement of this theorem in Section 0.3

normalize period to be one. This defines the mechanical Hamiltonian system with

$$H(q,p,t) = T_q(p) + U(q,t). \tag{3}$$

Mather's acceleration theorem I. *For a generic C^2-smooth metric ρ on $\mathbb{T}^2$ and a generic C^2-smooth function $U : \mathbb{T}^2 \times \mathbb{T} \to \mathbb{R}$, there exists a trajectory $q(t)$ of the Hamiltonian flow, defined by (3), with an unbounded speed $\dot q(t)$. Moreover, energy $E(t) = \rho_{q(t)}(p(t), p(t)) + U(q(t), t)$ tends to infinity as time goes to infinity.*

Remark 1. As the reader will see generic metrics on $\mathbb{T}^2$ and generic functions on $\mathbb{T}^2 \times \mathbb{T}$ form open dense sets in the corresponding spaces of C^2-metrics and C^2-functions.

Non-degeneracy Hypothesis on ρ and U.

Hypothesis 1. There is an indivisible homology class $h \in H_1(\mathbb{T}^2, \mathbb{Z})$ which has only one shortest hyperbolic periodic geodesic Γ;

Choose generators for the homology group $H_1(\mathbb{T}^2, \mathbb{Z})$. Let h_0 be the homology class corresponding to Γ. Lift $\mathbb{T}^2$ to the cylinder $\mathbb{A}$ with h_0 being the only nontrivial homology class of $\mathbb{A}$. Then Γ lifts to a countable collection of copies. Denote by Γ_0 and Γ_1 adjacent copies. Recall that a geodesic γ is called a *Morse Class A geodesic* if its lift $\hat\gamma$ to the universal cover $\mathbb{R}^2$ is globally length minimizing and γ is homoclinic to Γ, i.e. its α-limit set is Γ_0 and ω-limit set is Γ_1 (see [Ba1] (6.8) for more detailed discussion of Morse Class A geodesics).

Hypothesis 2. There is only one positive Morse Class A geodesic Λ.

Since Γ is hyperbolic it has stable and unstable manifolds $W^s(\Gamma)$ and $W^u(\Gamma)$ in $S\mathbb{T}^2$ respectively. Another way to view the geodesic Λ is as the intersection of invariant manifolds $W^s(\Gamma) \cap W^u(\Gamma)$.

Hypothesis 2′. Intersection of invariant manifolds $W^s(\Gamma) \cap W^u(\Gamma)$ is transversal.

This hypothesis is not necessary, but simplifies presentation. Let $s \mapsto \Gamma(s)$ and $s \mapsto \Lambda(s)$ be arclength parameterizations of both Γ and Λ with the same orientation. Then there are constants $a, b \in \mathbb{R}$ such that

$$\begin{aligned} &\rho(\Gamma(s), \Lambda(s+a)) \to 0 \quad \text{as} \quad s \to -\infty \\ &\rho(\Gamma(s), \Lambda(s+b)) \to 0 \quad \text{as} \quad s \to +\infty. \end{aligned} \tag{4}$$

Both quantities converge to zero exponentially fast by hyperbolicity of Γ which implies that one can define the *Melnikov Integral*

$$G(t) = \lim_{s\to\infty} \left[\int_{-s+a}^{s+b} U(\Gamma(\tau), t)\, d\tau - \int_{-s}^{s} U(\Lambda(\tau), t)\, d\tau \right]. \tag{5}$$

Hypothesis 3. $G(t)$ is not constant.

Fact. *In the space of C^2-smooth Riemannian metrics on $\mathbb{T}^2$ and C^2-smooth time periodic functions on $\mathbb{T}^2$ there is an open dense set of those who satisfy Hypothesis 1-3.*

Mather's acceleration theorem II. *If Hypotheses 1-3 are satisfied, there exists a trajectory of the Hamiltonian system (3) whose energy goes to infinity as time goes to infinity.*

Remark 2. The fact stated above implies that Mather's acceleration theorem version II implies version I.

This last theorem was originally proved by Mather [Ma6] using variational method. Later Bolotin-Treschev [BT] and Delshams-de la Llave-Seara [DLS1] gave in a sense similar proofs analogous theorems using the standard geometric approach. In [DLS2] the second group of authors extended this result to some manifolds different from the two-torus with apriori unstable geodesic flow. The proof which we present here is a mixture of geometric and variational approaches. Accelerating trajectories constructed in the present proof slightly differ from both given in Mather's work [Ma6] by the variational method and in [BT] an [DLS1] by geometric methods, even though in the spirit our proof uses ideas from both approaches.

0.3. **Birkhoff's invariant set theorem and Aubry-Mather sets as action-minimizing sets.** We need Birkhoff's invariant set theorem.

Theorem 2. [B2], [MF] *Let $C \subset \mathbb{A}$ be a BRI of an EAPT $\mathcal{P} : C \to C$. Suppose that $V \subset C$ is a closed connected set separating the cylinder and invariant under $\mathcal{P}$. Then V is equal C^-, or C^+, or contains both frontiers.*

For the purpose of completeness we formally describe Aubry-Mather as action-minimizing sets. Denote by $\tilde{C}$ the natural lift of $C \subset \mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$ to $\mathbb{R} \times \mathbb{R}$ and by $\tilde{\mathcal{P}} : \tilde{C} \to \tilde{C}$ the lift of the map $\mathcal{P}$. For a point $p \in C$ let $\tilde{p} \in \tilde{C}$ be its lift and let $(x_n(\tilde{p}), r_n(\tilde{p})) = \tilde{\mathcal{P}}^n \tilde{p}$. Then if the following limits

$$\rho_\alpha(p) = \lim_{n\to+\infty} x_n(\tilde{p})/n \quad \text{and} \quad \rho_\omega(p) = \lim_{n\to-\infty} x_n(\tilde{p})/n \tag{6}$$

exist and equal $\rho(p) = \rho_\alpha(p) = \rho_\omega(p)$, then is is called a *rotation number*. Geometric meaning of rotation number is average amount of rotation of the trajectory of p around the cylinder $\mathbb{A}$.

A fundamental property of an EAPT $\mathcal{P} : C \to C$ is that it can be globally described by a "generating" function $h : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$ is the set of points $(x, x') \in \mathbb{R}^2$ such that there is $r, r' \in \mathbb{R}$ with $(x, r) \in \tilde{C}$ and $\tilde{P}(x, r) = (x', r')$. Clearly, D is open. By twist condition for each $(x, x') \in D$ the pair r, r' as above is unique. Moreover, (r, r') depends continuously on (x, x'). See [MF] §5 or [Ba1] sec. 7 for more. If $\mathcal{P}$ is C^r-smooth, then h is C^{r+1} on D. A generating function $h : D \to \mathbb{R}$ can also be defined by

$$(7) \qquad \begin{cases} r = -\partial_1 h(x,x') \\ r' = \partial_2 h(x,x'). \end{cases}$$

By agreement extend h from D to the whole plane $\mathbb{R}^2$ by $h(x,x') = +\infty$. Sometimes such a generating function h is called *variational principle* (see e.g. [MF] §5).

Denote $\mathbb{R}^{\mathbb{Z}} = \{x : \ x : \mathbb{Z} \to R\}$ the space of sequences. Given an arbitrary sequence $(x_j,\dots,x_k), j<k$ of $x \in \mathbb{R}^{\mathbb{Z}}$ denote $h(x_j,\dots,x_k) = \sum_{i=j}^{k-1} h(x_i,x_{i+1})$. A segment $(x_j,\dots,x_k)$ is called *minimal* with respect to h if $h(x_j,\dots,x_k) \le h(x_j^*,\dots,x_k^*)$ for all $(x_j^*,\dots,x_k^*)$ with $x_j = x_j^*$ and $x_k = x_k^*$. A sequence $x \in \mathbb{R}^{\mathbb{Z}}$ is *h-minimal* if for every finite segment of x is h-minimal. Denote by $\mathcal{M}(h)$ the set of minimal sequences. If h is a generating function of an EAPT $\mathcal{P} : C \to C$, then using relation (7) each minimal sequence $x \in \mathbb{R}^{\mathbb{Z}}$ corresponds to a trajectory of $\mathcal{P}$. So the set $\mathcal{M}(h)$ corresponds to the set of points $\mathcal{M}(\mathcal{P}) \subset C$ (see [MF] §3 or [Ba1] sec. 7 for more). The set $\mathcal{M}(\mathcal{P})$ is called a set of *action-minimizing or h-minimal points* or, equivalently, $\mathcal{M}(\mathcal{P})$ the set of points whose trajectories are minimal.

Theorem 3. (Aubry [AL]-Mather [Ma1]) *With notations Theorem 1 for any rotation number $\omega \in [\omega_-,\omega_+]$ there is a nonempty invariant set $\Sigma_\omega \subset \mathcal{M}(\mathcal{P}) \subset C$ of points whose trajectories are h-minimal and have rotation number $\rho(p) = \omega$.*

Σ_ω is called an *action-minimizing* or *an Aubry-Mather set.*

0.4. **Structure of Aubry-Mather sets.** In this section we describe possible structures of Aubry-Mather sets. Recall that a point $p \in C$ is called *recurrent* with respect to $\mathcal{P}$ if its trajectory has p in the closure, i.e. $\overline{\{\mathcal{P}^n p\}_{n\in\mathbb{Z}\setminus 0}} \ni p$. Denote by $\Sigma_\omega^{rec} \subset \Sigma_\omega$ the set of minimal recurrent points with rotation number ω. Denote also by $\pi_1 : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1$ the natural projection onto the first component. The theorem about structure of Aubry-Mather sets, given below, follows e.g. from thms (4.3) and (5.3) [Ba1] and the graph theorem (thm 14.1) [MF].

Structure Theorem. (Irrational case $\omega \notin \mathbb{Q}$) *The Aubry-Mather set Σ_ω is*
- *either an invariant curve or*
- *Σ_ω^{rec} and its projection $\pi_1(\Sigma_\omega^{rec})$ are Cantor sets. Then Σ_ω^{rec} consists of three different types of trajectories.*

There is the set of $p \in \Sigma_\omega^{rec}$ whose projection $\pi_1(p)$ can be approximated from above and below by $\pi_1(\Sigma_\omega^{rec})$. This set had power continuum and corresponds to those points which are not endpoints of $\mathbb{S}^1 \setminus \pi_1(\Sigma_\omega^{rec})$.
There is the set of $p \in \Sigma_\omega^{rec}$ whose projection $\pi_1(p)$ can be approximated either

only from above or only from below by $\pi_1(\Sigma_\omega^{rec})$. *These sets are countable and correspond to the right resp. left endpoints of components of* $\mathbb{S}^1 \setminus \pi_1(\Sigma_\omega^{rec})$.[3]

To formulate the Structure Theorem in the case of rational rotation number introduce some sets. Let Σ_ω^{per} the set of periodic points. Two periodic points p^- and p^+ are neighboring elements of Σ_ω^{per} if projections $\pi_1(p^-)$ and $\pi_1(p^+)$ have a segment in $\mathbb{S}^1$ free from projection of all the other elements of $\pi_1(\Sigma_\omega^{per})$. For neighboring periodic points p^- and p^+ in Σ_ω^{per} let

$$\begin{aligned}\Sigma_\omega^+(p^-,p^+) = \{p \in \Sigma_\omega :\ & p \text{ is } \alpha - \text{asymptotic to } p^- \\ & \text{and } \omega - \text{asymptotic to } p^+\}, \\ \Sigma_\omega^-(p^-,p^+) = \{p \in \Sigma_\omega :\ & p \text{ is } \alpha - \text{asymptotic to } p^+ \\ & \text{and } \omega - \text{asymptotic to } p^-\}\end{aligned} \tag{8}$$

Let $\Sigma_\omega^\pm$ be the union of $\Sigma_\omega^\pm(p^-,p^+)$ over all neighboring periodic points p^- and p^+ in Σ_ω^{per}.

Structure theorem. (Rational case $\omega \in \mathbb{Q}$) *The Aubry-Mather set* Σ_ω *is a disjoint union of* Σ_ω^{per}, Σ_ω^+, *and* Σ_ω^-. *Moreover,* Σ_ω^{per} *is always non-empty and if* Σ_ω^{per} *is not a curve, then* Σ_ω^- *and* Σ_ω^+ *are non-empty too.*

1. A Proof of Mather's weak connecting theorem

The idea of the proof of Mather's weak connecting theorem is to choose two recurrent points p and p' in the starting Aubry-Mather sets Σ_ω and $\Sigma_{\omega'}$ respectively. Take an open ε-ball $V_\varepsilon(p)$ (resp. $V'_\varepsilon(p)$) about p (resp. p') and consider the union over forward (resp. backward) images of $V_\varepsilon(p)$ (resp. $V'_\varepsilon(p)$)[4]. It turns out that the following properties hold true.

Lemma 1. *With the notations above let* ω^- *and* ω^+ *be rotation numbers of the "top"* C_+ *and "bottom"* C_- *frontiers of a BRI* C. *Let* $\omega \in [\omega^-,\omega^+]$ *be a rotation number and* $\Sigma_\omega \subset C$ *be the corresponding Aubry-Mather set. Then for any recurrent point* $p \in \Sigma_\omega^{rec}$ *and any* $\varepsilon > 0$ *for the open* ε*-neighborhood* $V_\varepsilon(p)$ *of* p *the following holds true*

- *for some positive number* $n_+ = n_+(p,\varepsilon)$ *(resp.* $n_- = n_-(p,\varepsilon)$*) the union of images* $\cup_{j=0}^{n_+}\mathcal{P}^j\ V_\varepsilon(p)$ *(resp.* $\cup_{j=0}^{n_+}\mathcal{P}^{-j}\ V_\varepsilon(p)$*) separates the cylinder* $\mathbb{A}$.
- *the union over all forward (resp. backward) images* $V_\varepsilon^+(p) = \cup_{j\in\mathbb{Z}_+}\mathcal{P}^j\ V_\varepsilon(p)$ *(resp.* $V_\varepsilon^-(p) = \cup_{j\in\mathbb{Z}_+}\mathcal{P}^{-j}\ V_\varepsilon(p)$*) is connected and open.*
- *closure of* $V_\varepsilon^+(p)$ *(resp.* $V_\varepsilon^-(p)$*) contains both frontiers* $C^\pm$ *of* C.

[3] Two points $p \in \Sigma_\omega^{rec}$ and $p' \in \Sigma_\omega^{rec}$ are asymptotic, i.e. $\text{dist}(\mathcal{P}^n p, \mathcal{P}^n p') \to 0$ as $|n| \to \infty$, if and only if $\pi_1(p)$ and $\pi_1(p')$ are endpoints of some component of $\mathbb{S}^1 \setminus \pi_1(\Sigma_\omega^{rec})$. In this case we have $\sum_{n\in\mathbb{Z}} |\pi_1(\mathcal{P}^n p) - \pi_1(\mathcal{P}^n p')| \le 1$. Convergence in Σ_ω^{rec} is never uniform.

[4] In [Ha] topological arguments of very different nature are used to prove existence of Aubry-Mather sets

• *let* $V_\varepsilon^\infty(p)=\cup_{j\in\mathbb{Z}}\mathcal{P}^j\ V_\varepsilon(p)$*. Then* $V_\varepsilon^\infty(p)$ *is invariant and both* $V_\varepsilon^+(p)$ *and* $V_\varepsilon^-(p)$ *are open dense in* $V_\varepsilon^\infty(p)$*.*

Let $V_\varepsilon^+(p)$ and $V_\varepsilon^-(p')$ be the union of one-sided iterates of ε-neighborhoods of $p\in\Sigma_\omega$ and $p'\in\Sigma_{\omega'}$. This lemma implies that $V_\varepsilon^+(p)$ and $V_\varepsilon^-(p')$ have to have nonempty intersection as connected open sets both separating the cylinder and having frontiers C_+ and C_- of a BRI C in its closure. Intersection of two open sets is open so there is an open subset $V^\varepsilon_{\omega,\omega'}$ inside of the ε-neighborhood $V_\varepsilon(p)$ of p such that after a number of forward iterations, say n, the set $V^\varepsilon_{\omega,\omega'}$ is mapped into $V_\varepsilon(p')$, i.e. $V^\varepsilon_{\omega,\omega'}\subset V_\varepsilon(p)$ and $\mathcal{P}^n\ V^\varepsilon_{\omega,\omega'}\subset V_\varepsilon(p')$ for some $n\in\mathbb{Z}_+$. This proves the required statement in Mather's Weak Connecting Theorem. Now we prove the lemma.

1.1. **Union of iterates of a neighborhood of a recurrent point** p **separates the cylinder** $\mathbb{A}$**.** In this subsection we prove the first and the second parts of the lemma. Indeed, the set $V=\cup_{j\in\mathbb{Z}}\mathcal{P}^jV_\varepsilon(p)$ is invariant and open by definition. Show that V is connected.

Pick an arbitrary positive ε and consider an open ε-ball $V_\varepsilon(p)$ of p. Since p is recurrent, for an arbitrary ε there is $n\in\mathbb{Z}$ such that p and $\mathcal{P}^np$ are $\varepsilon/2$-close. It follows from simple properties of Aubry graphs (see Lemma 4.5 [Ba1] or Theorem 11.3 [MF]) that any recurrent point p of an Aubry-Mather set can be approximated by a periodic point r from another Aubry-Mather set $\Sigma_{\tilde\omega}$, $\tilde\omega=s/q\in\mathbb{Q}$. Therefore, if approximation is close enough then r is $\varepsilon/4$-close to p and $\mathcal{P}^nr$ is $\varepsilon/4$-close to $\mathcal{P}^np$. Thus, both $\mathcal{P}^nr$ and r are in $V_\varepsilon(p)$. We can also assume that q is prime. The projection onto the base circle of points of an action-minimizing orbit are cyclically ordered (see (4.1) and (5.1) [Ba1] or Theorem 12.3 [MF]). Recall also that ω is with prime numerator. All these remarks imply that the set $\cup_{j=0}^q\mathcal{P}^jV_\varepsilon(p)$ separates the cylinder $\mathbb{A}$. This union of neighborhoods reminds a "bicycle chain". This proves the first claim of the lemma.

Connectivity of $\cup_{j\in\mathbb{Z}_+}\mathcal{P}^j\ V_\varepsilon(p)$ can be shown as follows. Notice that $V_\varepsilon(p)$ has a periodic point r of period q inside. Therefore, the set $\cup_{s\in\mathbb{Z}_+}\mathcal{P}^{sq}\ V_\varepsilon(p)$ is connected. The sets $\cup_{s\in\mathbb{Z}_+}\mathcal{P}^{sq+j}\ V_\varepsilon(p)$ for different j's with $0\le j\le q-1$ are connected among each other because members of the orbit of r are connected by the "bicycle chain". This completes the proof of the second point.

The third point follows from Birkhoff's invariant set theorem. The last claim of s a direct corollary of area-preservation. This proves the lemma. Q.E.D.

2. Extension of Mather's Weak Connecting Theorem to Mather's Connecting Theorem for generic EAPTs

First we formulate genericity hypothesis for EAPTs:

Genericity Hypothesis: We say that an EAPT has KS property if linearization of any periodic point p has no eigenvalue 1 and transversal intersections of its stable and unstable manifolds.

We shall prove

Mather's Connecting Theorem for EAPTs with KS property. *With the notations of Mather's Connecting Theorem and an EAPT map $\mathcal{P} : \mathbb{A} \to \mathbb{A}$ satisfying KS property with a BRI $C \subset \mathbb{A}$ there is an open dense set $\mathcal{K} \subset [\omega_-, \omega_+]$ such that for any two pair $\omega, \omega' \in \mathcal{K}$ there is a heteroclinic trajectory with α-limit set in Σ_ω and ω-limit set in $\Sigma_{\omega'}$. Moreover, for any $\omega, \omega' \in [\omega_-, \omega_+]$ there is trajectory whose α-limit set contains Σ_ω and ω-limit set contains $\Sigma_{\omega'}$.*

Moreover, for any pair $\omega, \omega' \in [\omega_-, \omega_+]$ there is a heteroclinic trajectory whose α-limit set has nonempty intersection with Σ_ω and ω-limit set — with $\Sigma_{\omega'}$.

Remarks:

1. This property is an area-preserving analog of so called Kupka-Smale property that all periodic points are hyperbolic and their stable and unstable manifolds intersect transversally [Sm]. This property is generic in both sense topological (Baire residual) [Sm] and probabilistic/measure (prevalence) [Ka] for C^r-smooth (not necessarily area-preserving) diffeomorphisms of a compact manifold. It is also generic for area-preserving maps [Ro].

2. Direct computation [MF] show that an action-minimizing periodic point has linearization with trace at least two. Along with KS property and area-preservation this implies that *all* action-minimizing periodic points are saddles. So stable and unstable manifolds are 1-dimensional. By KS property intersection of stable and unstable manifolds of all periodic points transverse which implies that any rotational invariant curve can't have rational rotation number, i.e. ω_- and ω_+ are in $\mathbb{R} \setminus \mathbb{Q}$.

3. For any rational rotation number $p/q \in (\omega_-, \omega_+)$ the corresponding Aubry-Mather set $\Sigma_{p/q}$ has at least one periodic point x of period q. Hyperbolicity of x implies that it is isolated.

4. Existence of bi-asymptotic trajectories to $\Sigma^{per}_{p/q}$, i.e. trajectories from $\Sigma^{\pm}_{p/q}$ (see Structure Theorem), implies that the union of stable and unstable manifolds of periodic points in $\Sigma^{per}_{p/q}$ separates the cylinder. Thus, their closure contains the "top" and the "bottom" frontiers $C_\pm$ of C. This, in particular, implies that for any pair of rational numbers p/q and p'/q' in (ω_-, ω_+) there is a trajectory whose α-limit set is contained in $\Sigma_{p/q}$ and ω-limit set is in $\Sigma_{p'/q'}$. So we have

Lemma 2. *Hyperbolicity of periodic points along with Structure Theorem (non-emptiness of $\Sigma^{per}_{p/q}$ for all $p/q \in (\omega_-, \omega_+)$) implies Mather's Connecting Theorem for rational rotation numbers.*

To prove Mather's Connecting Theorem for irrational or mixed rotation numbers we need to know additional structure of Aubry-Mather sets with irrational rotation numbers.

Definition 4. An Aubry-Mather set Σ_ω is called *hyperbolic* if it admits two continuous line fields one is contracting the other is expanding.

The standard Hadamard-Perron theorem implies that if an Aubry-Mather set Σ_ω is hyperbolic, then from every point p in Σ_ω there are two smooth curves which stable and unstable manifolds of p. Since Aubry-Mather sets depend continuously on rotation number and cone property is open, it is not difficult to prove that hyperbolicity of periodic points leads to hyperbolicity of most of Aubry-Mather sets.

Theorem 5. *(e.g.* [L2] *Thm. 1.10) If a C^1-smooth EAPT $\mathcal{P} : \mathbb{A} \to \mathbb{A}$ has KS property, then there is an open dense set $\mathcal{K}$ of rotation numbers in $\mathbb{R}$ such that any Aubry-Mather with its rotation number from $\mathcal{K}$ is hyperbolic.*

Lemma 3. *If an Aubry-Mather set Σ_ω is hyperbolic, then the union of stable and unstable manifolds of points in Σ_ω separates the cylinder.*

Proof: In the rational case $\omega = p/q \in \mathbb{Q}$ this is in Remark 4 above. In the irrational case $\omega \notin \mathbb{Q}$ the idea is analogous. Let ω be irrational and suppose that $\mathcal{P}$ has a right twist, if not take the inverse. By Structure Theorem Σ_ω^{rec} consists of points of three types. The first type of p's in Σ_ω^{rec} is when the projection $\pi_1(p)$ can be approximated from both sides by $\pi_1(\Sigma_\omega^{rec})$. The other two types is when the projection $\pi_1(p)$ can only be approximated from one side.

Since stable and unstable line field have to be transversal with a separated from zero angle and continiuosly chanimg from a point to a point, stable and unstable manifolds of points of the first type have to intersect with unstable and stable manifolds respectively or nearby points of the same type. We need only to connect through the "wholes", i.e. through preimages of open intervals which are in the complement of $\pi_1(\Sigma_\omega^{rec})$. To do that we need to construct connecting trajectories of neighboring points of the second and third type.

Let p_r and p_l be points in Σ_ω^{rec} whose projections into $\mathbb{S}^1$ are neighbors, i.e. $(\pi_1(p_l), \pi_1(p_r)) \subset \mathbb{S}^1 \setminus \pi_1(\Sigma_\omega^{rec})$. By the footnote to Structure Theorem we know that for any $\varepsilon > 0$ there is $n \in \mathbb{Z}$ such that $\mathcal{P}^n\ p_r$ and $\mathcal{P}^n\ p_l$ are ε-close. Since ε is arbitrary, an angle between stable and unstable line fields is separated from zero, and stable and unstable manifolds of points in Σ_ω^{rec} are C^1 by Hadamard-Perron Theorem, we have that $W^u(p_l)$ and $W^s(p_r)$ intersect and their projection contains $(\pi_1(p_l), \pi_1(p_r))$. This implies that for any two neighbors of the second and third type their stable and unstable manifolds do intersect and the union of stable and unstable manifolds of points in Σ_ω^{rec} separates the cylinder. Let $W^s(\Sigma_\omega)$ be the union of stable manifolds of all points in Σ_ω which is a stable lamination. Similarly, one can define unstable lamination $W^u(\Sigma_\omega)$.

In a view of the above arguments and Birkhoff invariant set theorem, if $\omega, \omega' \in \mathcal{K}$ from Theorem 5 or, equivalently, Aubry-Mather sets Σ_ω and $\Sigma_{\omega'}$ are hyperbolic, then their stable and unstable laminations $W^s(\Sigma_\omega)$ and $W^u(\Sigma_{\omega'})$ intersect and the intersection contains at least one trajectory whose α-limit set is contained in Σ_ω and ω-limit set is in $\Sigma_{\omega'}$. So we proved the following

Proposition 6. *If $\mathcal{P} : \mathbb{A} \to \mathbb{A}$ is a C^1-smooth EAPT with KS property, $C = \mathcal{P}(C)$ is a BRI, C_+ and C_- are "top" and "bottom" frontiers of C respectively, ω_+ and ω_- are rotation numbers of $\mathcal{P}$ restricted to C_+ and C_- respectively, and $\mathcal{K} \cap [\omega_-, \omega_+]$ be an open dense set of rotation numbers with hyperbolic Aubry-Mather sets (from Theorem 5). Then for any ω and ω' from $\mathcal{K} \cap [\omega_-, \omega_+]$ the stable and unstable laminations $W^s(\Sigma_\omega)$ and $W^u(\Sigma_{\omega'})$ of corresponding Aubry-Mather sets do intersect and the intersection contains a heteroclinic trajectory going from Σ_ω to $\Sigma_{\omega'}$.*

This proves the first part of Mather's Connecting Theorem for EAPTs with KS property. To prove the "moreover" part of the Mather's connecting theorem from this section we use standard arguments of Arnold [Ar] usually called whiskered tori. In our case tori are replaced by periodci points. In our case whiskered tori are 0-dimensional and correspond to action-minimizing periodic points. By Remark 2 above they are hyperbolic saddles. Chose a bi-infinite sequence of rational numbers $\{\omega_n\}_{n\in\mathbb{Z}} \subset \mathcal{K} \subset [\omega_-, \omega_+]$ so that corresponding Aubry-Mather sets are hyperbolic and $\lim \omega_n = \omega$ (resp. ω') if $n \to -\infty$ (resp. $+\infty$). This implies that each periodic points $\{p_n \in \Sigma_{\omega_n}\}_{n\in\mathbb{Z}}$ is hyperbolic. By Proposition 6 for any $n \in \mathbb{Z}$ their stable $W^s(p_n)$ and unstable $W^u(p_{n+1})$ manifolds consequently cross each other, i.e. there is a point $q \in W^s(p_n) \cap W^u(p_{n+1})$ whose neighbourhood is separated by $W^s(p_n)$ in two parts and $W^u(p_{n+1})$ locally visits both.

Notice that if $W^s(p_{n-1})$ crosses $W^u(p_n)$ and $W^s(p_n)$ crosses $W^u(p_{n+1})$, then $W^s(p_{n-1})$ crosses $W^u(p_{n+1})$. This proves that by induction that for $k, n \in \mathbb{Z}$ we have $W^s(p_k)$ crosses $W^u(p_n)$. In other words, p_n's form a transition chain. Now choose a sequence of positive numbers ε_n which tends to 0 as $n \to \infty$. Choose a sequence of points $q_n \in W^s(p_{-n}) \cap W^u(p_n)$ so that q_n is ε_n-close to p_0, but $\varepsilon/\|\mathcal{P}\|_{C^1}$-away from p_0. By definition q_n its α-limit is the trajectory of p_{-n} and ω-limit — of p_n. Now recall that Aubry-Mather sets depend continiously on rotation number (Thm. 11.3 [MF]). So $\Sigma_{\omega_n} \to \Sigma_\omega$ as $\omega_n \to \omega$ in Hausdorff distance, in particular, $p_n \to p$ as $n \to -\infty$. Similarly, $p_n \to p'$ as $n \to +\infty$. Therefore, one can choose a subsequence p_{n_k} such that $q_{n_k} \to q$ and q is different from p_0 and ε-close to it. Moreover, α-limit (resp. ω-limit) set of q contains $p \in \Sigma_\omega$ (resp. $p' \in \Sigma_{\omega'}$). This proves Mather's Connecting Theorem for EAPTs with KS property. Q.E.D.

3. The Proof of Mather's Accelerating Theorem

Let's make several preliminary remarks in order to show connection of this problem with EAPTs of the cylinder discussed above.

The Hamiltonian phase space of the geodesic flow is the cotangent bundle of the torus $\mathbf{T}\mathbb{T}^2$ which is isomorphic to $\mathbb{R}^2 \times \mathbb{T}^2$. Recall that we denote by q

coordinates on $\mathbb{T}^2$ and by p coordinates in the cotangent space $\mathbf{T}^*_q\mathbb{T}^2$ which is isomorphic to $\mathbb{R}^2$. The geodesic flow is Hamiltonian with respect to the Darboux form $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ with the Hamiltonian function

$$H_0(p,q) = T_q(p) = \rho_q(p,p)/2,\ p \in \mathbf{T}^*_q\mathbb{T}^2.$$

Recall that ρ_q is the metric in $\mathbf{T}^*\mathbb{T}^2$. Denote by Φ_t the time t map of the geodesic flow. For each E, we denote by $\mathcal{L}^0_E = \{(p,q) : \ H_0(p,q) = E\}$ the corresponding energy level. In particular, $S^*\mathbb{T}^2 = \mathcal{L}^0_{1/2}$ is the unit energy level. Denote also $\hat{\mathcal{L}}^0_{E_0} = \cup_{E\ge E_0}\mathcal{L}^0_E$. We use energy as one of coordinates. It is not difficult to see that $\mathcal{L}^0_E$ is a 3-dimensional manifold invariant under the geodesic flow Φ_t and diffeomorphic to $\mathbb{T}^2 \times \mathbb{T}^1$. We can view any geodesic as the map "γ": $\mathbb{R} \to \mathbb{T}^2$ and denote by $\gamma_E(t) = (\gamma^q_E(t), \gamma^p_E(t))$ the orbit of the geodesic flow which belongs to the energy level $\mathcal{L}^0_E$ with a fixed origin $\gamma_E(0) \in \mathcal{L}^0_E$. These conditions determine the orbit $\{\gamma_E(t)\}_{t\in\mathbb{R}}$ uniquely.

Notice that an appropriate rescaling of time of an orbit $\{\gamma_E(t)\}_{t\in\mathbb{R}}$ on the energy level $\mathcal{L}^0_E$ leads to an orbit on the energy level $\mathcal{L}^0_{1/2}$.

$$(\gamma^q_E(t), \gamma^p_E(t)) = \left(\sqrt{2E}\gamma^q_{1/2}(\sqrt{2E}t)), \sqrt{2E}\gamma^p_{1/2}(\sqrt{2E}t\right). \tag{9}$$

Recall that $\Gamma \subset \mathcal{L}^0_{1/2}$ is the shortest hyperbolic periodic geodesic in the homology class $h \in H_1(\mathbb{T}^2, \mathbb{Z})$ from non-degeneracy Hypothesis 1. Then if $\Gamma = \Gamma_{1/2}$ has period T the period of the same geodesic with rescaled time $\Gamma_E(\sqrt{2E}t) \equiv \Gamma_{1/2}(t)$ is $T/\sqrt{2E}$.

3.1. **Hyperbolic Persistent Cylinder.** Let's give a rigorous statement of the fact stated after Non-degeneracy hypothesis in Section 0.2.

Theorem 7. (Morse, Anosov, Mather) *For a C^r generic*[5] *Riemannian metric ρ on $\mathbb{T}^2$ with $r \ge 2$, for a homology class $h \in H_1(\mathbb{T}^2, \mathbb{Z})$, and for any energy $H_0(p,q) = E > 0$ there is a periodic geodesic $\Gamma^0_{1/2} = \cup_{t\in\mathbb{R}}\Gamma^0_{1/2}(t)$ in the homology class h, whose time rescaled copies $\mathbb{A}^0_E = \cup_{E'\ge E}\Gamma^0_{E'}$, as in (9), form an invariant normally hyperbolic manifold (cylinder) in $\mathbb{A}^0_E \subset \mathbf{T}^*\mathbb{T}^2$. Its stable and unstable manifolds $W^s_{\mathbb{A}^0_E}$ and $W^u_{\mathbb{A}^0_E}$ respectively are 2-dimensional and on each energy level Γ^0_E there is a homoclinic orbit $\Lambda^0_E(t)$ given by*

$$\Lambda^0_E \subset (W^s_{\Gamma^0_E} \setminus \Gamma^0_E) \cap (W^u_{\Gamma^0_E} \setminus \Gamma^0_E), \tag{10}$$

whose projection into $\mathbb{T}^2$ corresponds to the Morse geodesic Λ. Moreover, this intersection of invariant manifolds restricted to any energy level $\mathcal{L}^0_E$ is transversal.

[5] Here generic means open dense set in the space of C^r metrices with the uniform C^r topology

Also on the energy level $E = 1/2$ for some a and b we have

$$\rho(\Lambda^0_{1/2}(s), \Gamma^0_{1/2}(s+a)) \to 0 \quad \text{as} \quad s \to -\infty$$
$$\rho(\Lambda^0_{1/2}(s), \Gamma^0_{1/2}(s+b)) \to 0 \quad \text{as} \quad s \to +\infty. \tag{11}$$

Transversal intersection of two 2-dimensional manifolds $W^s_{\Gamma^0_E}$ and $W^u_{\Gamma^0_E}$ in 3-dimensional energy level $\mathcal{L}_E$ is 1-dimensional and by the theorem on implicit function is locally isolated curve.

Now formalize meaning of normal hyperbolicity of the cylinder $\mathbb{A}^0_E$:

Lemma 4. *(e.g.* [DLS1]*) Let $\mathbb{A}^0_E = \cup_{E' \geq E} \Gamma^0_{E'}$ be the invariant cylinder diffeomorphic to $[E, +\infty) \times \mathbb{T}^1$ and the canonical symplectic form ω restricted to $\mathbb{A}^0_E$ is non-degenerate and invariant under the geodesic flow Φ_t. Moreover, for some $C, \alpha > 0$ and for all $x \in \mathbb{A}^0_E$ we have*

$$T_x \mathcal{L}^0_E = E^s_x \oplus E^u_x \oplus T_x \hat{\Gamma}^0_E \tag{12}$$

with $\|d\Phi_t(x)|_{E^s_x}\| \leq C \exp(-\alpha t)$ for $t \geq 0$, $\|d\Phi_t(x)|_{E^u_x}\| \leq C \exp(\alpha t)$ for $t \leq 0$, and $\|d\Phi_t(x)|_{T_x \Gamma^0_E}\| \leq C$ for all $t \in \mathbb{R}$.

3.1.1. *High energy motion.* Recall that the original Hamiltonian has the form $H(p, q, t) = \rho_q(p, p)/2 + U(q, t)$. So if energy is of order ε^{-2} for a sufficiently small $\varepsilon > 0$, then it is convenient to scale the Hamiltonian $\varepsilon^2 H(p/\varepsilon, q, t) = \rho_q(p, p)/2 + \varepsilon^2 U(q, t)$. Introduce new: time $\bar{t} = t/\varepsilon$, impulse $\bar{p} = \varepsilon p$, and symplectic form $\bar{\omega} = d\bar{p}_1 \wedge dq_1 + d\bar{p}_2 \wedge dq_2 = \varepsilon \omega$. Then the rescaled Hamiltonian can be written as $H_\varepsilon(\bar{p}, q, \varepsilon \bar{t}) = \rho_q(\bar{p}, \bar{p})/2 + \varepsilon^2 U(q, \varepsilon \bar{t})$.

Fix a sufficiently small ε. Notice that $H_\varepsilon(\bar{p}, q, \varepsilon \bar{t})$ is a small perturbation of the geodesic Hamiltonian system $H_0(\bar{p}, q)$. By Sacker-Fenichel theorem [Sa], [Fe], [HPS] the hyperbolic invariant cylinder $\mathbb{A}^0_E$ with a sufficiently large $E > \varepsilon^{-2}$ for the geodesic Hamiltonian system $H_0(\bar{p}, q)$ persists under a small perturbation and as smooth as H_0 is. Therefore, the rescaled Hamiltonian system $H_\varepsilon(\bar{p}, q, \varepsilon \bar{t})$ has a hyperbolic invariant cylinder $\mathbb{A}^0_{\bar{E}}$ which is close to the hyperbolic invariant cylinder $\hat{\Gamma}^0_{\bar{E}}$. Rescaling back to the initial Hamiltonian $H(p, q, t) = \rho_q(p, p)/2 + U(q, t)$ we see that $H(p, q, t)$ also has a hyperbolic invariant cylinder $\hat{\Gamma}_E$ close to $\hat{\Gamma}^0_E$ with $E = \bar{E}\varepsilon^{-2}$. This cylinder has to belong not to the phase space $\mathbf{T}^*\mathbb{T}^2$ of the geodesic Hamiltonian as $\mathbb{A}^0_E$ does, but the time *extended* phase space $\mathbf{T}^*\mathbb{T}^2 \times \mathbb{T}$. However, to avoid considering the extended phase space we take the time 1 map for the Hamiltonian system $H_0(p, q)$, denoted $\Phi^0 = \Phi^0_1 : \mathbf{T}^*\mathbb{T}^2 \to \mathbf{T}^*\mathbb{T}^2$ and for the initial Hamiltonian $H(p, q, t)$, denoted $\Phi = \Phi_1 : \mathbf{T}^*\mathbb{T}^2 \to \mathbf{T}^*\mathbb{T}^2$. Then Φ^0 has $\mathbb{A}^0_E$ as a hyperbolic invariant cylinder and for a sufficiently large E the map Φ also has a hyperbolic invariant cylinder, denoted by $\mathbb{A}_E \subset \mathbf{T}^*\mathbb{T}^2$. This cylinder has 3-dimensional stable and unstable manifolds $W^s(\mathbb{A}_E)$ and $W^u(\mathbb{A}_E)$ respectively which in a neighborhood of the cylinder Γ_E are small perturbation of stable and unstable manifolds $W^s(\mathbb{A}^0_E)$ and $W^u(\mathbb{A}^0_E)$ for the cylinder $\mathbb{A}^0_E$. By

Hypothesis 2′ $W^s(\mathbb{A}^0_E)$ and $W^u(\mathbb{A}^0_E)$ have transversal intersection. Therefore, $W^s(\mathbb{A}_E)$ and $W^u(\mathbb{A}_E)$ also have transversal intersection which is the family of homoclinic trajectories $\hat{\Lambda}_E$ to the cylinder $\mathbb{A}_E$[6].

3.2. **The inner and outer maps.** Consider $\Phi^0 : \mathbf{T}^*\mathbb{T}^2 \to \mathbf{T}^*\mathbb{T}^2$ the time 1 map for $H_0(p,q)$, restricted to the invariant cylinder $\hat{\Gamma}_E$. Wlog assume that period of the minimal periodic geodesic $\Gamma^0_{1/2}$ is 1, otherwise, rescaled the metric. Denote by $(\theta, r) \in \mathbb{S}^1 \times \mathbb{R} = \mathbb{A} \supset \mathbb{A}^0_E$ the natural coordinates on the invariant cylinder $\hat{\Gamma}_E$ chosen so that the time 1 map for $H_0(p,q)$ has the form $\mathcal{P}^0(\theta, r) = (\theta + r, r)$. It is possible by scaling arguments. It is easy to see that $\mathcal{P}_0$ is an EAPT. Denote by ω_Γ the restriction of the standard symplectic 2-form $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ to the invariant cylinder $\mathbb{A}^0_E$.

Similarly, consider $\Phi : \mathbf{T}^*\mathbb{T}^2 \to \mathbf{T}^*\mathbb{T}^2$ the time 1 map for $H(p,q,t)$. We know it has a hyperbolic invariant cylinder $\mathbb{A}_E$ for a sufficiently large E. Restrict $\Phi|_{\mathbb{A}_E}$ to this cylinder and denote it by

$$\mathcal{P}_E = \Phi|_{\mathbb{A}_E} : \mathbb{A}_E \to \mathbb{A}_E. \tag{13}$$

Using standard arguments one can check that restriction of the standard symplectic form ω to $\mathbb{A}_E$ gives an area form ω_E on $\mathbb{A}_E$. Moreover, $\mathcal{P}_E$ is an EAPT ω_Γ-preserving. Following terminology proposed in [DLS1] call $\mathcal{P}_E$ *the inner map*, because it acts within the invariant cylinder $\mathbb{A}_E$. Implicitly the inner map is used in [Ma6], but it is encoded into Mather's variational principle.

To define the *outer* map recall that the cylinder $\mathbb{A}_E$ has 3-dimensional stable and unstable manifolds $W^s(\mathbb{A}_E)$ and $W^u(\mathbb{A}_E)$ which intersect transversally in 4-dimensional space $\mathbf{T}^*\mathbb{T}^2$. Since contraction and expansion along transversal to $\mathbb{A}_E$ invariant directions dominate the inner dynamics on $\mathbb{A}_E$ for each point $x \in \mathbb{A}_E$ there is a unique trajectory containing $z_+(x)$ (resp. $z_-(x)$) in $W^s(\mathbb{A}_E) \cap W^u(\mathbb{A}_E)$ such that for some $\beta > 0$, a sufficiently large E, and all sufficiently large $n \in \mathbb{Z}_+$

$$\begin{aligned} \text{dist}\left(\Phi^{-n}x, \Phi^{-n}z_-(x)\right) &\le \exp(\beta n) \\ \text{dist}\left(\Phi^{n}x, \Phi^{n}z_+(x)\right) &\le \exp(\beta n) \end{aligned} \tag{14}$$

This defines the map

$$\mathcal{S}_E : \mathbb{A}_E \to \mathbb{A}_E \quad \mathcal{S}_E(x_-) = x_+, \tag{15}$$

where trajectories of $z_-(x_-)$ and $z_+(x_+)$ being the same. In other words, $\mathcal{S}_E$ sends x_- to x_+ if there is a heteroclinic trajectory from x_- to x_+. It is important for definition of the outer map that transversal to the cylinder $\mathbb{A}_E$ dynamics (expansion/contraction) dominates dynamics on the cylinder itself. It is certainly true in our case, because the inner map $\mathcal{P}_E$ is a perturbation of an integrable map

[6]If $W^u(\mathbb{A}^0_E)$ and $W^s(\mathbb{A}^0_E)$ are not transversal we need intersaction to be an isolated curve. The fact that this intersation is non-empty follows from existence of Morse Class A geodesics defined in sect. 0.1

with no expansion/contraction. Following terminology proposed in [DLS1] call $\mathcal{S}_E : \mathbb{A}_E \to \mathbb{A}_E$ *the outer map.* Again implicitly the outer map is used in [Ma6], even though it is encoded in action-minimizing trajectories. It is not difficult to see that $\mathcal{S}_E$ is continuous and even can be shown to be C^r-smooth area-preserving (see [DLS1]), but we don't use this fact.

4. Diffusion strategy

The idea of mixed diffusion following inner and outer dynamics goes back to the original work of Mather [Ma6]. Provided the Hamiltonian system (3) is sufficiently smooth Arnold's approach [Ar] of whiskered tori is also applicable (as shown in [BT] and [DLS1]). Our goal is to show existence of trajectories traveling indefinitely far along the invariant cylinder $\mathbb{A}_E$. Since the inner dynamics on $\mathbb{A}_E$, defined by the inner map $\mathcal{P}_E : \mathbb{A}_E \to \mathbb{A}_E$, is described by an EAPT, we can apply arguments and results from the first part of the paper. Namely, between any two adjacent invariant curves on $\mathbb{A}_E$ we have an EAPT acting inside of a BRI (Birkhoff region of instability). Therefore, there is a trajectory which goes from an arbitrary small neighborhood of the "bottom" invariant curve to an arbitrary small neighborhood of the "top" one. The problem which arises is that the inner map $\mathcal{P}_E$ is a small perturbation of the completely integrable map $\mathcal{P}^0(\theta, r) = (\theta + r, r)$. Indeed, every horizontal circle on $\mathbb{A}^0$ is $\mathcal{P}^0$-invariant, therefore, by KAM theory [He] after a small perturbation most of these invariant curves will survive. Thus, $\mathcal{P}_E$ has a lot of invariant curves. So, using the inner map $\mathcal{P}_E$ it is impossible to "jump" over an invariant curve Γ for $\mathcal{P}_E$ to increase energy. To overcome the problem we shall use the outer map $\mathcal{S}_E$. Mather [Ma6] shows that $\mathcal{S}_E(\Gamma)$ has a part strictly above Γ which provides the "jump" over Γ. Our strategy is to pick a neighborhood in $\mathbf{T}^*\mathbb{T}^2$ of a properly chosen point in $\mathbb{A}_E$ and iterate this neighborhood using alternating series of inner and outer maps.

To describe the strategy with more details introduce a bit of terminology: Let $\Gamma \subset \mathbb{A}_E$ be an invariant curve, which always means invariant for the inner map $\mathcal{P}_E$. Denote by $\mathbb{A}^+_\Gamma \subset \mathbb{A}_E$ an open topological annulus infinite on one side and bounded by Γ on the other. We say that an invariant curve $\Gamma \subset \mathbb{A}_E$ is top (resp. bottom) isolated if it has a neighborhood in $\mathbb{A}^+_\Gamma$ (resp. $\mathbb{A}_E \setminus \mathbb{A}^+_\Gamma$) free from invariant curves.

Notice also that for a large enough energy any invariant curve Γ is a small perturbation of the curve $\Gamma^0 = \{H_0 = \omega_\Gamma^2/2\} \cap \mathbb{A}_0$, where ω_Γ is the rotation number induced on Γ by $\mathcal{P}_E$. Thus, $H(\cdot, t)$ is almost constant on Γ.

It turns out that under non-degeneracy Hypothesis 3 for any $d > 1$ and any invariant curve $\Gamma \subset \mathbb{A}_E$ the outer image $\mathcal{S}_E(\Gamma) \subset \mathbb{A}_E$ deviates up and down from Γ so that there is a point $x_+ \in \mathcal{S}_E(\Gamma) \cap \mathbb{A}^+_\Gamma$ (resp. $x_- \in \mathcal{S}_E(\Gamma) \setminus \mathbb{A}^+_\Gamma$) such that we have

$$\operatorname{dist}(x_\pm, \Gamma) > \omega_\Gamma^{-d}. \tag{16}$$

Notice that for our method it does not matter an exact value of d in contract to the standard geometric method from [BT] or [DLS1]. Moreover, ω_Γ^{-d} can be replaced by a flat function, e.g. $\exp(-\omega_\Gamma)$. In [BT] and [DLS1] the value of d is crucial to overcome the so-called gap-problem about gaps between nearby invariant curves/tori. In [DLS3] modification of Arnold's whiskered tori approach is proposed to prove existence of diffusion in Arnold's example [Ar]. The authors use resonant tori in $\mathbb{A}_E$ as additional elements of transition chains there. In [X] extension of Mather's variational approach is given to show diffusion in this example. In future publicatons we shall extend mixed geometric/variational approach presented in this paper to include Arnold's example.

Let's distinguish two ways of diffusing: inner and outer.

Inner diffusion (or Birkhoff diffusion): Let x belong to a top isolated curve $\Gamma \subset \mathbb{A}_E$. Suppose $\Gamma' \subset \mathbb{A}_\Gamma^+$ is an adjacent invariant curve, i.e. $\mathcal{P}_E(\Gamma') = \Gamma'$ and the annulus between Γ and Γ' inside $\mathbb{A}_\Gamma^+ \setminus \mathbb{A}_\Gamma^- \subset \mathbb{A}_E$ is a BRI, i.e $\mathbb{A}_\Gamma^+ \setminus \mathbb{A}_\Gamma^-$ is free from other invariant curves (sect. 0.3). Then by Birkhoff invariant set theorem for any neighborhood U_x which has x in the closure and any neighborhood $U_{\Gamma'}$ of Γ' both in $\mathbf{T}^*T^2$ there is $n_0 \in \mathbb{Z}_+$ depending on all the above quantities such that $\mathcal{P}_E^n(U_x \cap \mathbb{A}_E) \cap U_{\Gamma'} \neq \emptyset$. Moreover, $\cup_{n\in\mathbb{Z}_+}\mathcal{P}_E^n(U_x \cap \mathbb{A}_E)$ contains Γ' in the closure.

Outer diffusion: Let $\Gamma \subset \mathbb{A}_E$ be an invariant curve which is bottom isolated and $U \subset \mathbf{T}^*\mathbb{T}^2$ is an open set containing Γ in the closure. Consider the outer image $\mathcal{S}_E(\Gamma)$. There are two case:

· the first — $\mathbb{A}_\Gamma^+$ has a topological annulus C^{inv} close to Γ foliated by invariant curves and $\mathcal{S}_E(\Gamma)$ does not intersect a top isolated invariant curve Γ above C^{inv} in $\mathbb{A}_E$ and

· the second — $\mathcal{S}_E(\Gamma)$ intersects a top isolated invariant curve Γ' and does not intersect any topological annulus $C^{inv} \subset \mathbb{A}_{\Gamma'}^+$ separating $\mathbb{A}_E$ and foliated by invariant curves.

In the first case of outer diffusion we need the following

Lemma 5. *Let $\mathcal{P}_E : \mathbb{A} \to \mathbb{A}$ be a C^1-EAPT of the annulus $\mathbb{A}$. Suppose there is a topological annulus $C \subset \mathbb{A}$ separating $\mathbb{A}$, and consisting of invariant curves, i.e. every $x \in C$ belongs to a rotational curve $\Gamma = \mathcal{P}(\Gamma)$. Then for any neighborhood $U \subset C$ there is $n \in \mathbb{Z}_+$ so that $\cup_{k=0}^n \mathcal{P}^k U$ contains an invariant curve Γ^*.*

We shall prove this lemma at the end of this section for completeness. Clearly such a situation is easily destroyable by a pertubation, but for us it is easier to prove this lemma. In the outer case under consideration there is an open set $V \subset \mathbb{A}_E$ sufficiently close to Γ so that $U = \mathcal{S}_E(V)$ intersects only those invariant curves $\tilde{\Gamma}$ in $\mathbb{A}_\Gamma^+$ that for the point $x_+ \in \mathcal{S}_E(\Gamma)$ satisfy (16) we have

$$\operatorname{dist}(x_+, \tilde{\Gamma}) > \omega_\Gamma^{-2}/2. \tag{17}$$

Then application of the above lemma shows that for some $n \in \mathbb{Z}_+$ we get $\cup_{k=0}^n \mathcal{P}_E U$ contains an open neighborhood of some invariant curves $\tilde\Gamma \subset \mathbb{A}_\Gamma^+$ satisfying (17). In other words, first we send a neighborhood $V \subset \mathbb{A}_E$ below Γ by the outer map $\mathcal{S}_E(V)$ first and then by iterate by the inner map so that its images contain a neighborhood of an invariant $\tilde\Gamma$ which is "higher" than Γ by at least $\omega_\Gamma^{-d}/2$. If $\mathbb{A}_{\tilde\Gamma}^+$ is foliated by invariant circles, then iterating this procedure of applying the outer map and than the inner map a finite number of times we can increase energy indefinitely. If $\mathbb{A}_{\tilde\Gamma}^+$ is not foliated by invariant curves, then a number of iterations if this procedure will brings as to the second case above.

In the second outer case we know that $\mathcal{S}_E(\Gamma)$ intersects Γ' and has a part in $\mathbb{A}_{\Gamma'}^+$. Thus, if an open subset $U \subset \mathbb{A}_E$ has Γ in the closure its outer image $\mathcal{S}_E(U)$ closure has nonempty intersection with Γ' and there is a neighborhood U' in $\mathcal{S}_E(U) \cap \mathbb{A}_{\Gamma'}^+$ which has Γ' in the closure and fits to apply the inner diffusion. We also either require Γ' to satisfy (17) with $\tilde\Gamma = \Gamma'$ or pick the lowest invariant curve Γ^* in $\mathbb{A}_{\tilde\Gamma}^+$ to satisfy (17) with $\tilde\Gamma = \Gamma^*$. This completes the proof of Mather's acceleration theorem based on lemma 5 and estimate (16) about oscillations of the outer map. Now we proof lemma 5 and in the next section describe Mather's variational approaches to prove estimate (16) which is *the key.*

Proof of lemma 5: By Birkhoff invariant curve theorem ([MF] Thm.15.1) every rotation invariant curve Γ of an EAPT $\mathcal{P}$ is a Lipschitz graph over the base $\mathbb{S}^1$. This naturally induces a coordinate system on each invariant curve and gives that $\mathcal{P}|_\Gamma$ is a homeomorphism on each invariant curve Γ.

Notice that by area-preservation of $\mathcal{P}$ and invariance of C for some $n \in \mathbb{Z}_+$ we have $U \cap \mathcal{P}^k U \neq \emptyset$. U consists of intersections with invariant Lipschitz curves $U_\Gamma = U \cap \Gamma$. So for an open set of them $U_\Gamma \cap \mathcal{P}^k U_\Gamma \neq \emptyset$ and by area-preservation and twist conditions $\mathcal{P}^k U_\Gamma \setminus U_\Gamma \neq \emptyset$. Pick a curve Γ with irrational rotation number. Since $\mathcal{P}^k U_\Gamma \cap U \neq \emptyset$, the same is true for $\mathcal{P}^{rk} U_\Gamma$ and $\mathcal{P}^{(r+1)k} U_\Gamma$ with $r \in \mathbb{Z}$. Therefore, $\cup_{r=0}^s \mathcal{P}^{rk} U_\Gamma$ is always connected and by irrationality of rotation number of Γ should cover Γ for some s. This completes the proof of the lemma. Q.E.D.

5. The Jump Lemma

In this section we outline Mather's variational approach to prove the oscillation property (16) of the outer map $\mathcal{S}_E : \mathbb{A}_E \to \mathbb{A}_E$, defined in (15). The general idea of Mather's method is to construct trajectories of a Hamiltonian system as solutions to a variational problem. In our case time 1 map of the Hamiltonian system (3), denoted by $\Phi : \mathbf{T}^*\mathbb{T}^2 \to \mathbf{T}^*\mathbb{T}^2$, possesses an invariant cylinder $\mathbb{A}_E \subset \mathbf{T}^*\mathbb{T}^2$ and restriction of $\Phi|_{\mathbb{A}_E} = \mathcal{P}_E$ is a C^2-EAPT. Thus, by Aubry-Mather theory [AL], [Ma1], [MF], or [Ba1] for every rotation number ω there is an action-minimizing (Aubry-Mather) invariant set $\Sigma_\omega \subset \mathbb{A}_E$ (Theorem 3). By twist condition if $\Gamma = \mathbb{A}_E$ is an invariant curve with rotation number $\omega = \omega_\Gamma$, then $\Sigma_\omega \subseteq \Gamma$ and for any $\omega' > \omega$ the corresponding Aubry-Mather $\Sigma_{\omega'}$ belongs to $\mathbb{A}^+_\Gamma$, i.e. is above Σ_ω. Similarly, if $\omega' < \omega$, then $\Sigma_{\omega'} \subset \mathbb{A}_E \setminus \mathbb{A}^+_\Gamma$. Moreover, the map from Aubry-Mather sets to $\mathbb{R}$ according to their rotation numbers can be extended to a Lipschitz map (see e.g. [Do]). Since the EAPT $\mathcal{P}_E$ induced on the invariant cylinder $\mathbb{A}_E$ is a small perturbation of the completely integrable map $\mathcal{P}(\theta, r) = (\theta + r, r)$ for large rotation numbers, to prove existence of orbits with *arbitrarily increasing energy* it suffices to prove existence of orbits which consequently visit neighborhoods of invariant sets with *arbitrarily increasing rotation numbers* on $\mathbb{A}_E$. Therefore, *if for some $\omega' < \omega < \omega''$ with $|\omega' - \omega''| > \omega^{-d}$ there is a trajectory whose α-limit set is $\Sigma_{\omega'}$ and ω-limit set is $\Sigma_{\omega''}$*[7] or, almost equivalently, $\mathcal{S}_E(\Sigma_{\omega'}) \cap \Sigma_{\omega''} \neq \emptyset$, then this along with the arguments from the previous section proves Mather's acceleration theorem. The rest of the paper is devoted to a formal statement of the Jump lemma and outline of Mather's variational approach to prove it.

5.1. **Duality between Hamiltonian and Lagrangian systems.** Recall that $U : \mathbb{T}^2 \times \mathbb{T} \to \mathbb{R}$ be a C^2-smooth time periodic function on $\mathbb{T}^2$ which we use as the potential energy. We associate the kinetic energy to the metric ρ

$$T^q(v) = \rho^q(v, v)/2, \quad v \in \mathbf{T}_q\mathbb{T}^2, \tag{18}$$

where $\mathbf{T}_q\mathbb{T}^2$ denotes the tangent bundle of $\mathbb{T}^2$ at q.

Definition 8. The *Lagrangian* L is a real valued function, defined on the phase space $\mathbf{T}\mathbb{T}^2 \times \mathbb{T}$ by

$$L(q, v, t) = T^q(v) - U(q, t). \tag{19}$$

Define the *Euler-Lagrange* flow associated to L. This is a flow on the phase space $\mathbf{T}\mathbb{T}^2 \times \mathbb{T}$ such that trajectories are associated to the variational condition

$$\delta \int_a^b L(d\gamma(t), t)dt = 0, \tag{20}$$

where $\gamma : [a, b] \to \mathbb{T}^2$ is a C^1 curve, $d\gamma(t)$ stands for the 1-jet $(\gamma(t), \dot\gamma(t))$, and the variation is taken relative to fixed endpoints. If γ satisfies this variational condition, then

$$\xi(t) = (d\gamma(t), t \bmod 1) \tag{21}$$

is a trajectory of the Euler-Lagrange flow. All trajectories, defined on $[a, b]$ are of this form. Equivalently, trajectories of the Euler-Lagrange flow are solutions of the Euler-Lagrange equation

$$\frac{d}{dt}(L_{\dot\theta}) = L_\theta, \tag{22}$$

where $\theta = (\theta_1, \theta_2)$ is the standard angular coordinate system on $\mathbb{T}^2$.

[7] we shall call this statement "the Jump Lemma"

Lemma 6. *(see e.g.* [Fa] *sect. 2.3) In the setting above trajectories of the Hamiltonian system with the Hamiltonian H, defined by (3), are in one-to-one correspondence with trajectories of the Euler-Lagrange flow for the Lagrangian L, given by (19). Moreover, a trajectory for H is mapped into a trajectory for (22) by the Legendre transform*

$$H(q,p,t) = \langle p, v\rangle_q - L(q,v,t), \quad \text{where} \;\; p = \frac{\partial L}{\partial v}(q,v) \quad \text{and} \quad \langle\ ,\ \rangle_q : \mathbf{T}_q\mathbb{T}^2 \times \mathbf{T}^*_q\mathbb{T}^2 \to \mathbb{R} \quad \text{is a dual pairing.} \tag{23}$$

5.2. **Action-Minimizing Probabilities and Mather sets.** In this section we discuss Mather's theory of minimal or action minimizing measures. This theory can be considered as an extension of KAM theory. Namely, it provides a large class of invariant sets for an Euler-Lagrange flow (or the dual Hamiltonian flow). KAM invariant tori is an example of *Mather sets*. Mather sets are also generalization of Aubry-Mather sets from two degrees of freedom to arbitrary number of degrees of freedom.

Following Mather [Ma4] we say that μ is a probability if it is a Borel measure and of total mass one. A probability on $\mathbf{T}\mathbb{T}^2 \times \mathbb{T}$ is *invariant* if it is invariant under Euler-Lagrange flow. If η is a one form on $\mathbb{T}^2 \times \mathbb{T}$, we may associate to it a real valued function $\hat{\eta}$ on $\mathbf{T}\mathbb{T}^2 \times \mathbb{T}$, as follows: express η in the form

$$\eta = \eta_1 d\theta_1 + \eta_2 d\theta_2 + \eta_\tau d\tau \tag{24}$$

with respect to the standard angular coordinates (θ_1, θ_2) on $\mathbb{T}^2$ and τ on $\mathbb{T}^1$ and set

$$\hat{\eta} = \eta_{\mathbb{T}^2} + \eta_\tau \circ \pi, \tag{25}$$

where $\pi : \mathbf{T}\mathbb{T}^2 \times \mathbb{T} \to \mathbb{T}^2 \times \mathbb{T}$ denotes the natural projection. This function has the property

$$\int_a^b \hat{\eta}(d\gamma(t), t)dt = \int_{\gamma,\tau} \eta, \tag{26}$$

for every C^1 curve $\gamma : [a,b] \to \mathbb{T}^2$ with the right side being the usual integral and $(\gamma, \tau) : [a,b] \to \mathbb{T}^2 \times \mathbb{T}$ defined by $(\gamma,\tau)(t) = (\gamma(t), t \bmod 1)$.

If μ is an invariant probability on $\mathbf{T}\mathbb{T}^2 \times \mathbb{T}$, its *average action* is defined as

$$A(\mu) = \int L d\mu. \tag{27}$$

Since L is bounded below, this integral is defined, although it may be $+\infty$. If $A(\mu) < +\infty$, one can define the *rotation vector* $\rho(\mu) \in H_1(\mathbb{T}^2, \mathbb{R})$ of μ by

$$\langle \rho(\mu), [\eta]_{\mathbb{T}^2}\rangle + [\eta]_{\mathbb{T}} = \int \hat{\eta} d\mu \tag{28}$$

for every C^1 one form η on $\mathbb{T}^2 \times \mathbb{T}$, where

$$[\eta] = ([\eta]_{\mathbb{T}^2}, [\eta]_{\mathbb{T}}) \in H^1(\mathbb{T}^2 \times \mathbb{T}, \mathbb{R}) = H^1(\mathbb{T}^2, \mathbb{R}) \times \mathbb{R} \tag{29}$$

denotes de Rham cohomology class and $\langle \ , \ \rangle$ denotes dual pairing $H_1(\mathbb{T}^2, \mathbb{R}) \times H^1(\mathbb{T}^2, \mathbb{R}) \to \mathbb{R}$. Mather introduced this concept in [Ma4] in the case of time independent one forms, but in time dependent case arguments are the same. In [Ma4] using Krylov-Bogoliuboff arguments Mather proved that

Lemma 7. *For every homology class $h \in H_1(\mathbb{T}^2, \mathbb{R})$ there exists an invariant probability μ such that $A(\mu) < +\infty$ and $\rho(\mu) = h$.*

Such a probability is called *minimal* or *action-minimizing* if

$$A(\mu) = \min\{A(\nu) : \rho(\nu) = \rho(\mu)\}, \tag{30}$$

where ν ranges over invariant probabilities such that $A(\nu) < +\infty$. If $\rho(\mu) = h$, we also say that μ is *h-minimal*.

The rotation vector has a natural geometric interpretation as an asymptotic direction of motion. More exactly, for a μ-generic trajectory of the Euler-Lagrange flow $\gamma : \mathbb{R} \to \mathbb{T}^2$ for $T > 0$ let z_T be the closed curve consisting of two parts: $\gamma|_{[-T,T]}$ and the shortest geodesic connecting $\gamma(-T)$ and $\gamma(T)$ on $\mathbb{T}^2$. Then

$$\rho(\mu) = \lim_{T \to +\infty} \frac{1}{2T}[z_T]. \tag{31}$$

We say that an invariant probability is *c-minimal* (for $c \in H^1(\mathbb{T}^2, \mathbb{R})$) if it minimizes $A(\nu) - \langle \rho(\nu), c \rangle$ over all invariant probabilities. Mather [Ma4] also proved

Lemma 7′. *For every cohomology class $c \in H^1(\mathbb{T}^2, \mathbb{R})$ there exists an invariant c-minimal probability μ such that $A(\mu) < +\infty$.*

We say that an invariant probability μ is minimal if and only if there is a one form η on $\mathbb{T}^2 \times \mathbb{T}$ such that μ minimizes $\int (L - \hat{\eta}) d\nu = A(\nu) - \langle \rho(\nu), [\eta]_{\mathbb{T}^2} \rangle - [\eta]_{\mathbb{T}}$ over invariant probabilities ν (see [Ma6]).

Definition 9. We call the function

$$\beta : H_1(\mathbb{T}^2, \mathbb{R}) \to \mathbb{R}, \quad \beta(h) = A(\mu), \quad \text{where } \mu \text{ is } h\text{-minimal}. \tag{32}$$

Mather's β-function and we call

$$\alpha : H^1(\mathbb{T}^2, \mathbb{R}) \to \mathbb{R} \quad \alpha(c) = \inf_{h \in H_1(\mathbb{T}^2, \mathbb{R})} \{\beta(h) - \langle h, c \rangle\}. \tag{33}$$

Mather's α-function. It is well defined by lemma 7.

Thus, the α-function is conjugate to the β-function by the Legendre transform. It follows that both functions are convex. By definition

$$\beta(h) + \alpha(c) \geq \langle h, c \rangle, \quad h \in H_1(\mathbb{T}^2, \mathbb{R}), \ c \in H^1(\mathbb{T}^2, \mathbb{R}). \tag{34}$$

The β-Legendre transform

$$\mathcal{L}_\beta : H_1(\mathbb{T}^2, \mathbb{R}) \to \{\text{compact, convex, non-empty subsets of } H^1(\mathbb{T}^2, \mathbb{R})\}$$

is defined by putting $\mathcal{L}_\beta(h)$ as the set of $c \in H^1(\mathbb{T}^2, \mathbb{R})$ for which the inequality above becomes equality. The α-Legendre transform

$$\mathcal{L}_\alpha : H^1(\mathbb{T}^2, \mathbb{R}) \to H_1(\mathbb{T}^2, \mathbb{R}) \tag{35}$$

as the inverse of $\mathcal{L}_\beta$. In what follows, we shall identify a h-minimal invariant probability with a c-minimal invariant probability, provided that $c \in \mathcal{L}_\beta(h)$. We call a *Mather set* $\mathcal{M}_h = supp\ \mu \subset \mathbf{T}\mathbb{T}^2$ with a rotation vector $h \in H_1(\mathbb{T}^2, \mathbb{R})$ support of an h-minimal probability μ.

We say that an absolutely continuous curve $\gamma : [a, b] \to \mathbb{T}^2$ is *c-minimal* if for some one form $[\eta] = c$ the curve minimizes

$$\int_a^b (L - \hat{\eta})(d\gamma(t), t)dt \tag{36}$$

subject to a fixed endpoint condition. In other words, we require that

$$\int_a^b (L - \hat{\eta})(d\gamma(t), t)dt \le \int_a^b (L - \hat{\eta})(d\gamma_1(t), t)dt \tag{37}$$

for any absolutely continuous curve $\gamma_1 : [a, b] \to \mathbb{T}^2$ such that $\gamma(a) = \gamma_1(a)$ and $\gamma(b) = \gamma_1(b)$. Note that γ and γ_1 does not have to be homologous and c-minimality is independent of the choose of the closed one form. We say that an absolutely continuous curve $\gamma : \mathbb{R} \to \mathbb{T}^2$ is *c-minimal* if every segment $[a, b] \subset \mathbb{R}$ gives a c-minimal curve subject to a fixed point condition $\gamma(a)$ and $\gamma(b)$.

5.3. **Mather sets in the phase space and Aubry-Mather sets on the invariant cylinder $\mathbb{A}_E$.** By duality between the Hamiltonian flow (3) and the Euler-Lagrange flow (22) we have that the time one map of the latter flow has an invariant cylinder in $\mathbf{T}\mathbb{T}^2$ which for simplicity we also denote by $\mathbb{A}_E$ as the dual one for the former flow. Denote by $\tilde{\mathbb{A}}_E \subset \mathbf{T}\mathbb{T}^2 \times \mathbb{T}$ the suspension of $\mathbb{A}_E$ by the flow (22). It is easy to see that $\tilde{\mathbb{A}}_E$ is locally diffeomorphic to $\mathbb{A}_E \times \mathbb{R}$.

Recall that $h_0 \in H_1(\mathbb{T}^2, \mathbb{R})$ denotes the homology class of the unique hyperbolic minimal geodesic $\Gamma \subset \mathbb{T}^2$ for the metric ρ chosen in Hypothesis 1. Let $\mathcal{N} \subset \mathbb{T}^2$ be a neighborhood of Γ. Denote by $l_{h_0} \subset H_1(\mathbb{T}^2, \mathbb{R})$ the ray in the space of homologies given by $\{h = Eh_0 :\ E \in \mathbb{R}_+\}$. Denote by $r_{h_0} = \mathcal{L}_\beta(l_{h_0})$ the image of this ray under the Legendre transform. Mather [Ma6] proved that

Lemma 8. *The Legendre image $r_{h_0} \subset H^1(\mathbb{T}^2, \mathbb{R})$ of the ray l_{h_0} contains a truncated cone, i.e. a cone centered at the origin with non-empty interior and without a large ball around the origin.*

If $h = E'h_0$ for some sufficiently large $E' \in \mathbb{R}_+$ and $c \in \mathcal{L}_\beta(h)$, then for any $c \in \mathcal{L}_\beta(h)$ any c-minimal curve $\gamma : \mathbb{R} \to \mathbb{T}^2$ has its image in $\mathcal{N}$.

This lemma can also be deduced from Bangert's result [Ba2] on (non-)differentiability on β-function or so-called stable norm. This lemma implies the following

Corollary 10. *For a cohomology class $c \in \mathcal{L}_\beta(l_{h_0})$ as above any c-minimal curve belongs to the suspended cylinder $\tilde{\mathbb{A}}_E \subset \mathbf{T}\mathbb{T}^2 \times \mathbb{T}$. The natural projection $\pi : \mathbf{T}\mathbb{T}^2 \times \mathbb{T} \to \mathbb{T}^2$ of support of a c-minimal probability μ is contained in $\tilde{\mathbb{A}}_E$.*

Proof: If we consider the time one map $\Phi : \mathbf{T}\mathbb{T}^2 \to \mathbf{T}\mathbb{T}^2$ of the Euler-Lagrange flow, then $\mathbb{A}_E$ is a hyperbolic invariant cylinder. Projection of this cylinder into $\mathbb{T}^2$ by scaling arguments (sect. 3.1.1) has to belong to a neighborhood $\mathcal{N}$ of Γ. Therefore, a c-minimal trajectory belongs to a neighborhood of the cylinder $\mathbb{A}_E$ in $\mathbf{T}\mathbb{T}^2$. But $\mathbb{A}_E$ is hyperbolic with 1-dimensional stable and unstable direction at every point of $\mathbb{A}_E$, so the only trajectories which stay in a neighborhood $\pi^{-1}(\mathcal{N})$ of $\mathbb{A}_E$ for all time are those which belong to $\mathbb{A}_E$. This proves the first part of the Corollary. To prove the second part recall support of a c-minimal probability is contained in a union of all c-minimal trajectories and, therefore, has to belong to $\tilde{\mathbb{A}}_E$. Q.E.D.

Corollary 11. *If $c, c' \in r_{h_0} \subset H^1(\mathbb{T}^2, \mathbb{R})$ and $\langle h_0, c\rangle = \langle h_0, c'\rangle$, then c-minimal orbits defined on all $\mathbb{R}$ are the same as c'-minimal orbits.*

Proof: This immediately follows from the fact that c and c'-minimal curves belong to $\mathcal{N}$. Q.E.D.

5.4. **A Variational Principle.** Choose a smooth curve S embedded in $\mathbb{T}^2$ that
$\cdot$ S does not intersect the periodic geodesic Γ for the metric ρ, i.e. S is topologically parallel to Γ, and
$\cdot$ S intersects homoclinic (Morse) geodesic Λ transversally at one point.

Denote $G = \{t \in \mathbb{T}^1 : G(t) = \min_{s\in\mathbb{T}^1} G(s)\}$ the set of moments of time when Melnikov integral (5) takes its minimal value.

We say that η (or $[\eta] \in H^1(\mathbb{T}^2, \mathbb{R}) \times \mathbb{R}$) is *subcritical, critical, or supercritical* according as $\int (L - \hat{\eta}) d\mu$ is positive, zero, or negative respectively. In other words, η is subcritical when $[\eta]_\mathbb{T} < (A(\mu) - [\eta]_{\mathbb{T}^2})$, critical when $[\eta]_\mathbb{T} = (A(\mu) - [\eta]_{\mathbb{T}^2})$, and supercritical when $[\eta]_\mathbb{T} > (A(\mu) - [\eta]_{\mathbb{T}^2})$ holds.

Given a closed one form η on $\mathbb{T}^2 \times \mathbb{T}$, which is critical and its homology class on $\mathbb{T}^2$ satisfies $\langle h_0, [\eta]_{\mathbb{T}^2}\rangle$, we define the variational principles $h^\pm_\eta : (S \times \mathbb{T}) \to \mathbb{R} \cup \{-\infty\}$ as follows:

$$h^+_\eta(\sigma, \tau) = \inf_\gamma \left\{ \liminf_{T\to+\infty} \int_{t_0}^T (L - \hat{\eta})(d\gamma(t), t)dt \right\}, \tag{38}$$

where the infimum is taken over the set of all absolutely continuous curves $\gamma : [t_0, +\infty) \to \mathbb{T}^2$ such that $t_0 \equiv \tau (\mathrm{mod}\ 1)$, $\gamma(t_0) = \sigma$, and $\gamma(t) \in \mathcal{N}$ for all large t.

Similarly,

$$h^-_\eta(\sigma,\tau)=\inf_\gamma\left\{\liminf_{T\to+\infty}\int_{-T}^{t_0}(L-\hat\eta)(d\gamma(t),t)dt\right\}, \tag{39}$$

where the infimum is taken over the set of all absolutely continuous curves $\gamma : (-\infty,t_0]\to\mathbb{T}^2$ such that $t_0\equiv\tau(\text{mod }1)$, $\gamma(t_0)=\sigma$, and $\gamma(-t)\in\mathcal{N}$ for all large t.

Mather's Fundamental Lemma. [Ma6] *The functions $h^\pm_\eta$ are finite and continuous and minima are achieved by minimizing trajectories $\gamma^+ : [0,+\infty)\to\mathbb{T}^2$ and $\gamma^- : (-\infty,0]\to\mathbb{T}^2$ respectively.*

Let η and η' be critical closed one forms such that $[\eta],[\eta']\in\mathcal{L}_\beta(l_{h_0})$, $|[\eta]|\,,|[\eta']|$ are sufficiently large, $|[\eta]-[\eta]|<|[\eta]|^{-2}$,[8] and η,η' coincide in a small neighborhood of $\{S\cap\Lambda\}\times G\subset\mathbb{T}^2\times\mathbb{T}$. Consider the following variational problem

$$h_{\eta,\eta'}=\inf_{(\sigma,\tau)\in S\times\mathbb{T}^1}\left(h^+_\eta(\sigma,\tau)+h^-_{\eta'}(\sigma,\tau)\right). \tag{40}$$

Then there is a trajectory $\gamma:\mathbb{R}\to\mathbb{T}^2$ of the Euler-Lagrange flow (22) such that it realizes the minimum for $h_{\eta,\eta'}$, $[\eta]$-minimal for any segment in $t\geq 0$ and $[\eta']$-minimal for $t\leq 0$ and

$$\begin{aligned}\operatorname{dist}(d\gamma(t),\mathcal{M}_{[\eta]})\to 0 \quad &\text{as } t\to+\infty\\ \operatorname{dist}(d\gamma(t),\mathcal{M}_{[\eta']})\to 0 \quad &\text{as } t\to-\infty.\end{aligned} \tag{41}$$

Moreover, γ intersects the Poincare section S only once and $\gamma(t)$ belongs to the neighbourhood $\mathcal{N}$ of Γ for all $|t|$ sufficiently large.

Consider the time one map $\Phi:\mathbf{T}\mathbb{T}^2\to\mathbf{T}\mathbb{T}^2$ of the Euler-Lagrange flow (22). Recall that by definition if a closed one form $\eta^*\in r_{h_0}\subset H^1(\mathbb{T}^2,\mathbb{R})$ is critical and $c^*=[\eta^*]_{\mathbb{T}^2}$, then the corresponding Mather set $\mathcal{M}_{c^*}$ belongs to the suspended invariant cylinder $\tilde{\mathbb{A}}_E\subset\mathbf{T}\mathbb{T}^2\times\mathbb{T}$. If we consider the time one map Φ, i.e. discretize (22), then the invariant set for Φ is the cylinder $\mathbb{A}_E\subset\mathbf{T}\mathbb{T}^2$. Moreover, if $h=\mathcal{L}_\alpha(c^*)$ is a dual homology class to c^*, then the restriction of the Mather set $\mathcal{M}_{c^*}$ to the time one map Φ gives an invariant set for the inner map $\mathcal{P}_E:\mathbb{A}_E\to\mathbb{A}_E$ which we denote by $\Sigma_\omega\subset\mathbb{A}_E$ with $h=\omega h_0$. So for each Mather set $\mathcal{M}_{c^*}$ there is an invariant set $\Sigma_{\omega(c^*)}$, which can be proved to be an Aubry-Mather for $\mathcal{P}_E$ (see [Ma4]), but we are not going to use this fact in general. We shall use it only in the case $\Sigma_{\omega(c^*)}$ is an invariant curve and $\mathcal{M}_{c^*}$ is an invariant two-torus, when this is straightforward.

Corollary 12. (Jump Lemma) *With the notations of Mather's Fundamental lemma and above we suppose that for some critical closed one forms η and η' with $c=[\eta]_{\mathbb{T}^2}$ and $c'=[\eta']_{\mathbb{T}^2}$ from r_{h_0} and $|c-c'|<|c|^{-2}$. Suppose also that η*

[8] 2 can be replaced by any $d\geq 1$

and η' coincide in a small neighborhood of $S \cap \Lambda \times G \subset \mathbb{T}^2 \times \mathbb{T}^1$ and there is $c^* \in r_{h_0}$ such that $\mathcal{M}_{c^*}$ is an invariant two-torus and

$$\langle h_0, c\rangle < \langle h_0, c^*\rangle < \langle h_0, c'\rangle. \tag{42}$$

Then there is a trajectory $\gamma : \mathbb{R} \to \mathbb{T}^2$ of the Euler-Lagrange flow (22) which
· crosses a small neighborhood of $\{S \cap \Lambda\} \times G$;
· is c-minimal for any segment $t \geq 0$ and c'-minimal for any segment $t \leq 0$;
· $\gamma(t)$ belongs to $\mathcal{N}$ for any $|t|$ sufficiently large.

Moreover, either the invariant set $\Sigma_{\omega(c')} \subset \mathbb{A}_E$ corresponding to $\mathcal{M}_{c'}$ belongs to a nonempty BRI $C \subset \mathbb{A}_E$ with the "top" frontier $\Sigma_{c''}$ satisfying $|c'' - c^*| > |c|^{-2}/2$ or $\Sigma_{\omega(c')}$ is an invariant curve itself.

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Current: Institute for Advanced Study, Einstein Drive, Princeton, NJ, 08540

On leave: Department of Mathematics 253-37, Caltech, Pasadena, CA 91125
E-mail address: `kaloshin@math.mit.edu, kaloshin@math.princeton.edu`