Invent. math. 151, 451–512 (2003)
DOI: 10.1007/s00222-002-0244-9

Inventiones mathematicae

The Existential Hilbert 16-th problem and an estimate for cyclicity of elementary polycycles

V. Kaloshin[⋆]

Department of Mathematics, MIT, BLDG 2-244, 77 Massachusetts Ave, Cambridge, MA 02139, USA (e-mail: kaloshin@math.mit.edu)

Oblatum 12-IX-2001 & 22-V-2002
Published online: 18 December 2002 –

Dedicated to the memory of my grandmother Eskina Anisia Urievna

1. Introduction

The original Hilbert 16-th problem consists of two parts: a "part a" about the classification of ovals defined by a polynomial equations $\{H(x, y) = 0\}$ and a "part b" about the limit cycles of of polynomial vector fields (see e.g. [R2]). In this paper we shall talk only about problems related to the second part. Consider a polynomial line field on the real (x, y)-plane

$$(1) \qquad \frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}, \ P, Q - \text{ polynomials }, \deg P, Q \le n.$$

The Existential Hilbert 16-th Problem (EHP). *Prove that for any $n \ge 2$ there exists a number $H(n) < \infty$ such that any polynomial line field of degree $\le n$ has at most $H(n)$ limit cycles (isolated periodic solutions).* [1]

The Existential Hilbert Problem is a weak version of the part b of the Hilbert 16-th problem which also asks not only about the number, but also about position of limit cycles of (1). The problem about finiteness of number of limit cycles for an *individual* polynomial line field (1) is called *Dulac problem*, since the pioneering work of Dulac [Du], who claimed in 1923 to solve this problem, but an error was found. The Dulac problem was solved by two independent and rather different proofs given independently

⋆ The author is partially supported by the Sloan Dissertation Fellowship and the American Institute of Mathematics Five-Year Fellowship

[1] For $n = 1$ linear vector fields have no limit cycles and $H(1) = 0$. Existence of $H(2)$ is an open question. See the series of papers [DRR,DMR,GR,RSZ,DIR] for partial results.

by Ilyashenko [I1] and Écalle [E]. However, both proofs do not allow any generalization to solve the Existential Hilbert Problem.

Consider the equation (1) for different polynomials $(P(x, y), Q(x, y))$ as the family of line fields on $\mathbb{R}^2$ depending on parameters of the polynomials. Using the central projection $\pi : \mathbb{S}^2 \to \mathbb{R}^2$ and homogeneity with respect to parameters of the equation (1) (line fields $\lambda P(x, y)/\lambda Q(x, y)$ and $P(x, y)/Q(x, y)$ for any $\lambda \neq 0$ are the same) one can construct a *finite parameter family of analytic line fields on the sphere* $\mathbb{S}^2$ *with a compact parameter base B (see e.g. [IY2] for details).* Usually this is called *Poincaré compactification*. After this reduction the Existential Hilbert Problem becomes a particular case of the following

Global finiteness Conjecture (GFC). *For any analytic family of line fields on* $\mathbb{S}^2$ *with a compact parameter base B the number of limit cycles is uniformly bounded over all parameter values.*

We refer the reader to the volumes [S], [IY2], the book [R2], and the recent survey [I3], where various development of these and related problems are discussed. Families of analytic fields are extremely difficult to analyze because of their rigidity and non-genericity. In the middle of 80's Arnold [AAIS], §3.2.8 proposed to consider generic families of smooth vector fields on $\mathbb{S}^2$ and stated several conjectures. One of them gave rise to a smooth analog of Global Finiteness Conjecture, so-called Hilbert-Arnold Problem. This problem is motivated in [I2] (see also [IK]) and can be formulated in the following way

Hilbert-Arnold Problem (HAP). *Prove that in a generic finite-parameter family of vector fields on the sphere* $\mathbb{S}^2$ *with compact base B, the number of limit cycles is uniformly bounded.*

Assume for a moment that a generic smooth vector field on the sphere $\mathbb{S}^2$ has an infinite number of limit cycles. By the Poincare-Bendixon Theorem, any limit cycle should surround an equilibrium point and, since our vector field has at most finitely many equilibria, there should be an infinite "nested" sequence around one of equilibria. Then those "nested" limit cycles have to accumulate (in the sense of Hausdorff metric) to a certain contour (polygon) consisting of equilibria (as vertexes) and separatric curves (as sides of that polygon) connecting them. Such objects are called *polycycles*. It turns out that a possible solution to Hilbert-Arnold Problem reduces to investigation of bifurcation of polycycles. Let us give several definitions.

Definition 1. *A polycycle* γ *of a vector field on the sphere* $\mathbb{S}^2$ *is a cyclically ordered collection of equilibrium points* $p_1, \dots, p_k$ *(with possible repetitions) and of distinct arcs* $\gamma_1, \dots, \gamma_k$ *(integral curves of the vector field) connecting them in the specific order: the j-th arc* γ_j *connects* p_j *with* $p_{j+1(mod\ k)}$ *for* $j = 1, \dots, k$.

Definition 2. *Let* $\{\dot{x} = v(x, \varepsilon)\}_{\varepsilon \in B^n},\ x \in \mathbb{S}^2$ *be an n-parameter family of vector fields on* $\mathbb{S}^2$ *having a polycycle* γ *for a critical parameter value* $\varepsilon_* \in B$. *The polycycle* γ *has cyclicity* μ *within the family* $\{v(x, \varepsilon)\}_{\varepsilon \in B^n}$ *if there exist neighborhoods U and V such that* $\mathbb{S}^2 \supseteq U \supset \gamma,\ B \supseteq V \ni \varepsilon_*$ *and for any* $\varepsilon \in V$ *the field* $v(\cdot, \varepsilon)$ *has no more than* μ *limit cycles inside U and* μ *is the minimal number with this property.*

Examples: 1) In a generic n-parameter family, the maximal multiplicity of a degenerate limit cycle does not exceed $n + 1$, e.g. in codimension 1 a semistable limit cycle has cyclicity 2. Thus, the cyclicity of a trivial polycycle (a polycycle without singular points) in a generic n-parameter family does not exceed $n + 1$.

2) (Andronow-Leontovich, 1930s; Hopf, 1940s). A nontrivial polycycle of codimension 1 has cyclicity at most 1.

3) (Takens, Bogdanov, Leontovich, Mourtada, Grozovskii, early 1970s-1993 (see [G], [KS] and references there)). A nontrivial polycycle of codimension 2 has cyclicity at most 2.

Definition 3. *The bifurcation number* $B(k)$ *is the maximal cyclicity of a nontrivial polycycle occurring in a generic k-parameter family.*

The definition of $B(k)$ does not depend on a choice of the base of the family, it depends only on the number of parameters k.

Local Hilbert-Arnold Problem (LHAP). ([I2]) *Prove that for any finite k, the bifurcation number* $B(k)$ *is finite and find an upper estimate for* $B(k)$.

Compactness arguments due to Roussarie [R1] show that a solution to Local Hilbert-Arnold Problem implies a solution to Hilbert-Arnold Problem.

Similarly to the generic smooth vector fields, in the case of analytic vector fields one can define so-called *limit periodic set* [FP], [R1], [P], [IY1]. A limit periodic set might be either a polycycle, or have arcs γ_j's of equilibrium points or even more comlicated. In general structure of limit sets even for analytic families is not known (see [R2], Chap. 6 for more). Notice a generic vector fields can not have an arc of equilibrium points. Similarly to Definition 2 one can define cyclicity of a limit set and formulate

Finite Cyclicity Conjecture (FCC). ([R1]) *Prove that any limit periodic set of an analytic family of line fields on* $\mathbb{S}^2$ *has a finite cyclicity in this family*.

Smooth vector fields are more flexible then analytic vector fields and easier to analyze. A strategy to attack Existential Hilbert Problem proposed by Ilyashenko [I2] (see also [IK]) is first understand generic smooth vector fields and then try to apply developed methods to analytic vector fields.

We stress out that a solution to (Local) Hilbert-Arnold Problem would not necessarily solve (Finite Cyclicity) Global Finiteness Conjecture, because study of generic vector fields does not include study of non-generic vector fields and non-generic limit sets which are unavoidable in analytic setting. However, it might give some insight. Now we shall formulate the Main Result of the paper which is a solution of a particular case of Local Hilbert-Arnold Problem.

Definition 4. *A singular (equilibrium) point of a vector field on the two-sphere is called elementary if at least one eigenvalue of its linear part is nonzero. A polycycle is called an elementary polycycle if all its singularities are elementary.*

The elementary bifurcation number $E(k)$ is the maximal cyclicity of a nontrivial elementary polycycle occurring in a generic k-parameter family.

A k-parameter family of smooth vector fields on the two-sphere is called generic if it belongs to an open set of k-parameter families on the two-sphere.

From examples 2) and 3) above it follows that

$$E(1) = 1, \quad E(2) = 2.$$

Trifonov [Tr2] (see also [IK]) summarizes activity of a number of mathematicians which lead to the estimate $E(3) = 3$. The main result of the paper is a bound on the rate of growth of $E(k)$ as k tends to infinity. The first crucial step was done by Ilyashenko and Yakovenko:

Finiteness Theorem. ([IY3]) *For any k the elementary bifurcation number $E(k)$ is finite.*

Corollary 1. *Under the assumption that families of vector fields have elementary singularities only the global Hilbert-Arnold conjecture is solved, i.e. any generic finite parameter family of vector fields on the sphere $\mathbb{S}^2$ with a compact base and only elementary singularities has a uniform upper bound for the number of limit cycles.*

Main Theorem. *For any $k \in \mathbb{Z}_+$*

$$E(k) \leq 2^{25k^2}. \tag{2}$$

This is the first known estimate for cyclicity of polycycles of arbitrary large codimension and was announced in [Ka3] and [IK]. The case of a polycycle consisting of only one singular point with no arcs at all, is well known. An elementary equilibrium point can generate limit cycles in its small neighborhood if it is a slow focus, that is the linearization matrix has a pair of two imaginary eigenvalues. This bifurcation was investigated by Takens [Ta] and cyclicity is proved to be bounded by codimension.

Corollary 2. *Under the assumption that all the polycycles are elementary the Main Theorem gives a solution to the Local Hilbert-Arnold problem.*

The Main Theorem is an improvement of Ilyashenko-Yakovenko Finiteness Theorem. It is a great pleasure for the author to say that the paper of Ilyashenko-Yakovenko [IY3] was a cornerstone for the present paper. In [IY3] the authors made an extremely important step: they found a pass from *bifurcation theory* to *singularity theory* using the Khovanski reduction method [Kh]. Earlier application of the Khovanski reduction method for this kind of problems was proposed by Moussu-Roche [MR]. We follow this pass at the beginning and then using some new ideas get the estimate (2) for cyclicity of polycycles. To make this paper readable we have to reproduce some fragments from [IY3].

1.1. **From the Main Theorem to the Local Hilbert-Arnold Problem (Resolution of Singularities (RS) and Singular Perturbations (SP)).** The Finiteness and Main Theorems seems to be an important step toward the Local Hilbert-Arnold Problem is a view of the following theorem. A C^∞ smooth vector field on $\mathbb{S}^2$ has an equilibria of finite order or satisfying *a Lojasiewicz condition* at zero, i.e. $\dot{x} = v(x)$, $v(0) = 0$ if $\|v(x)\| \geq c\|x\|^k$ for some $c > 0$, $k \in \mathbb{Z}_+$, and all small x. It is shown in [D] that any generic finite-parameter family of vector fields on $\mathbb{S}^2$ has only vector fields with equilibrium points satisfying a Lojasiewicz condition. For these kind of equilibrium points Bendixon-Seidenberg-Dumortier Theorem [D] says that after a finite sequence of *resolution of singularities (RS) or blow ups* one gets a vector field with only *elementary equilibria*. This reduces *an individual* vector field, occurring in a generic finite-parameter family, with arbitrary equilibria to an individual vector field with only *elementary* equilibria. However, this Theorem is for an *individual* vector field, not for a *family* of vector fields which makes an application to families impossible[2].

Different approaches to extending Bendixon-Seidenberg-Dumortier Theorem to families of vector fields were proposed by Denkowska-Roussarie [DeR] and by Trifonov [Tr1]. An approach proposed by Trifonov leads to the dynamical phenomenon called *Singular Perturbation* (SP): in the simplest case one needs to analyze families of vector fields on the plane, which for some values of parameters have a curve of equilibria. Certainly, a generic finite-parameter family of vector fields has no curve of equilibria, however, after even one step of blow-up such a curve can occur even for a generic family [Tr1]. Appearance of curves of equilibria after a desingularization in a family now seems to be *the main obstacle* between the Main Theorem and Local Hilbert-Arnold Problem (see [Tr1], [IY2], and [R2] for more). Let us summarize the discussion in the form of the following diagram:

[2] A quantitative version of Bendixon-Seidenberg-Dumortier Theorem with estimates number of necessary blow-ups was obtained by Kleban [Kl]

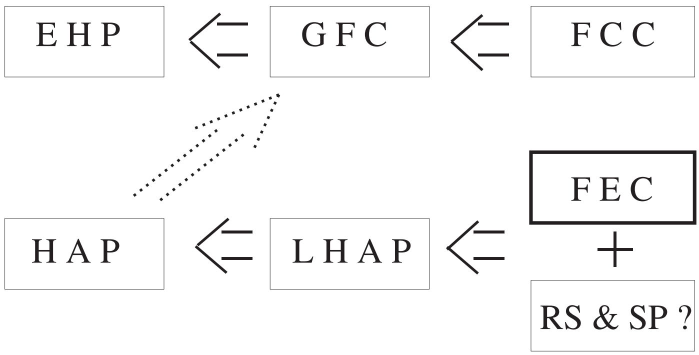

Fig. 1. Problems around Existential Hilbert 16-th

1.2. **Geometric Multiplicity of generic germs.** As the byproduct of the machinery developed to prove the Main Theorem we obtain an estimate on multiplicity of germs of generic maps, which might have an independent interest and gives an answer to the problem about multiplicity of germs by Arnold [A].

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a generic C^k–smooth map, $k \geq n+1$. Fix a point $a \in \mathbb{R}^n$ and denote $F(a)$ by b.

Definition 5. *The geometric multiplicity of a map germ $F : (\mathbb{R}^n, a) \to (\mathbb{R}^n, b)$ at a, denoted by $\mu_a^G = \mu_a^G(F)$, is the maximal number of regular preimages $F^{-1}(\tilde{b})$ close to a:*

$$\mu_a^G(F) = \limsup_{r\to 0} \sup_{\tilde{b}\in\mathbb{R}^n} \#\{x \in B_r(a) : F(x) = \tilde{b},\ \text{rank}\ dF(x) = n\}. \tag{3}$$

For example, the geometric multiplicity of the real-valued function $f : x \to x^2$ at 0 is two, but the geometric multiplicity of the real-valued function $f : x \to x^3$ at 0 is one, even though 0 is a degenerate point of the second order.

In the complex case the geometric multiplicity equals the usual multiplicity (see [AGV]). In the real case the first is no greater than the second. Define geometric multiplicity of n-dimensional germs, $\mu^G(n)$, as follows.

Definition 6. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a generic map. The geometric multiplicity of F equals the least upper bound of geometric multiplicities of $\mu_a^G(F)$ taken for all points $a \in \mathbb{R}^n$. Then the geometric multiplicity of n–dimensional germs is the maximum of the geometric multiplicities of all generic maps F from $\mathbb{R}^n$ to $\mathbb{R}^n$*

$$\mu^G(n) = \sup_{F-generic,\ a\in\mathbb{R}^n} \mu_a^G(F). \tag{4}$$

It turns out that the geometric multiplicity of n-dimensional germs is finite for all positive integer n and depends only on the dimension n.

Remark 1. For example, if $n = 1$, then $\mu^G(1) = 2$, because a generic function has only non-degenerate critical points. If $n = 2$ the Whitney Theorem about maps of surfaces states that a generic map of two dimensional manifolds $F : M^2 \to N^2$ can have only three different types of germs: 1-to-1, a fold, and a pleat (see e.g. [AGV]). This implies that $\mu^G(2) = 3$.

A natural problem, posed by Arnold [A] is
Give estimates for the geometric multiplicity $\mu^G(n)$ of n-dimensional germs.
As the reader will see the problem on an upper estimate is closely related to the problem to estimate cyclicity of elementary polycycle. The analytic case was considered by [GK]. The upper bound for the geometric multiplicity for n-dimensional smooth germs of generic maps is given by the following

Theorem 1. *The geometric multiplicity of germs of a generic C^k smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$, $k \geq n+1$ admits the following upper estimate:*

$$\mu_a^G(F) \leq 2^{n(n-1)/2+1} n^n, \ \forall\, a \in \mathbb{R}^n. \tag{5}$$

Using the same method one can prove

Theorem 2. *Let $F : \mathbb{R}^n \to \mathbb{R}^N$ be a generic C^k smooth map with $k \geq n+1$, $N \geq n$ and $P : \mathbb{R}^N \to \mathbb{R}^n$ be a polynomial of degree d. Then the geometric multiplicity of germs of a chain map $P \circ F : \mathbb{R}^n \to \mathbb{R}^n$ admits the following upper estimate:*

$$\mu_a^G(P \circ F) \leq 2^{n(n-1)/2+1} (dn)^n, \ \forall\, a \in \mathbb{R}^n. \tag{6}$$

First, notice that the outer polynomail P has to be nontrivial, i.e. having an open image $P(\mathbb{R}^N)$ or having regular points, where rank $dP = n$. Otherwise, rank $dP \geq$ rank $d(P \circ F)$ and $P \circ F$ can't have regular points at all. An interesting feature of this theorem is that the geometric multiplicity does not depend on dimension N of the intermediate space and the outer polynomial P. Only of degree of P. Notice also that these estimates are a little better than announced in [Ka3].

1.3. **Three stages of the proof.** The proof of the Main Theorem consists of three steps. Relation to the proof of the Finiteness Theorem [IY3] is discussed after this short description.

Step 1. Normal forms for local families of vector fields and their integration. In Sect. 2 we use normal forms to establish an explicit form for the Poincare correspondence map near equilibrium points on the polycycle under consideration. In [IY3] it is shown that these maps satisfy Pfaffian (polynomial differential) equations with polynomial coefficients depending smoothly on the parameters of the family. As the result a *basic system* of equations for determination of limit cycles is obtained.

Step 2. The Khovanski reduction method. In Sect. 3 we discuss a variation of the Khovanski method [Kh]. This method allows us to investigate systems of equations that involve singular functions satisfying Pfaffian equations. In Sect. 4 we present a formal reduction from the basic system to a *mixed functional-Pfaffian* system[3], which is also done in [IY3], and extract upper bounds for the degrees of the involved into the procedure polynomials. After application of the Khovanski method to the mixed functional-Pfaffian system we obtain several *chain maps*, the maps of the form

$$x \mapsto (P_1, \ldots, P_n) \circ (x, f(x), f'(x), \ldots, f^{(n)}(x)), \tag{7}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is a point, $P = (P_1, \ldots, P_n)$ is a vector-polynomial given by its coordinate functions of known degree, and

$$\begin{aligned} f(x) &= (f_1(x_1), \ldots, f_n(x_n)), \\ f^{(k)}(x) &= \big(f_1^{(k)}(x_1), \ldots, f_n^{(k)}(x_n)\big) \text{ for } k = 1, \ldots, n \end{aligned} \tag{8}$$

is a generic vector-function of the form (8, line 1) along with its jet of order n. The problem of estimating the number of limit cycles reduces to estimating the number of regular preimages of some special points by the chain map. Special points form an open cone-like semialgebraic set $\mathbf{K}^n$ in the image $\mathbb{R}^n$.

Denote by F the map $F : x \mapsto (x, f(x), f'(x), \ldots, f^{(n)}(x))$ which is called the n-th jet of f. Denote by L_F the linearization of F at the point $x = 0$.

Step 3. Bezout's Theorem for the Chain maps. In Sect. 7 we construct an algebraic set Σ_P in the image of F (in the space of n-jets). If F is transversal to Σ_P, then the number of preimages of any point a from the set of special points $\mathbf{K}^n$ is *the same* for F and its linearization L_F at zero, namely,

$$\#\{x : P \circ F(x) = a\} = \#\{x : P \circ L_F(x) = a\} \le \prod_{j=1}^{n} \deg P_j. \tag{9}$$

If L_F is a linear map transversal to Σ_P, then one can apply Bezout's Theorem to estimate the right-hand side of the equality. This observation completes the proof of the Main Theorem.

First of all we point out that this paper can be considered as a new independent proof of Ilyashenko-Yakovenko Finiteness Theorem [IY3] by modulo of deriving functional-Pfaffian system (Step 1). Application of the Khovanski method at Step 2 here compare to [IY3] also has new important features. In [IY3] after application of the Khovanski method the authors obtain the same collection of chain maps of the form (7). However, they investigate the number of regular preimages of points in the image by the chain maps *without any restriction* on those points. In the present proof,

[3] Application of the Khovanski method to this kind of problems was first proposed in [MR]

using additional arguments in the Khovanski method, we reduce consideration to only preimages of *special* points, i.e. points from an open cone-like set $\mathbf{K}^n$ in the image. At this point our proof goes independently, because investigation of the number of regular preimages of special points is a more concrete problem and it allows application of stratification theory.

Let us present a more detailed description of each step of the proof.

1.4. **Normal forms of local families and their integration.** This step is done in [IY3] § 0.3 and § 1. We just say several words about it.

It turns out that in a small neighborhood of an elementary equilibrium point there exists a finitely differentiable normal coordinates (in the Cartesian product of the phase space and the parameter space), so-called normal forms of an equilibrium point. The list of finitely differentiable normal forms was obtained in [IY1]. The main feature of the list: all normal forms are *polynomial and integrable.* The smaller is the neighborhood of a normal form, the higher is its smoothness. So smoothness can be chosen arbitrary large. All normal forms are summarized in Table 1 Sect. 2.

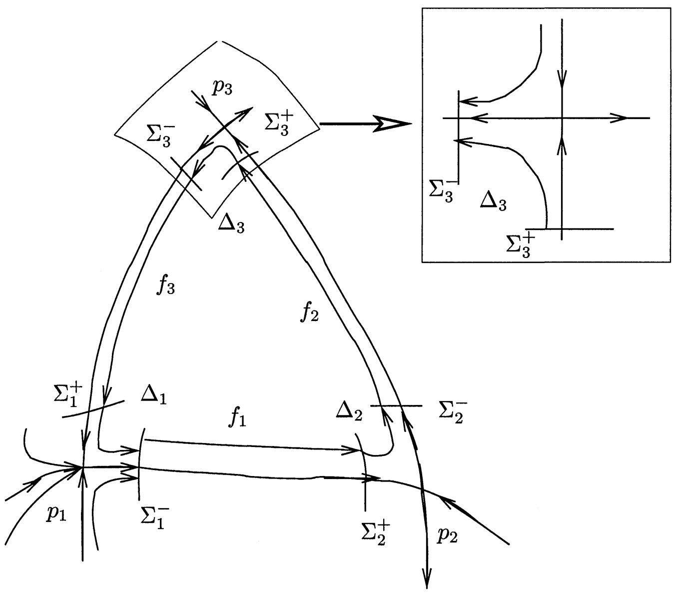

Fig. 2. Construction of entrance and exit transversals

In a small neighborhood of an elementary equilibrium point one can choose two small segments, say Σ^+ and Σ^-, transversal to the vector field for the critical value of parameter and explicitly calculate the Poincare (correspondence) map which maps a point from one segment, say Σ^+, along the corresponding phase curve to a point from the other segment Σ^- (see Fig. 2). For an appropriate choice of segments Σ^+, Σ^- and coordinate functions x, y in Σ^+, Σ^- respectively, and a smooth function $\lambda(\varepsilon)$ in the original parameter ε of the family of the Poincare return map $\Delta_\varepsilon : x \to y$ can be explicitly computed. Moreover, there is a Pfaffian 1-form ω (with polynomial coefficients) of the form

$$\omega = P(x, y, \lambda(\varepsilon))\, dx + Q(x, y, \lambda(\varepsilon))\, dy \tag{10}$$

which vanishes on the graph $y = \Delta_\varepsilon(x)$. For example, in the case of a nonresonant saddle $\Delta_\varepsilon(x) = x^{\lambda(\varepsilon)}$ and $\omega = x\, dy + \lambda(\varepsilon) y\, dx$. See Table 1 for the other cases.

1.5. **Singular-regular systems determining the number of limit cycles.** We present a description of a system of equations determining the number of limit cycles. For a detailed description we refer to [IY3] § 0.4 and § 1.4.

Let γ be a polycycle, occurring in a generic k-parameter family, with equilibrium points $p_1, \dots, p_n$ (possibly with repetitions) and connecting phase curves $\gamma_1, \dots, \gamma_n$ (without repetitions) such that γ_j connects equilibria p_j with $p_{j+1 \pmod n}$ respectively. For each $1 \le j \le n$ endow the point p_j with a *C^r-normal coordinate charts U_j*. Consider transversal segments "entrance" Σ_j^+ and "exit" Σ_j^- which are parallel to coordinate axis of the normal chart. The phase curve γ_{j-1} enters the neighborhood U_j through Σ_j^+ and the phase curve γ_j exists U_j through Σ_j^-. The normal coordinates induce coordinates x_j and y_j on Σ_j^+ and Σ_j^- respectively. For some parameter values the corresponding vector field defines the following collection of the Poincare maps:

$$\begin{aligned} &\Delta_j(\cdot, \varepsilon) : x_j \to y_j = \Delta_j(x_j, \varepsilon), \quad j = 1, \dots, n \\ &f_j(\cdot, \varepsilon) : y_j \to x_{j+1} = f_j(y_j, \varepsilon),\ j = 1, \dots, n \pmod n, \end{aligned} \tag{11}$$

where $\Delta_j(\cdot, \varepsilon)$ is a local Poincare map from the "entrance" segment Σ_j^+ to the "exit" segment Σ_j^- and $f_j(\cdot, \varepsilon)$ is a semilocal Poincare map along the phase curve γ_j from the "exit" segment Σ_j^- to the "entrance" segment $\Sigma^+_{j+1 (\mathrm{mod}\ n)}$.

Now we decompose the monodromy map (the Poincare first return map) along the polycycle γ into the chain of the local singular maps Δ_j and the semilocal regular maps f_j of the total length $2n$. Limit cycles correspond to the fixed points of the monodromy. But instead of writing one equation for the fixed points of the monodromy we consider a system of $2n$ equations, which will be called the preliminary basic system:

$$\begin{cases} y_j = \Delta_j(x_j, \varepsilon), & j = 1, \dots, n \\ x_{j+1} = f_j(y_j, \varepsilon), & j = 1, \dots, n \pmod n. \end{cases} \tag{12}$$

Recall that x_j's are C^r-normal coordinates on Σ_j^+ and y_j's are C^r-normal coordinates on Σ_j^-. Thus the system involves C^r-smooth regular functions f_j's and the maps Δ_j from Table 1 below (modulo reparametrization $\varepsilon \to \lambda(\varepsilon)$), that are essentially singular. The problem now is to estimate the number of solutions uniformly over all sufficiently small parameter values.

1.6. **The Khovanski reduction method.** The system (12) is not easy to analyze, because it has the singular functions Δ_j. Singularity of Δ_j's appears e.g. in the fact that derivatives of $\Delta_j(0, \varepsilon)$'s might tend to infinity as ε tends to zero. Also some Δ_j's are given by implicit formulas, which are difficult to compute explicitly (see e.g. Table 1 the case S_μ). The first key idea of the second step is to replace these singular equations $y_j = \Delta_j(x_j, \varepsilon)$ in (12) by the Pfaffian (polynomial differential) equations of the form (10). As a result we obtain the *mixed* functional-Pfaffian system of the form

$$
\begin{gathered}
\begin{cases} \omega_j = 0 \\ F_j(x, y, \varepsilon) = 0 \qquad\qquad j = 1, \dots, n \end{cases} \\
\omega_j = P_j\, dx_j + Q_j\, dy_j, \quad F_j(x, y, \varepsilon) = x_{j+1} - f_j(y_j, \varepsilon) \\
(x, y) = (x_1, y_1, \dots, x_n, y_n) \in (\mathbb{R}^{2n}, 0),\ \varepsilon \in (\mathbb{R}^k, 0),
\end{gathered} \tag{13}
$$

where ω_j are Pfaffian forms in the form (10). This system can be interpreted as follows: one has to take an integral manifold Γ for the Pfaffian equations of the system (13) and compute its intersection with the level set $F^{-1}(0)$, where $F : (\mathbb{R}^{2n}, 0) \to \mathbb{R}^n$ is the map with the coordinate functions F_j. In order to estimate the number of isolated solutions to (12) one needs to estimate the number of isolated points in the intersection. It turns out that *it is sufficient to analyze only transversal intersections* of Γ with a generic level set $F^{-1}(b)$ for b sufficiently close to the origin in $\mathbb{R}^n$. Since the integral manifold and the level sets have complementary dimensions, a transversal intersection always consists of isolated points, which are called *regular solutions* to the system (13). What we are interested in is an upper estimate for their number, uniform over all the integral manifolds Γ and all sufficiently small values of the parameters.

The method suggested by Khovanski [Kh] allows us to replace a mixed functional-Pfaffian system of the form (13) by two auxiliary systems of a similar form with the number of Pfaffian equations reduced by 1. Moreover, the number of regular solutions to the initial system is bounded from above by the sum of the number of regular solutions to these two auxiliary systems.

1.7. a_P**-stratification and Bezout's Theorem for a chain map** $P \circ F$ **with a generic** F**.** In this section we shall discuss the formula (9). The problem of estimating geometric multiplicity (the maximal number of small isolated preimages) is equally difficult for a chain map $P \circ F : \mathbb{R}^n \to \mathbb{R}^n$ with a generic map $F : \mathbb{R}^n \to \mathbb{R}^N$, $N \geq n$ and for a chain map $P \circ j^n F : \mathbb{R}^n \to \mathbb{R}^n$ with the n-jet of a generic map. We shall prove that if the map F (resp. $j^n F$) satisfies a transversality condition in an appropriate space, then F (resp. $j^n F$) can be replaced by its linear part L_F (resp. $L_{j^n F}$), which means that geometric multiplicity of $P \circ F$ (resp. $P \circ j^n F$) and $P \circ L_F$ (resp. $P \circ L_{j^n F}$) is the same for all special points in the image $\mathbf{K}^n \subset \mathbb{R}^n$. After F (resp. $j^n F$) is replaced by its linear part we can apply Bezout's Theorem and get an estimate for geometric multiplicity of the chain map $P \circ F$ (resp. $P \circ j^n F$) uniformly over all special points. Special points form a cone-like set $\mathbf{K}^n = \{(a_1, \dots, a_n) \in \mathbb{R}^n : 1 \gg |a_1| \gg \cdots \gg |a_n| \geq 0\}$. This set is mentioned in Sect. 1.3 and will be defined in Sect. 6, Def. 28. So, to simplify notations in this section we shall consider only a chain map of the form of composition of a polynomial P and a map F (not its jet) $P \circ F : \mathbb{R}^n \to \mathbb{R}^n$.

1.7.1. A Heuristic description. Consider a chain map $P \circ F : \mathbb{R}^2 \to \mathbb{R}^2$, where $F : \mathbb{R}^2 \to \mathbb{R}^N$ is a generic C^k-smooth map, $k > 2$ and $P = (P_1, P_2) : \mathbb{R}^N \to \mathbb{R}^2$ is a polynomial of degree d with $P(0) = 0$. Fix a small positive r. We would like to estimate the maximal number of small regular preimages

$$\#\{x \in B_r(0) : P_1 \circ F(x) = \varepsilon,\ \ P_2 \circ F(x) = 0\} \tag{14}$$

for a small enough ε.

To exhibit the idea we put $N = 3$, $P_1(x, y, z) = x^2 + y^2$, and $P_2(x, y, z) = xy$. Assume also that $F(0) = 0$. Denote the level set by $V_\varepsilon = \{P_1 = \varepsilon, P_2 = 0\}$. The level set V_ε for $\varepsilon > 0$ consists of 4 parallel lines (see Fig. 3).

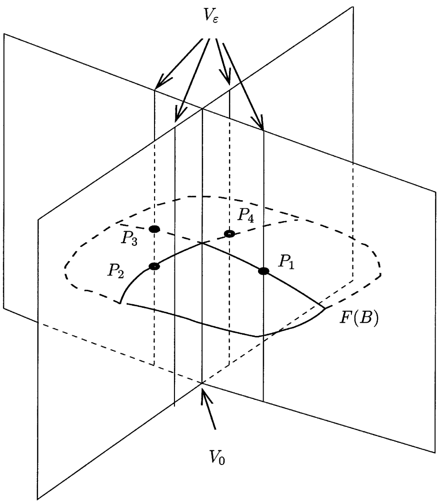

Fig. 3. The idealistic example

Notice that in our notation the number of intersections of $F(B_r(0))$ with V_ε equals the number of preimages of the point $(\varepsilon, 0)$ (14).

It is easy to see from Fig. 3 that if F is transversal to V_0, then is to V_ε too for any small $\varepsilon > 0$. Moreover, the number of intersections $F(B_r(0))$ with V_ε equals 4 (see the points $P_1, \dots, P_4$ in Fig. 3).

Another way to calculate the same number is as follows. Let us replace F by its linear part L_F at zero. Then $\#\{x \in B_r(0) : P_1 \circ F(x) = \varepsilon, P_2 \circ F(x) = 0\} = \#\{x \in B_r(0) : P_1 \circ L_F(x) = \varepsilon,\ P_2 \circ L_F(x) = 0\}$ and solving this polynomial system also yields 4.

The idea behind this picture is the following: Consider an arbitrary $N > 2$ and a polynomial $P = (P_1, P_2) : \mathbb{R}^N \to \mathbb{R}^2$ of degree at most d. Define the 1-parameter family semialgebraic varieties $V_\varepsilon = (P_1, P_2)^{-1}(\varepsilon, 0)$ as the level set of P for $\varepsilon \neq 0$ and $V_0^* = \lim_{\varepsilon \to 0} V_\varepsilon$ as the Hausdorff limit of V_ε's. By the Tarsky-Seidenberg principle V_0^* is semialgebraic [J].

Assume for simplicity that for any small $\varepsilon \neq 0$ the level set V_ε is a manifold of codimension 2. We shall get rid of this assumption later (see Theorem 15 b). It turns out that there exists a partition of V_0 into disjoint smooth semialgebraic parts $V_0^* = \cup_{\alpha \in \mathcal{A}} V_\alpha,\ \mathcal{V}_0 = \{V_\alpha\}_{\alpha \in \mathcal{A}}$ (see Definition 23), depending only on P only with the following property. If F is transversal to each part V_α for $\alpha \in \mathcal{A}$, denoted by F is transversal to $(V_0^*, \mathcal{V}_0)$, then F is transversal to V_ε for any small $\varepsilon \neq 0$.

$$\text{(15)} \quad \boxed{F \quad \text{transverse} \quad (V_0^*, \mathcal{V}_0)} \qquad \Longrightarrow \qquad \boxed{F \quad \text{is transversal to} \quad V_\varepsilon}$$

Condition (15) is written for 2-dimensional image $n = 2$, but it works in general for any n. We present a Proposition which is one of the keys to the proof of Bezout's Theorem for chain maps.

Proposition 1. *Let $B_r(0)$ be the r-ball centered at the origin $0 \in \mathbb{R}^2$ and let L_F denote the linearization of F at 0. Then F satisfying (15) for a sufficiently small r the number of intersections of the image $F(B_r(0))$ with V_ε coincides with the number of intersections of the image $L_F(B_r(0))$ with V_ε, i.e.*

$$\begin{aligned} \#\{x \in B_r(0) : (P_1, P_2) \circ F(x) = (\varepsilon, 0)\} = \\ \#\{x \in B_r(0) : (P_1, P_2) \circ L_F(x) = (\varepsilon, 0)\}. \end{aligned} \tag{16}$$

Remark 2. Argument below is independent of codimension of $V_\varepsilon = (P_1, P_2)^{-1}(\varepsilon, 0)$. We only need condition (15) and the fact that codimension of V_ε coincides with dimensions of the image and the preimage of a chain map $P \circ F$.

Proof. Consider the 1-parameter family of maps $F_t = tF + (1 - t)L_F$ deforming the linear part of F into F. Clearly, $F_1 \equiv F$ and $F_0 \equiv L_F$. Fix a small $r > 0$. Since F is transversal to V_0 at 0 all F_t are transversal to V_0 at 0. Condition (15) implies that for any sufficiently small ε and all $t \in [0, 1]$ we have F_t is transversal to V_ε. Since the segment $[0, 1]$ is compact, simple arguments by contradiction show that we have uniformity in t.

Therefore, *the number of intersections of $F_t(B_r(0))$ with V_ε is independent of t.* Indeed, assume that $\#\{F_{t_1}(B_r(0)) \cap V_\varepsilon\} \neq \#\{F_{t_2}(B_r(0)) \cap V_\varepsilon\}$ for some $t_1 < t_2$. Then as t_1 goes up to t_2 there is a point t^* where the number of intersections drops or jumps. At this point t^* the condition of transversality of F_{t^*} and V_ε must fail. This contradiction completes the proof of the proposition. Q.E.D.

Now we can formulate Bezout's Theorem for chain maps. Let $B_r = \{x \in \mathbb{R}^n : |x_j| < r\}$ be the r-ball, $B = B_1$, $f : B \to \mathbb{R}^s$ be a C^k-smooth map, and $j^m f : B \to J^m(\mathbb{R}^n, \mathbb{R}^s)$ be the m-jet of f for some positive integers m, n, s, and $k > n + m$. Fix a coordinate system in $\mathbb{R}^n$. For any $\delta > 0$ and $d \in \mathbb{Z}_+$ we call a cone-like semialgebraic set

(17)

$$K^n_{d,\delta} = \left\{a \in \mathbb{R}^n : 0 < a_1 < \delta,\ 0 < a_{j+1} < (a_1 \dots a_j)^d \text{ for } 1 \le j \le n-1\right\}$$

the (d, δ)-cone. We also call a polynomial $P = (P^1, \dots, P^n) : J^m(\mathbb{R}^n, \mathbb{R}^s) \to \mathbb{R}^n$ nontrivial if the image $P(J^m(\mathbb{R}^n, \mathbb{R}^s))$ contains an open set in $\mathbb{R}^n$.

Theorem 3. *Consider a nontrivial polynomial* $P = (P^1, \dots, P^n) : J^m(\mathbb{R}^n, \mathbb{R}^s) \to \mathbb{R}^n$. *Then for any* $k > n+m$ *there is an open dense subset of maps* $\mathsf{F}_P \subset \mathbf{C}^k(U, \mathbb{R}^s)$ *and an integer* $d = d(P)$ *such that for each* $f \in \mathcal{F}_P$ *there exists a characteristic size* $r > 0$ *and* $\delta > 0$ *for which*

A) if the m*-jet* $j^m f(x)$ *is replaced by its linearization* $L_{0,j^m f}(x)$ *at zero, then*

$$\#\{x \in B_r : P \circ j^m f(x) = a\} = \#\{x \in B_r : P \circ L_{0,j^m f}(x) = a\} \tag{18}$$

for any $a \in K^n_{d,\delta}$.

B) part A) along with Bezout's Theorem implies estimate on multiplicity of the composition $P \circ j^m f : \mathbb{R}^n \to \mathbb{R}^s$ *of the form:*

$$\sup_{a \in K^n_{d,\delta}} \#\left\{x \in B_r : P \circ j^m f(x) = a\right\} \le \prod_{i=1}^n \deg P^i. \tag{19}$$

1.8. **Guide to the proof.** In this section we describe order and relation among different Theorems proved in this paper and what is the strategy to prove the Main Theorem.

Recall that the Main Theorem gives an estimate (2) for maximal cyclicity of an elementary polycycle of codimension k, denoted by $E(k)$. To prove this result let's fix an elementary polycycle γ of codimension k. In Sect. 2 we exhibit how to encode information about γ into symbols $(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}; r_0)$ and define a basic system, highlighted in Sects. 1.4, 1.5, and (12) and rigorously defined in Definition 8. The number of isolated solutions of the basic system (12) is denoted by $\mathcal{B}_k(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}; r_0)$. Theorem 5 states that

$$E(k) \le \max_\gamma \mathcal{B}_k(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}; r_0), \tag{20}$$

where the maximum is taken over all elementary polycycles of codimension k.

In Chap. 3 we describe the Khovanski method and in Chap. 4 we apply this method to the basic system, defined by the symbol $(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}_\gamma; r_0)$. The first main result of such an application is Theorem 10 is that the number of isolated solutions of the basic system is bounded by the Khovanski number (or the number of isolated solutions) of mixed functional-Pfaffian system satisfies

$$(21)\qquad \mathcal{B}_k(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}_\gamma; r_0) \le \mathcal{K}_{M(r)}\{\Omega_\gamma, F_\gamma; \mathcal{F}_{\mathbf{f}_\gamma} = a\} + k,$$

where $\Omega_\gamma = 0, F_\gamma = b; \mathcal{F}_{\mathbf{f}_\gamma} = a$ denotes a mixed functional-Pfaffian system, defined in Sect. 3.1. The second main result is Theorem 11. It gives an estimate for the Khovanski number of $\mathcal{K}_{M(r)}\{\Omega_\gamma, F_\gamma; \mathcal{F}_{\mathbf{f}_\gamma} = a\}$ by the sum of the Khovanski numbers of the chain maps, defined by (70), i.e.

$$(22)\qquad \mathcal{K}_{M(r)}\{\Omega_\gamma, F_\gamma; \mathcal{F}_{\mathbf{f}_\gamma} = a\} \le \sum_{i=0}^{\mathbf{m}} \mathcal{K}_{M(r)}\left(\mathbf{P}^i\left(\mathbf{j}^{\mathbf{m}}_{\mathcal{I}}\mathbf{f}_\gamma\right) = a^*\right),$$

where $\mathbf{m} \le 6k$ and $a^* = (a_1, \dots, a_{\mathbf{m}})$ satisfies $1 \gg |a_1| \gg \cdots \gg |a_{\mathbf{m}}| \ge 0$.

Theorem 12 (Bezout's Theorem for chain maps) provides a necessary estimate for the Khovanski numbers in the last formula. Namely, writing each $\mathbf{P}^i$ using its coordinate functions $(\mathbf{P}^i_1, \dots, \mathbf{P}^i_{\mathbf{m}})$ we have

$$(23)\qquad \mathcal{K}_{M(r)}\left(\mathbf{P}^i\left(\mathbf{j}^{\mathbf{m}}_{\mathcal{I}}\mathbf{f}_\gamma\right) = a^*\right) \le \prod_{j=1}^{\mathbf{m}} \deg \mathbf{P}^i_j.$$

Theorems 10, 11, and 12 combined into the Main Corollary after Theorem 12 which implies the Main Result: estimate for maximal cyclicity $E(k)$ given by (2).

Finally, Chaps. 5 through 7 devoted to the proof of Bezout's Theorem for chain maps (Theorem 12 consequently reformulated to Theorems 13, 15, and 16). In Chap. 5 we introduce necessary notions from stratification theory and, in particular, so-called a_P-stratification. Then using arguments very similar to Proposition 1 we deduce Theorem 13 from Theorem 15 about existence of a_P-stratification. Chap. 7 is devoted to the proof of Theorem 16.

In Chap. 8, based on developed machinery, we prove Theorems 1 and 2.

2. Normal forms for local families and their applications

In this section we present a functional–Pfaffian system whose number of solutions bounds from above the number of limit cycles. This system was obtained in [IY3]. The idea of writing Poincare return map in Pfaffian form is first proposed in [MR].

2.1. **Local families and polynomial normal forms.** A local family of planar vector fields is the germ of a map,

$$v : (\mathbb{R}^2, 0) \times (\mathbb{R}^k, 0) \to (\mathbb{R}^2, 0), \qquad (x, y, \varepsilon) \mapsto v(x, y, \varepsilon).$$

A C^r-smooth conjugacy between two local families v and w of the above form is a map

$$H : (\mathbb{R}^2, 0) \times (\mathbb{R}^k, 0) \to (\mathbb{R}^2, 0), \qquad (x, y, \varepsilon) \mapsto H(x, y, \varepsilon),$$

such that

$$H_* v(x, y, \varepsilon) = w(H(x, y, \varepsilon), \varepsilon),$$

where H_* stands for the Jacobian matrix with respect to the variables x, y. This definition does not yet allow reparametrization of a local family. Two families are finitely smoothly equivalent, if for any $r < \infty$ there exists a C^r-conjugacy between them. The two families v, w are orbitally equivalent, if there exists the germ of a nonvanishing function $\phi : (\mathbb{R}^2, 0) \times (\mathbb{R}^k, 0) \to \mathbb{R}^1$ such that v is equivalent to $\phi \cdot w$.

To allow for a reparametrization of local families, we say that a family $v(\cdot, \varepsilon)$ is induced from another family $w(\cdot, \lambda)$, $\lambda \in (\mathbb{R}^m, 0)$, if $v(\cdot, \varepsilon) = w(\cdot, \lambda(\varepsilon))$, where $\lambda(\varepsilon)$ is the germ of a smooth map $(\mathbb{R}^k, 0) \to (\mathbb{R}^m, 0)$. The number of new parameters m may be different from k.

Assume that the family $w(\cdot, \lambda)$ is *global* (i.e. the expression $w(x, y, \lambda)$ makes sense for all $(x, y, \lambda) \in \mathbb{R}^{m+2}$); this happens in particular when w is *polynomial* in all its arguments. Restricting the parameters λ onto a small neighborhood of a certain point $(0, 0, \mathbf{c}) \in \mathbb{R}^2 \times \mathbb{R}^m$, we obtain a *localization* of the global family w, which formally becomes a local family after the parallel translation $\lambda \mapsto \lambda - \mathbf{c}$.

Definition 7. *1. A local family $v = v(\cdot, \lambda)$ is finitely smooth orbital versal unfolding (in short, versal unfolding) of the germ $v(\cdot, 0)$, if any other local family unfolding this germ is finitely smoothly orbitally equivalent to a family induced from v.*

2. A polynomial family $w(\cdot, \lambda)$, $\lambda \in \mathbb{R}^m$, is a global finitely smooth orbital versal unfolding (in short, global versal unfolding) for a certain class of germs of families of vector fields, if any local family from this class is finitely smoothly orbitally equivalent to a local family induced from some localization of w.

To investigate a versal unfolding means to investigate at the same time all smooth local finite-parametric families which unfold the same germ $v(\cdot, 0)$. The main result describing versal unfoldings of germs of elementary singularities on the plane, is given by the following

Theorem 4. *[IY1] Suppose that a generic finite-parameter family of smooth vector fields on the plane possesses an elementary singular point for a certain value of the parameters. If this point has at least one hyperbolic sector, than the family is finitely differentiable orbitally equivalent to a family induced from some localization of one of the families given in the second column of Table 1.*

Table 1. Unfolding of elementary equilibrium points on the plane

Type	Normal forms	Poincare Correspondence maps	Pfaffian equations
S_0	$\dot{x} = x,$ $\dot{y} = -\lambda y.$ $\lambda = \lambda_0 \in \mathbb{R}^1$	$y = x^\lambda,$ $x > 0,\ y > 0$	$x\,dy - \lambda y\,dx = 0$
S_μ	$\dot{x} = x\left(\frac{n}{m} + P_\mu(u,\lambda)\right),$ $\dot{y} = -y.$ $u = u(x,y) = x^m\ y^n,$ $P_\mu(u,\lambda) = \pm u^\mu(1+\lambda_\mu u^\mu)$ $+W_{\mu-1}(u,\lambda),$ $\lambda = (\lambda_1,\dots,\lambda_\mu)$	$0 = m\log y +$ $\int_{x^m}^{y^n} \frac{du}{u P_\mu(u,\lambda)}.$ $x > 0,\ y > 0$	$y\,P_\mu(y^n,\lambda)\,dx -$ $\left(\frac{n}{m} + P_\mu(y^n,\lambda)\right) \times$ $xP_\mu(x^m,\lambda)\,dy = 0$
D_μ^c	$\dot{x} = Q_\mu(x,\lambda),$ $\dot{y} = -y.$ $Q_\mu(x,\lambda) = \pm x^{\mu+1}(1+\lambda_\mu x^\mu)$ $+W_{\mu-1}(x,\lambda),$ $\lambda = (\lambda_1,\dots,\lambda_\mu)$	$y = C(\lambda)x,$ $C = \int_{-1}^{1} \frac{du}{Q_\mu(u,\lambda)},$ $x,\ y \in \mathbb{R}^1$	$x\,dy - y\,dx = 0$
D_μ^h		$0 = \log y +$ $\int_x^1 \frac{du}{Q_\mu(u,\lambda)}$ $y > 0,\ x \in \mathbb{R}^1$	$Q_\mu(x,\lambda)\,dy -$ $y\,dx = 0$

In what follows we use the following notations for elementary equilibria (the subscript indicates the degree of degeneracy):

S_0— Nonresonant saddle;

S_μ— Resonant saddle whose quotient equation (the differential equation for $u = x^m\ y^n$ below) has the singular point of multiplicity $\mu + 1$ at the origin, $\mu \geq 1$; if we want to specify explicitly the resonance between the eigenvalues, we use the extended notation $S_\mu^{(n:m)}$ assuming that the natural numbers m, n are mutually prime;

D_μ— Degenerate saddlenode of multiplicity μ, superscript D_μ^c or D_μ^h depends on a position of "entrance" Σ^+ and "exit" Σ^- transversals: if both of them intersect the central manifold, then the Poincare return map is along the central manifold and this type is denoted by D_μ^c, otherwise it is denoted by D_μ^h;

$W_{\mu-1}(z, \lambda) = \lambda_0 + \lambda_1 z + \cdots + \lambda_{\mu-1} z^{\mu-1}$ is a Weierstrass polynomial of degree $\mu - 1$.

The localization of the universal families from Table 1 always satisfies the following additional condition: the vector $\mathbf{c} \in \mathbb{R}^{\mu+1}$ has all coordinates except for the last one equal to zero. This is a way to say that in the reparametrization mentioned in Definition 7 the parameters of the versal unfoldings are smooth functions of the original parameters of the local family,

$$\lambda = \lambda(\varepsilon) = (\lambda_0(\varepsilon), \ldots, \lambda_\mu(\varepsilon)), \tag{24}$$

and the map $\varepsilon \mapsto \lambda(\varepsilon)$ has the following properties: it is smooth and all the values $\lambda_j(0)$ except for the last one $\lambda_\mu(0)$, vanish, while the last one $\lambda_\mu(0)$ is the formal invariant of the germ $v(\cdot, 0)$.

In particular this means that the global versal unfolding for a nonresonant saddle (which has only one parameter) should be localized at a point $\mathbf{c} \in \mathbb{R}^1$ which is the hyperbolicity ratio of the nonperturbed germ $v(\cdot, 0)$.

Different technical remarks concerning other aspects of this table can be found in [IY3] § 1.1. We just briefly describe each column.

The first two columns do not need extra words. In the third column of the table the Poincare correspondence maps $y = \Delta(x, \lambda)$ for the polynomial normal forms are given. They are implicitly defined by the equations relating x and y, these equations depend explicitly on the parameters λ and thus, implicitly, on the original parameters ε. The choice of segments transversal to the phase curves of the family is described in Fig. 4. In the fourth column Pfaffian equations vanishing on the graph $y = \Delta(x)$.

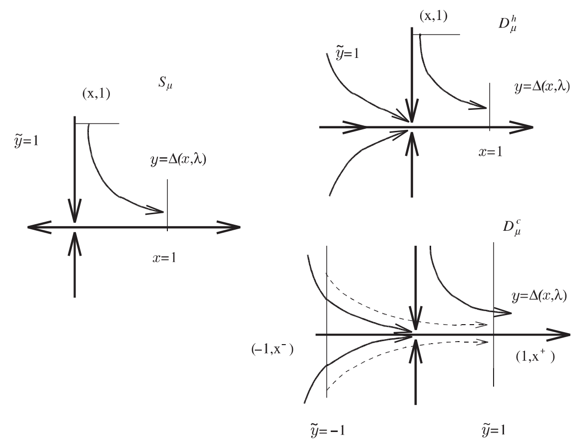

Fig. 4. The Poincare correspondence maps

2.2. **Basic system.** Here we describe the system of equations which will be analyzed from now on. Assume that a polycycle occurs in a generic k-parameter family of vector fields, and all the vertexes of the polycycle are elementary.

Remark 3. [IY3] § 1.4 Under the assumption that a polycycle γ is elementary and occurs in a generic k-parameter family of vector fields the number of vertexes n is bounded by k. Moreover, if each vertex of γ is of one of types $S_{\mu_j},\ \mu_j \geq 0$, or $D^c_{\mu_j},\ \mu_j \geq 1$, or $D^h_{\mu_j},\ \mu_j \geq 1$, then $\sum \mu_j \leq k$.

Next, we proceed with introducing the normalizing C^{p}-smooth local coordinates near each elementary vertex, as this is described above (the exact order of smoothness will be specified later on). Then a pair of C^{p}-smooth transversals may be chosen near each vertex, and endowed with local C^{p}-smooth charts x_j, y_j in such a way that the correspondence map taking a point with a coordinate x_j on the "entrance" transversal to a point with a coordinate y_j on the "exit" transversal, will be of one of the standard types listed in Table 1.

More precisely, for each vertex $j = 1, \dots, n$ Theorem 4 yields the *localization point* $\mathbf{c}_j = (0, \dots, 0, c_j) \in \mathbb{R}^{\mu_j+1}$, where $c_j \in \mathbb{R}^1$ is the formal invariant of the unperturbed singular point, and also if j-th vertex is a resonant saddle, then the rational hyperbolicity ratio $n : m$ is explicitly specified.

Denote by $\Delta_l(x, \lambda)$ the correspondence map for each of the four types of singularities from Table 1, where $l = S_0, S_\mu, D^c_\mu$ or D^h_μ, with the corresponding index $\mu \in \mathbb{Z}_+$ (for $l = S_0$ by definition $\mu = 0$). In case S_μ when $\mu > 0$ we consider the mutually prime pair of natural numbers as an additional parameter of the corresponding map, so in this case the rigorous notation would be $\Delta_{S_\mu}(x, \lambda; [n, m])$.

Definition 8. *1. The unspecified basic system for determination of limit cycles occurring in k-parametric families of vector fields is the system of n regular and n singular functional equations in 2n variables x_j, y_j, depending on parameters $\lambda^j, n_j, m_j, \varepsilon$. In what follows*

$$
j = 1, \dots, n \ (\mathrm{mod}\, n), \quad l_j \in \{S_0, S_\mu, D^c_\mu, D^h_\mu\},
$$

$$
n_j, m_j \in \mathbb{Z}_+, \quad \mu_j \in \mathbb{Z}_+, \quad \sum_{j=1}^{n} \mu_j \leq k, \quad n \leq k, \tag{25}
$$

$$
\Delta_{l_j} \textit{ depends on } n_j, m_j \textit{ only if } l_j = S_\mu \textit{ and } \mu > 0.
$$

Then the unspecified basic system is:

$$
\begin{cases} y_j = \Delta_{l_j,\mu_j}\big(x_j, \lambda^j; [n_j, m_j]\big), & \lambda^j \in \mathbb{R}^{\mu_j+1}, \\ x_{j+1} = f_j(y_j, \varepsilon), & \varepsilon \in (\mathbb{R}^k, 0). \end{cases} \tag{26}
$$

2. A specified basic system is one of a finite number of unspecified basic systems together with an explicit indication of specification, which by definition is the collection of:

- *localization points* $\mathbf{c}_j = (0, \dots, 0, c_j) \in \mathbb{R}^{\mu_j+1}$*; in particular this means that hyperbolicity ratios of all nonresonant saddles are explicitly given;*
- *hyperbolicity ratios* $n_j : m_j$ *for all resonant saddles;*
- *smooth functions* $f_j(x, \varepsilon)$ *depending on the parameters* ε*, are defined in some open neighborhoods* $(\mathbb{R}^{k+1}, 0)_j$ *and* $f_j(0, 0) = 0$*;*
- *characteristic size, that is, the value* $r > 0$ *which determines the domain of the specified basic system as follows:*

$$(27)\qquad \begin{gathered} (x, y) \in I_r = \{|x_j| < r,\ |y_j| < r, \quad j = 1, \dots, n\} \subset \mathbb{R}^{2n}; \\ (\underline{\lambda}, \varepsilon) \in B_r = \left\{\|\lambda^j - \mathbf{c}_j\| < r,\ \|\varepsilon\| < r\right\} \subset \mathbb{R}^{k+\mu_1+\dots+\mu_n}, \end{gathered}$$

where $\underline{\lambda}$ *is the tuple of all parameters of all normal forms from Table 1,* $\lambda = (\lambda^1, \dots, \lambda^n)$*; the characteristic size must be so small that all functions* f_j *were defined for the corresponding values of their arguments.*

Notations related to Definition 8: There is only a finite number of unspecified basic systems, each one being completely characterized by the string of discrete data

$$(28)\qquad \mathcal{T} = (l_1, \dots, l_n)$$

subject to the total restriction $n \le k$, $\sum_{j=1}^{n} \mu_j \le k$, where μ_j's are orders of degeneracies of the corresponding equilibria. We call the data $\mathcal{T}$ the *combinatorial type of the unspecified basic system*.

Let $j_1, \dots, j_s$ be the set of indices corresponding to equilibrium points of the type of resonant saddles: $l_{j_\alpha} = S_{\mu_\alpha}$. Then the string

$$(29)\qquad \begin{gathered} \mathcal{S}_{alg} = (c_1, \dots, c_n, \dots, m_{j_\alpha}, n_{j_\alpha}, \dots) \in \mathbb{R}^{n+2s} \\ r > 0,\ c_j \in \mathbb{R}^1,\ m_{j_\alpha}, n_{j_\alpha} \in \mathbb{Z}_+,\ f_j \in \mathbb{C}^{\mathsf{p}}(\mathbb{R}^{k+1}, 0). \end{gathered}$$

will be referred to as the *algebraic part of the specification* (for reasons to be clarified later), while the string of functions

$$(30)\qquad \mathbf{f} = (f_1, \dots, f_n)$$

is called the *functional part* of the specification. The functions f_j are defined on the domain $I_r \times B_r$, where r is the characteristic size introduced earlier.

Definition 9. *Let* $\mathcal{B}_k(\mathcal{T}, \mathcal{S}_{alg}, \mathbf{f}, r_0)$ *be the maximal number of isolated solutions to the specified basic system* $(\mathcal{T}, \mathcal{S}_{alg}, \mathbf{f})$ *in the domain* I_{r_0}.

One can check that $\mathcal{B}_k(\mathcal{T}, \mathcal{S}_{alg}, \mathbf{f}, r_0)$ is defined in such a way that it bounds cyclicity of a polycycle with such a specification.

After all these notions (or rather the language) being introduced, we may formulate the problem of estimating cyclicity of elementary polycycles occurring in generic k-parameter families as follows.

Theorem 5. *For any elementary polycycle γ occurring in a generic k-parameter family, for any type $\mathcal{T}_\gamma$ of an unspecified basic system associated γ and any choice of the algebraic part $\mathcal{S}_{alg,\gamma}$ corresponding to γ one may choose an order of smoothness $\mathsf{p}_0 > 6k+1$ and an open dense subset $\mathsf{F}_\gamma = \mathsf{F}_{\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, r_0}$ in the space of C^{p_0}-smooth functions $C^{\mathsf{p}_0}(I_{r_0} \times B_{r_0}, \mathbb{R}^n)$ such that for every $\mathbf{f}_\gamma = (f_1, \dots, f_n) \in \mathsf{F}_\gamma$ and a sufficiently small characteristic size $r_0 = r_0(\mathbf{f})$ the maximal number of isolated solutions $\mathcal{B}_k(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}_\gamma; r_0)$ to the specified basic system $(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}_\gamma)$ in the domain I_{r_0} is uniformly bounded over all parameter values $(\lambda, \varepsilon) \in B_{r_0}$:*

$$(31) \quad \mathcal{B}_k(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}_\gamma; r_0) = \sup_{(\varepsilon,\lambda) \in B_{r_0}} \#\{(x, y) \in I_{r_0} \text{ satisfy (26)}\} \le 2^{25k^2}$$

and, therefore, by definition of $\mathcal{B}_k(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}_\gamma; r_0)$ we have

$$(32) \qquad E(k) \le \max_\gamma \mathcal{B}_k(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}_\gamma; r_0) \le 2^{25k^2},$$

where the maximum is taken over all elementary polycycles of codimension k.

3. The Khovanski reduction method

In this section we describe the method of reducing of a functional–Pfaffian system to a chain map of the form (7). The construction in its full generality is described in the book [Kh]. Our exposition relies on the one in [IY3], but has new important features so we can't just refer to neither [Kh], nor [IY3].

3.1. **Pfaffian systems and their separating solutions.** Let M be a smooth orientable n-dimensional manifold, not necessarily compact or connected, and ω be a smooth 1-form on M.

Definition 10. *A codimension 1 smooth submanifold $\Gamma \subset M$ is the separating solution for the Pfaffian equation $\omega = 0$, if:*

a) Γ is the integral manifold, that is, the restriction of ω to the tangent bundle of Γ is identically zero:

$$\forall x \in \Gamma,\ \forall v \in T_x\Gamma \quad \omega(v) = 0;$$

b) Γ does not pass through singular points of ω:

$$\forall x \in \Gamma,\ \exists v \in T_xM \quad \omega(v)|_{T_xM} \neq 0;$$

c) Γ is the boundary of a domain $D \subseteq M$ and the coorientation induced on Γ by ω, coincides with its coorientation as the boundary. In other words, on any vector pointing outward from D, the form ω is positive.

Let now $\omega_1, \dots, \omega_k$ be an ordered k-tuple of smooth 1-forms on M. Consider the system of Pfaffian equations

$$\omega_1 = 0, \quad \dots \quad , \omega_k = 0. \tag{33}$$

Definition 11. *A submanifold* Γ *is the separating solution for the system of Pfaffian equations, if there exists an increasing chain of smooth submanifolds*

$$\Gamma = \Gamma_k \subset \Gamma_{k-1} \subset \cdots \subset \Gamma_1 \subset \Gamma_0 = M \tag{34}$$

such that for any $j = 1, \dots, k$ *the submanifold* Γ_j *is a separating solution for the Pfaffian equation on* Γ_{j-1}*, determined by the restriction of the form* ω_j *on the latter submanifold.*

Let $\mathcal{F} : M \to \mathbb{R}^s$ be a smooth map and $s < \dim M - k$. Recall that a point $a \in \mathbb{R}^s$ is called *a regular value* for the map $\mathcal{F}$ if the linearization matrix, denoted by $d\mathcal{F}(x)$, has full rank for any $x \in \mathcal{F}^{-1}(y)$. By the Rank Theorem the level set $V_a = \mathcal{F}^{-1}(a)$ of a regular value $a \in \mathbb{R}^s$ is a smooth manifold of dimension $n - s$ (see e.g. [GG]).

Definition 12. *Let V be a smooth submanifold of M of dimension at least k, $\omega_1, \dots, \omega_k$ be a k-tuple of smooth* 1*-forms, and* $\Omega = \omega_1 \wedge \cdots \wedge \omega_k$ *be the wedge product k-form. The restriction of* Ω *on V is called nondegenerate if the set of points x in V, where the restriction of* Ω *to* T_xV *is identically zero, have measure zero on* V.

A point $a \in \mathbb{R}^s$ *is called a regular value for* $\mathcal{F}$ *with respect to Pfaffian equations (33) if a is a regular value of* $\mathcal{F}$ *and the restriction of* Ω *to* V_a *is nondegenerate.*

Consider a pair of smooth maps $F : M \to \mathbb{R}^{n-k-s}$ and $\mathcal{F} : M \to \mathbb{R}^s$. Now we add to a Pfaffian system (33) two types of functional equations. The first type consists of functional equations $F = b$, where $b \in \mathbb{R}^{n-k-s}$ is *a variable*. The second type consists of functional equations $\mathcal{F} = a$, where $a \in \mathbb{R}^s$ is *a fixed regular value of* $\mathcal{F}$ *with respect to a Pfaffian system (33).* We call equations $F = b$, with a varying $b \in \mathbb{R}^{n-s-k}$, *loose* and $\mathcal{F} = a$, with a fixed $a \in \mathbb{R}^s$, *rigid*.

Definition 13. *Let* $\omega_1, \dots, \omega_k$ *be a k-tuple of smooth 1-forms,* $\Omega = \omega_1 \wedge \cdots \wedge \omega_k$ *be the wedge product,* $F : M \to \mathbb{R}^{n-k-s}$ *and* $\mathcal{F} : M \to \mathbb{R}^s$ *be smooth maps, and* $a \in \mathbb{R}^s$ *be a regular value for* $\mathcal{F}$ *with respect to* Ω. *Let* $L_b \subseteq M$ *be the preimage* $F^{-1}(b)$ *of some* $b \in \mathbb{R}^{n-k-s}$ *and* Γ^a *be a separating solution for the Pfaffian system* $\Omega = 0$*, restricted to* $V_a = \mathcal{F}^{-1}(a)$. *Then solutions to the mixed functional–Pfaffian system*

$$\Omega = 0, \quad F = b, \quad \mathcal{F} = a, \quad b \in \mathbb{R}^{n-k-s}, \quad a \in \mathbb{R}^s \tag{35}$$

are isolated points which belong to the intersection $\Gamma^a \cap L_b$.

The solution is *regular*, if Γ^a is a separating solution for the restriction of Pfaffian equations to V_a and b is a regular value for the restriction of the map G on Γ^a. If (Γ^a, L_b) is a regular solution, then Γ^a and L_b are transversal and the intersection consists of isolated points.

Definition 14. *The Khovanski number* $\mathcal{K}\{\Omega, F; \mathcal{F} = a\}$ *for the mixed system (35) is the upper bound for the cardinalities* $\#\{\Gamma^a \cap L_b\}$ *over all regular solutions of the system (35).*

Remarks. 1. The Khovanski number is also defined if $k = 0$ (resp. $s = 0$), i.e., there are no Pfaffian (resp. rigid) equations at all. In this case one may put formally $\Gamma = M$ (resp. $V_a = M$), and $\mathcal{K}\{\emptyset, F; \mathcal{F} = a\}$ (resp. $\mathcal{K}\{\Omega, F; \emptyset\}$) is equal to the upper bound of the cardinality of preimages of *regular values* of $\mathcal{F}|_{V_a} : V_a \to \mathbb{R}^{n-k-s}$, i.e. cardinality for $\#\{L_b \cap V_a\}$ (resp. $\#\{L_b \cap \Gamma\}$).

2. If we want to stress in the notation the phase space M of the functional–Pfaffian system, we use the notation $\mathcal{K}_M\{\Omega, F; \mathcal{F} = a\}$. Usually this is necessary when F, $\mathcal{F}$, and Ω are defined on the Euclidean space $\mathbb{R}^n$, while we are interested only in solutions belonging to some open semialgebraic set.

3. If we fix a coordinate system in $\mathbb{R}^{n-k-s}$, denote by $F_1, \ldots, F_s$ coordinate functions of the map $F : M \to \mathbb{R}^{n-k-s}$, and introduce the $(n-k-s)$-tuple of 1-forms $\Omega_F = (dF_1, \ldots, dF_s)$, then we can consider the following mixed system

$$\Omega = 0, \quad \Omega_F = 0, \quad \mathcal{F} = a. \tag{36}$$

Regularity in the definition of the Khovanski number $\mathcal{K}\{\Omega, F; \mathcal{F} = a\}$ implies that $\mathcal{K}\{\Omega, F; \mathcal{F} = a\} = \mathcal{K}\{(\Omega, \Omega_F), \emptyset; \mathcal{F} = a\}$.

The goal is to estimate the Khovanski number for the mixed functional-Pfaffian system by a linear combination of the Khovanski number of a certain finite number of entirely rigid functional systems. To reach this goal we apply the Khovanski method.

- Using Remark 3.1 Part 3 we replace all loose functional equations for Pfaffian equations so that we have only Pfaffian and rigid equations. This substitution might only increase the Khovanski number.
- Application of the Khovanski method allows to estimate the Khovanski number for a given mixed system by a linear combination of the Khovanski numbers of two auxiliary systems *containing a reduced by one number of Pfaffian equations and an increased by one number of rigid equations*. Therefore, if we apply the reduction principle to the mixed system consisting of $(n-s)$ Pfaffian equations (Ω, Ω_F) and s rigid equations $(n-s)$ times we obtain a finite collection of entirely rigid functional systems.

3.2. **The Reduction principle for one Pfaffian equation.** As before M is a smooth orientable n-dimensional manifold. We show how to eliminate the Pfaffian equation from the mixed system with $(n-s-1)$ loose equations, s rigid functional equations, and one Pfaffian equation.

$$(37)\quad \omega = 0, \quad F = b, \quad \mathcal{F} = a, \quad F : M \to \mathbb{R}^{n-s-1}, \quad \mathcal{F} : M \to \mathbb{R}^s,$$

We shall outline only the key ideas.

Definition 15. *A smooth positive function $\rho : M \to \mathbb{R}_+$ is called covering, if it tends to zero along any discrete sequence of points in M. In other terms, ρ vanishes "at infinity" on M, so that all level hypersurfaces of the covering function are compact subsets of M.*

Suppose that the manifold M is endowed with the Riemann volume. Since it is orientable, one may use the duality between functions and n-forms on M. Denote by the asterisk the operator taking an n-form into the function (dividing by the volume form).

Fix Euclidean structures in $\mathbb{R}^{n-s-1}$ and $\mathbb{R}^s$. Let $F_1, \dots, F_{n-s-1}$ and $\mathcal{F}_1, \dots, \mathcal{F}_s$ be the coordinate functions of the maps F and $\mathcal{F}$ in (37) respectively.

Definition 16. *The contact function for the mixed system (37) is*

$$(38)\qquad \mathcal{F}_{s+1} = *(\omega \wedge dF_1 \wedge \cdots \wedge dF_{n-s-1} \wedge d\mathcal{F}_1 \wedge \cdots \wedge d\mathcal{F}_s).$$

The operator taking the mixed system $(\omega, F; \mathcal{F})$ into the corresponding contact function, will be denoted by $\sigma : (\omega, F; \mathcal{F}) \mapsto \sigma(\omega, F; \mathcal{F}) = \mathcal{F}_{s+1}$.

Define the two maps by their coordinate functions,

$$(39)\qquad \mathcal{F}^c = (\mathcal{F}_1, \dots, \mathcal{F}_s, \mathcal{F}_{s+1}), \qquad \mathcal{F}^\infty = (\mathcal{F}_1, \dots, \mathcal{F}_s, \rho),$$

both taking M to $\mathbb{R}^{s+1}$, where $\mathcal{F}_{s+1}$ is the contact function (38), and ρ is the covering function.

Theorem 6. *Suppose that the system (37) admits regular solutions in the sense of definition 13. Then for any sufficiently small regular ε*

(40)
$$\mathcal{K}\{\omega, F; \mathcal{F} = a\} \le \frac{1}{2}\mathcal{K}\{\emptyset, F; \mathcal{F}^\infty = (a, \varepsilon)\} + \mathcal{K}\{\emptyset, F; \mathcal{F}^c = (a, \varepsilon)\},$$

where regularity of ε means that (a, ε) is a regular value for both $\mathcal{F}^\infty$ and $\mathcal{F}^c$. This inequality makes sense only when the right-hand side is finite.

Before proving this theorem recall a variation of the Rolle lemma from an elementary calculus.

Lemma 1. *Consider C^1-smooth functions $f : S^1 \to \mathbb{R}^1$ on the circle and $g : [0,1] \to \mathbb{R}^1$ on the segment, i.e., functions f and g have only finitely many critical points. Then for each $a \in \mathbb{R}$*

$$\begin{aligned} \#\{x : f(x) = a\} &\leq \#\{x : f'(x) = \varepsilon\} \\ \#\{x : g(x) = a\} &\leq \#\{x : g'(x) = \varepsilon\} + 1 \end{aligned} \tag{41}$$

for any sufficiently small ε.

Similarly, consider also C^1-smooth 1-forms ω_f, defined on the oriented circle, and ω_g, defined on the oriented segment. Denote by $v_x \in T_x S^1$ (or $T_x[0,1]$) the positively oriented unit vector. Then for any separating solutions Γ_f and Γ_g of ω_f and ω_g respectively we have

$$\begin{aligned} \#\{\Gamma_f\} &\leq \#\{x : \omega_f(x, v_x) = \varepsilon\} \\ \#\{\Gamma_g\} &\leq \#\{x : \omega_g(x, v_x) = \varepsilon\} + 1 \end{aligned} \tag{42}$$

for any sufficiently small ε.

Proof. Prove the formula for $f : S^1 \to \mathbb{R}^1$. For a sufficiently small ε the number of local maxima and minima equals $\#\{x : f'(x) = \varepsilon\}$. Between any two consecutive preimages x_1 and x_2 of a point a, i.e., $f(x_1) = f(x_2) = a$, there exists a local minimum or maximum.

Proof in the case of 1-forms is the same. Q.E.D.

When ε is fixed formula (41) transfers a loose equation into a rigid one and formula (42) transfers a Pfaffian equation into a rigid one too.

Proof of Theorem 6. Take a regular solution (Γ^a, L_b) for (37), where $L_b = F^{-1}(b)$, and suppose that the intersection $\Gamma^a \cap L_b$ consists of isolated, say $d < \infty$, points. If d the right-hand side of (40) is unbounded we shall prove that the left-hand side of (40) is unbounded too. Since b is regular value of the restriction $F|_{\Gamma^a}$, any small variation of b may only increase the number of intersections. Take b to be a regular value of the restriction $F|_{V_a}$ or equivalently (b, a) to be a regular value of the map $(F, \mathcal{F})$ (rather than of the restriction of $\mathcal{F}$ to Γ^a).

Then any level set L_b is a 1-dimensional smooth manifold, intersecting Γ transversally. By the classification Theorem for 1-dimensional manifolds, L_b is the union of compact (circles) and noncompact (lines) components. Fix some orientation on each circle and each noncompact curve in L_b. Consider the restricted 1-form function $\omega|_{L_b}$ on L_b.

Fix a connected component, denoted by $\gamma \subset L_b$. The intersection of γ and Γ is a separating solution of $\omega|_{L_b}$ by definition. Therefore, between any two consecutive intersections x and y of L_a with Γ values $\omega|_{L_b}(x, v_x)$ and $\omega_f(y, v_y)$ must have different signs. Now we can apply the second part of (Rolle) Lemma 1 with $\omega|_{L_b} = \omega_f$, when γ is a circle, and $\omega|_{L_b} = \omega_g$, when γ is a noncompact curve.

Each point x where $\omega|_{L_b}(x, v_x) = 0$ (resp. $\omega|_{L_b}(x, v_x)$ is small) is the point where the linear functionals $dF_1(x), \dots, dF_{n-s-k}(x), d\mathcal{F}_1(x), \dots, d\mathcal{F}_s(x)$ and $\omega(x)$ are linear dependent (resp. almost dependent), i.e. $\mathcal{F}_{s+1}(x) = 0$ (resp. $\mathcal{F}_{s+1}(x) = \varepsilon$). This completes the proof of the theorem. Q.E.D.

Corollary 3. *If M is compact, then for any sufficiently small regular ε*

$$\mathcal{K}\{\omega, F; \mathcal{F} = a\} \leq \mathcal{K}\{\emptyset, F; \mathcal{F}^c = (a, \varepsilon)\}, \tag{43}$$

where regularity of ε means that (a, ε) is a regular value for $\mathcal{F}^c$.

Proof. Indeed, in this case the first term in (37) disappears. Q.E.D.

Remark 4. The choice of the Riemann volume form is not essential for the above construction. Indeed, if the volume form vol^n is replaced by a new one $b \cdot vol^n$, where b is a positive function, then the function $\mathcal{F}_{s+1}$ will be replaced by $\tilde{\mathcal{F}}_{s+1} = b^{-1}\mathcal{F}_{s+1}$, and the map $\tilde{\mathcal{F}}^c = (\mathcal{F}_1, \dots, \mathcal{F}_s, \tilde{\mathcal{F}}_{s+1})$ will have the same zero set.

3.3. **The Khovanski reduction in the general case.** Consider now the general case of the mixed system (35) with the number of Pfaffian equations $k > 1$. Suppose that Γ^a is a separating solution for the Pfaffian system $\Omega = 0$ restricted to V_a. By definition, this means that for $\Omega' = (\omega_1, \dots, \omega_{k-1})$ there exists a separating solution $\Gamma^a_k \subset V_a$ to the Pfaffian equation $\omega_k = 0$ on a separating solution $\Gamma^a_{k-1} \subset V_a$ to the Pfaffian system $\Omega' = 0$ restricted to V_a. Note that if ρ is a covering function on the manifold M, then its restriction on Γ^a_{k-1} is the covering function for the latter submanifold. Next, one can endow V_a (resp. Γ^a_{k-1}) by the Riemann $(n-s)$-volume (resp. $(n-k-s+1)$-volume) form $vol^{n-s}_{V_a}$ (resp. $vol^{n-k-s+1}_{\Gamma^a_{k-1}}$) in such a way that

$$\begin{gathered} \omega_1 \wedge \dots \wedge \omega_{k-1} \wedge \mathrm{vol}^{n-k+1}_{\Gamma^a_{k-1}} = \mathrm{vol}^{n-s}_{V_a} \\ d\mathcal{F}_1 \wedge \dots \wedge d\mathcal{F}_s \wedge \mathrm{vol}^{n-s}_{V_a} = \mathrm{vol}^n_M \,. \end{gathered} \tag{44}$$

Since the forms $\omega_j,\ j = 1, \dots, k-1$ are linear independent in a neighborhood of Γ^a_{k-1}, these formulas define volume forms near V_a and Γ^a_{k-1} respectively. As this was mentioned before, the choice of the Riemann volume form does not affect the assertion of Theorem 6.

Thus one can apply Theorem 6 to the mixed system

$$\omega_k = 0, \quad F = b, \quad \mathcal{F} = a \tag{45}$$

on the manifold $\Gamma^a_{n-k} \subset V_a$. To describe the result, we introduce the following two maps from M to $\mathbb{R}^{n-k+1}$,

$$\mathcal{F}^c = (\mathcal{F}_1, \dots, \mathcal{F}_s, \rho), \qquad \mathcal{F}^\infty = (\mathcal{F}_1, \dots, \mathcal{F}_s, \mathcal{F}_*), \tag{46}$$

where ρ is the covering function on the manifold M, and $\mathcal{F}_* : M \to \mathbb{R}$ is the smooth function obtained as

(47)
$$\mathcal{F}_* = \sigma(\Omega, F; \mathcal{F}) = *(\omega_1 \wedge \ldots \wedge \omega_k \wedge dF_1 \wedge \ldots \wedge dF_{n-k-s} \wedge d\mathcal{F}_1 \wedge \ldots \wedge d\mathcal{F}_s).$$

The above choice of the Riemann volume on Γ^a_{k-1} implies that the asterisk operator in the ambient manifold M agrees with the asterisk operator relevant to Γ^a_{k-1}, therefore the formula (47) defines the same function as the formula (38): $\omega_k \wedge dF_1 \wedge \cdots \wedge dF_{n-k-s} \wedge d\mathcal{F}_1 \wedge \cdots \wedge d\mathcal{F}_s = \mathcal{F}_* \cdot \mathrm{vol}^{n-k+1}_{\Gamma^a_{k-1}}$.

Theorem 7. *Let $\Omega, \Omega', F, \mathcal{F}^c$, and $\mathcal{F}^\infty$ be as above. Then for any sufficiently small regular ε*

(48)
$$\mathcal{K}\{\Omega, F; \mathcal{F} = a\} \le \tfrac{1}{2}\mathcal{K}\{\Omega', F; \mathcal{F}^\infty = (a, \varepsilon)\} + \mathcal{K}\{\Omega', F; \mathcal{F}^c = (a, \varepsilon)\},$$

where regularity of ε means that (a, ε) is a regular value for both $\mathcal{F}^\infty$ and $\mathcal{F}^c$.

Corollary 4. *If either V_a is compact or the restriction $F|_{V_a} : V_a \to \mathbb{R}^{n-k-s}$ is a proper map, i.e. preimage of any point is compact, then for any sufficiently small regular ε*

(49)
$$\mathcal{K}\{\Omega, F; \mathcal{F} = a\} \le \mathcal{K}\{\Omega', F; \mathcal{F}^c = (a, \varepsilon)\},$$

where regularity of ε means that (a, ε) is a regular value for $\mathcal{F}^c$.

Proof. Straightforward application of Theorem 6.

Iterating the above two statements, one can replace one by one the Pfaffian equations by the rigid functional ones, obtaining new systems whose Khovanski numbers estimate from above that of the initial one, by virtue of the inequalities (48) and its compact counterpart (49). On each step one has two possibilities, either to replace a Pfaffian equation by the contact function, or by the covering function. But once the covering function appears among the rigid functional equations, the level sets $F^{-1}(\cdot) \cap V_a$ becomes compact as a submanifold of a compact V_a, hence on the next steps Corollary 4 applies to Theorem 7 rather than Theorem 7 itself.

Denote by T^c_ε and T^∞_ε the two operators, transforming the mixed system $\{\Omega, F; \mathcal{F} = a\}$ into the mixed systems $\{\Omega', F; \mathcal{F}^c = (a, \varepsilon)\}$ and $\{\Omega', F, \mathcal{F}^\infty = (a, \varepsilon)\}$ respectively, where the maps $\mathcal{F}^c$ and $\mathcal{F}^\infty$ are given by (46) and (47):

(50)
$$T^c_\varepsilon\{\Omega, F; \mathcal{F} = a\} = \{\Omega', F; (\mathcal{F}, \sigma\{\Omega, F, \mathcal{F}\}) = (a, \varepsilon)\},$$
$$T^\infty_\varepsilon\{\Omega, F; \mathcal{F} = a\} = \{\Omega', F; (\mathcal{F}, \rho) = (a, \varepsilon)\}.$$

If we start with the mixed functional–Pfaffian system $\{(\Omega, \Omega_F), \emptyset; \mathcal{F} = a\}$, with (Ω, Ω_F) being an $(n - s)$-tuple $(\omega_1, \ldots, \omega_k, dF_1, \ldots,$

dF_{n-k-s}), and eliminate subsequently the forms $\omega_k, \omega_{k-1}, \ldots, \omega_1, dF_1, \ldots, dF_{n-k-s}$, then the following maps from M to $\mathbb{R}^n$ arise:

a) the map $\mathcal{F}_{[0]}$, if on each step the contact function was used,

$$\{\mathcal{F}_{[0]} = (a, \varepsilon^{n-s})\} = \left(T^c_{\varepsilon_{n-s}} \circ \cdots \circ T^c_{\varepsilon_1}\right)\{\Omega, F; \mathcal{F} = a\}, \tag{51}$$

where $\varepsilon^{n-s} = (\varepsilon_1, \ldots, \varepsilon_{n-s})$;

b) the maps $\mathcal{F}_{[j]}$, if on the jth step the covering function was used, while on all other steps the contact ones were, $j = 1, \ldots, n-s$,

$$\left\{\mathcal{F}_{[j]} = (a, \varepsilon^{n-s})\right\} = \left(T^c_{\varepsilon_{n-s}} \circ \cdots \circ T^c_{\varepsilon_{j+1}} \circ T^\infty_{\varepsilon_j} \circ \circ T^c_{\varepsilon_{j-1}} \circ \cdots \circ T^c_{\varepsilon_1}\right)\{\Omega, F; \mathcal{F} = a\}. \tag{52}$$

Then inductive application of Theorem 7 immediately yields the following fundamental result.

Theorem 8. *Let M be a manifold with a covering function ρ. Then the Khovanski number for the mixed system (35) on M for any sufficiently fast decaying to zero sequence $|\varepsilon_1| \gg \cdots \gg |\varepsilon_{n-s}| \geq 0$ admits an upper estimate by a linear combination of Khovanski numbers of some $(n-s+1)$ auxiliary systems, each of them containing only rigid equations and no Pfaffian equations at all:*

$$\mathcal{K}\{\Omega, F; \mathcal{F} = a\} \leq \mathcal{K}\left\{\emptyset; \mathcal{F}_{[0]} = (a, \varepsilon^{n-s})\right\} + \frac{1}{2}\sum_{j=1}^{k} \mathcal{K}\{\emptyset; \mathcal{F}_{[j]} = (a, \varepsilon^{n-s})\}, \tag{53}$$

where the maps $\mathcal{F}_{[j]}$ are defined by the formula (50–52).

Condition $|\varepsilon_1| \gg \cdots \gg |\varepsilon_{n-s}| \geq 0$ of sufficiently fast decaying to zero parameters is necessary to apply Theorem 7 inductively. Namely, consider the equation

$$T^c_{\varepsilon'} \circ T^c_{\varepsilon}\{\Omega, F; \mathcal{F} = a\} = T^c_{\varepsilon'}\{\Omega', F; \mathcal{F} = a, \sigma\{\Omega, F, \mathcal{F}\} = \varepsilon\}, \tag{54}$$

where ε and ε' are sufficiently small. If ε' were zero, then by definition covectors $\omega_1, \cdots, \omega_{k-1}, dF_1, \cdots, dF_{n-k-s}, d\mathcal{F}_1, \cdots d\mathcal{F}_s, d\sigma\{\Omega, F, \mathcal{F}\}$ are linearly dependent. The idea of iterative application of (Rolle) Lemma 1 is that these covectors should be almost linearly dependent for the Khovanski method to work. When ε is sufficiently small, $d\sigma\{\Omega, F, \mathcal{F}\}$ could be small too and oriented volume $\sigma\{\Omega', F; \mathcal{F}, \sigma\{\Omega, F, \mathcal{F}\}\}$ is also small, but *not because of almost linear dependence*. However, choosing ε' much smaller than ε assures that $\sigma\{\Omega', F; \mathcal{F}, \sigma\{\Omega, F, \mathcal{F}\}\}$ is small because of almost linear dependence and application of (Rolle) Lemma 1 is possible. See also [IK] for more detailed explanation.

3.4. **Applications.** The Khovanski reduction process is constructive. This leads to the result, which will be formulated now.

Assume that the set M where a functional-Pfaffian system is defined is an open domain in $\mathbb{R}^n$ and admits a polynomial covering function ρ. The main example is the unit ball $B = \left\{x \in \mathbb{R}^n : \sum_j x_j^2 < 1\right\}$, for which one may take $\rho(x) = 1 - \sum_j x_j^2$. Then the Riemann volume form can be chosen algebraic, $dx_1 \wedge \cdots \wedge dx_n$.

Assume also that all the forms $\omega_i,\ i = 1, \dots, k$ are polynomial (i.e. with polynomial coefficients), and maps F and $\mathcal{F}$ are at least C^n-smooth. Since the operators T_ε^c and T_ε^∞ introduced above, involve only algebraic operations and differentiation of functions, the following statement holds.

Theorem 9. *If the system (35) is defined on a semialgebraic open subset $M \subseteq \mathbb{R}^n$, all coefficients the covering function ρ are polynomial of degrees $\leq D$ and the sum of degrees of coefficients of the Pfaffian forms over the forms is $\sum \deg \omega_j \leq d$, then all the maps $\mathcal{F}_{[\alpha]} : M \to \mathbb{R}^n$, $\alpha = 0, 1, \dots, n - s$ constructed in Theorem 8 (see formulas (50–52)) are of the form*

$$\mathcal{F}_{[\alpha]} = P^\alpha \circ j^{n-s}(F, \mathcal{F}), \tag{55}$$

where $j^{n-s}(F, \mathcal{F})$ is the $(n - s)$-jet extension of the map $(F, \mathcal{F}) : \mathbb{R}^n \to \mathbb{R}^{n-k}$, and $P^\alpha = (P_1^\alpha, \dots, P_n^\alpha)$ are certain vector-polynomials defined on the jet space $J^{n-s}(\mathbb{R}^n, \mathbb{R}^{n-k})$. For each $\alpha = 0, \dots, n - s$ and each $j = 0, 1, \dots, s$ degree $\deg P_j^\alpha$ *of the j-th component of P^α is* 1. *For each $\alpha = 0, \dots, n-s$ and $j = s+1, \dots, n$ we have* $\deg P_j^\alpha \leq 2^{j-s-1} \max(D, d+n)$. *Moreover, each map $\mathcal{F}_{[\alpha]}$ has a regular point for a generic map $(F, \mathcal{F})$.*

Proof. The reduction procedure of elimination of a Pfaffian equation boils down to consecutive application $n - s$ times of one of the operators T_ε^c or T_ε^∞, defined in (50). Consider the first step. Define the function

$$\begin{aligned} \mathcal{F}_*^{1,\alpha} = \sigma\{(\Omega, \Omega_F), \emptyset; \mathcal{F})\} = *(\omega_1 \wedge \cdots \wedge \omega_k \wedge \\ dF_1 \wedge \cdots \wedge dF_{n-k-s} \wedge d\mathcal{F}_1 \wedge \cdots \wedge d\mathcal{F}_s) = P^{s+1,\alpha} \circ j^1(F, \mathcal{F}), \end{aligned} \tag{56}$$

where $P_{s+1}^\alpha : J^1(\mathbb{R}^n, \mathbb{R}^{n-k}) \to \mathbb{R}$ is a polynomial of degree at most $\max(D, d + n)$ and is defined on the space of 1-jets $J^1(\mathbb{R}^n, \mathbb{R}^{n-k})$.

Denote by $\Omega_0^* = (\Omega, \Omega_F)$ the n-tuple of the 1-forms and for each $m = 1, \dots, n-s$ denote dF_m by ω_{k+m}. Denote also by Ω_r^* the $(n-r)$-tuple of the 1-forms, which consists of all of 1-forms of Ω^* except of the first r.

By the construction $\mathcal{F}_*^{r,\alpha} = \mathcal{F}_r$ for each $r = 1, \dots, s$ and $\alpha = 1, \dots, n - s$. For $r = s + 1, \dots, n - s$ functions $\{\mathcal{F}_*^{r,\alpha}\}_{r,\alpha}$ are defined inductively by $\mathcal{F}_*^{r,\alpha} = \sigma\{\Omega_{r-1}^*, \omega_r; \mathcal{F}^{r-1,\alpha}\}$ and $\mathcal{F}^{r,\alpha} = (\mathcal{F}^{r-1,\alpha}, \mathcal{F}_*^{r,\alpha})$ for $r = 1, \dots, n - s$. It is easy to see from the definition of $\sigma\{\dots\}$ that $\mathcal{F}_*^{r,\alpha}$ has the form $\mathcal{F}_*^{r,\alpha} = P_r^\alpha \circ j^{r-s}(F, \mathcal{F})$.

Now we estimate degrees of the corresponding P_r^α's. For $r = s+1$ we have $\deg P_{s+1}^\alpha \le \max(D, d+n)$. For $r > s+1$ we have $\deg P_r^\alpha \le 2^{r-s-1}\max(D, d+n)$ which can be directly verified by induction in r. This completes the proof. Q.E.D.

4. Functional-Pfaffian system for limit cycles

In this section we consider a specified basic system $(\mathcal{T}, \mathcal{S}_{alg}, \mathbf{f}; r)$ obtained from the unspecified basic system (26), that is we consider a system (26) together with a collection of formal invariants $(c_1, \dots, c_n)$ of all singularities (which determines a point in the λ-space), a collection of hyperbolicity ratios $n_{j_\alpha} : m_{j_\alpha}$ of all resonant saddles and an n-tuple of sufficiently smooth functions f_j, on a sufficiently small open cube $I_r \times B_r$ in the (ε, λ)-space.

Our local goal is to reduce this system to a functional–Pfaffian system having the form described in Sect. 3, with the following properties:

- the new system has the form allowing application of Theorems 7–9;
- the number of *regular solutions* to the functional–Pfaffian system is greater or equal to the number of *isolated* solutions to (26), up to k, where k is the number of parameters of the original family.

After application of Theorem 7 we will obtain a number of *chain maps* with controlled degrees of the exterior polynomial parts.

4.1. **Upper estimate of the number of solutions for the basic system: statement of results.** First of all we make the following remark. The algebraic part of the specification can be identified with a point

$$\mathcal{S}_{alg} = (c_1, \dots, c_n, n_{j_1}, m_{j_1}, \dots, n_{j_s}, m_{j_s}) \in \mathbb{R}^{n+2s}, \tag{57}$$

where $s \le n$ is the number of resonant saddles on the polycycle: the fact that the numbers $n_{j_\alpha}, m_{j_\alpha}$ are in fact natural will become inessential for our constructions.

Theorem 10. *(reduction from basic to functional–Pfaffian system) Consider an unspecified basic system (26) of a certain type $\mathcal{T}$ in codimension k, together with an arbitrary specification*

$$\mathcal{S} = (S_{alg}, \mathbf{f}; r), \quad \mathcal{S}_{alg} \in \mathbb{R}^{n+2s}, \quad \mathbf{f} \in C^{\mathrm{p}}(I_r \times B_r, \mathbb{R}^n), \quad r > 0.$$

Then one can explicitly construct a functional–Pfaffian system of the following form $\{\Omega, F; \mathcal{F}\}$, where $\Omega = (\Omega_1, \dots, \Omega_{n+2s})$ is $(n+2s)$-tuple of 1*-forms, $F = (F_1, \dots, F_{k+m})$ is $(n+m)$-tuple of functions with $m = n + \sum \mu_j$, $\mathbf{f} = (f_1, \dots, f_n)$ is the functional part of the unspecified basic system (26), and $\mathcal{F}_{\mathbf{f}} = (x_2 - f_1, \dots, x_n - f_{n-1}, x_1 - f_n)$. In a more traditional notation, the mixed system consists of loose functional F, rigid functional $\mathcal{F}_{\mathbf{f}}$, and Pfaffian Ω equations respectively*

$$\Omega = 0, \quad F = b; \quad \mathcal{F}_{\mathbf{f}} = a \tag{58}$$

defined in a certain open bounded semialgebraic subset

$$M = M(r) \subset I_r \times B_r \times \mathbb{R}^{2s}$$

(see Definition 8)*, such that the following conditions hold:*

- *For any choice of the parameters $(\varepsilon, \lambda) \in B_r$ the number of isolated (x, y)-solutions, denoted by $\mathcal{B}_k(\mathcal{T}, \mathcal{S}_{alg}, \mathbf{f}; r)$ of the specified basic system $(\mathcal{T}, \mathcal{S}_{alg}, \mathbf{f}; r)$ admits an estimate by the Khovanski system (58) on the manifold M:*

  $$\mathcal{B}_k(\mathcal{T}, \mathcal{S}_{alg}, \mathbf{f}; r) \leq \mathcal{K}_{M(r)}\{\Omega, F; \mathcal{F}_{\mathbf{f}} = a\} + k, \tag{59}$$

  provided that components of $a = (a_1, \ldots, a_n)$ decay sufficiently fast with n, i.e. $1 \gg |a_1| \gg \cdots \gg |a_n| \geq 0$*;*
- *The forms Ω_k have coefficients which are polynomial in all their arguments, and also in coordinates of the point $\mathcal{S}_{alg} \in \mathbb{R}^{n+2s}$; the degrees of those polynomials do not exceed* $6\mu + 1$*, where μ is the order of degeneracy of the corresponding equilibrium point, thus, the sum of degrees of degrees of all the forms is bounded by* $7k$*;*
- *The covering function $\rho(\cdot\,; r)$ for the phase space $M(r)$ is polynomial in all its arguments and also in r, of the total degree not exceeding* $14k$*;*
- *The coordinate functions of the maps $\mathcal{F}_\beta$ (resp. F_β) are explicitly given as linear on the* 0*-jet space of functions $J^0(I_r \times B_r, \mathbb{R}^n)$ (resp. in ε's and λ's) with coefficients* ± 1.

The proof of this theorem is completely constructive and given in mainly [IY3]. We derive estimates on degrees of involved into the procedure polynomials, which are not given in [IY3]. A little difference with [IY3] is that using some additional argument, presented in the next section, to move the functional part of the unspecified system (26) into the rigid part of the functional-Pfaffian system (58) in order not to use the Khovanski reduction method additional n times. This allows a little improvement of the constant for the main estimate (2).

4.2. **Moving the functional part the unspecified system (26) into the rigid part of the corresponding mixed functional system.** Consider the unspecified basic system (26). To simplify notations we write it in this system in the form (12). As we pointed out in Sect. 1.5 the Poincare return map $\Delta_\gamma(\cdot, \varepsilon)$ along the polycycle γ for parameter ε is given as the composition of maps involved into the right-hand sides of (12). The Poincare map $\Delta_\gamma(\cdot, \varepsilon)$, when exists, is defined on an open subset of $\mathbb{R}$. Limit cycles correspond to isolated fixed point of $\Delta_\gamma(x, \varepsilon) = x$. Suppose that for $\varepsilon = \varepsilon^*$ the number of limit cycles is the maximal possible. Then by choosing a_1 being small positive or negative we get that all solutions $\Delta_\gamma(x, \varepsilon^*) = x + a_1$ are regular and the number of those for the shifted equation is at least the number of limit cycles. To see that consider the graph of

$$\Delta_\gamma(x, \varepsilon) = x. \tag{60}$$

It is shown in [IY3], Lm. 3.3 that regular solutions of (60) correspond to regular solutions of the shifted system

$$(61)\qquad \begin{cases} y_j = \Delta_j(x_j,\varepsilon), & j=1,\dots,n \\ x_{j+1} = f_j(y_j,\varepsilon), & j=1,\dots,n-1 \ (\mathrm{mod}\ n) \\ x_1 - f_n(y_n,\varepsilon) = a_1. \end{cases}$$

By Theorem on implicit function the number of regular solutions is locally constant in the image. Therefore, we can choose $a_2,\dots,a_n$ so that $(a_1,\dots,a_n)$ is a regular point for the map F from (13), which is made from functional equations of (61), $(a_1,\dots,a_n)$ satisfies the condition $1 \gg |a_1| \gg \dots \gg |a_n| \geq 0$, and the number of regular solutions of

$$(62)\qquad \begin{cases} y_j = \Delta_j(x_j,\varepsilon) & j=1,\dots,n \\ x_{j+1} - f_j(y_j,\varepsilon) = a_{j+1}, & j=1,\dots,n-1 \ (\mathrm{mod}\ n). \end{cases}$$

for $\varepsilon = \varepsilon^*$ is the same as the one for (61). After we found $(a_1,\dots,a_n)$ with the above properties we fix it and consider equations $\mathcal{F}_j(x,y,\varepsilon) = x_{j+1} - f_j(y_j,\varepsilon) = a_j$ as rigid. From now on in this chapter we apply the Khovanski method to (62) as it is done in Sect. 3.

4.3. **Pfaffian systems associated with unfolding of elementary equilibrium points.**

Table 2. Separating solutions for Pfaffian systems associated with unfolding of elementary equilibrium points

Type	Submanifold γ	Domain M_r, Covering function ρ	Pfaffian system $\Omega = 0$
S_0	$y = \Delta(x,\lambda)$	$0 < x,\ y < r$, $\lambda \in L_r$, $\rho_{S_0} = xy(r-x)(r-y)\tilde{\rho}$	$x\,dy - \lambda y\,dx = 0$
S_μ	$y = \Delta(x,\lambda)$ $z = x^m$ $w = y^n$	$0 < x,\ y,\ z,\ w < r$, $\lambda \in L_r$, $P_\mu(z,\lambda) \neq 0$, $\rho_{S_\mu} = xyzw(r-x)(r-y)\times$ $(r-z)(r-w)P_\mu^2(z,\lambda)\tilde{\rho}$	$x\,dz - m\,z\,dx = 0,\ (1)$ $y\,dw - n\,w\,dy = 0,\ (2)$ $m\,P_\mu(w,\lambda)\times$ $y\,P_\mu(z,\lambda)\,dx -$ $(mP_\mu(w,\lambda)+n)\times$ $x\,P_\mu^2(z,\lambda)\,dy = 0\ (3)$
D_μ^c	$y = \Delta(x,\lambda)$	$\|x\|, \|y\| < r,\ x \neq 0$, $\lambda \in L_r$, $\rho_{D_\mu^c} = (r^2-x^2)(r^2-y^2)x^2\tilde{\rho}$	$x\,(x\,dy - y\,dx) = 0$
D_μ^h	$y = \Delta(x,\lambda)$	$0 < y < r,\ \|x\| < r$, $\lambda \in L_r$ $Q_\mu(\cdot,\lambda)\|_{[x,1]} > 0$, $\rho_{D_\mu^h} = y(r-y)(r^2-x^2)\times$ $Q_\mu(x,\lambda)\tilde{\rho}$	$Q_\mu(x,\lambda)\,dy - y\,dx = 0$

Notes to the table: Here we use the same notation as in Table 1 (and in fact this table continues Table 1). In particular, $n : m$ is the hyperbolicity ratio in the resonant saddle case S_μ.

In the third column of the table the symbol L_r stands for a small r-cube in the $(\mu+1)$-dimensional space of the parameters λ, centered at the localization point $\mathbf{c} = (0, \dots, 0, c) \in \mathbb{R}^{\mu+1}$, corresponding to the unperturbed system:

$$L_r = \{\lambda \in \mathbb{R}^{\mu+1} : |\lambda_i| < r,\ i = 0, \dots, \mu - 1,\ |\lambda_\mu - c| < r\}.$$

Everywhere in the table the function $\tilde{\rho} = \tilde{\rho}(\lambda)$ is the covering function for the set L_r, defined as

$$\tilde{\rho}(\lambda) = \left(r^2 - \lambda_1^2\right) \cdots \left(r^2 - \lambda_{\mu-1}^2\right) \cdot \left(r^2 - (c - \lambda_\mu)^2\right).$$

This is a polynomial of degree 2μ in all variables λ, r, c. Recall that $\deg P_\mu = 2\mu$ and $\deg Q_\mu = 2\mu + 1$ (see Table 1). Each covering function ρ is, therefore, a polynomial (explicitly written in the Table). Thus, we obtain the following degree estimates:

Type S_0: $\deg \Omega = 1$ and $\deg \rho_{S_0} = 2\mu + 4$.

Type S_μ, $\mu > 0$: $\deg \Omega \le 6\mu + 1$ and $\deg \rho_{S_\mu} = 6\mu + 8$.

Type D_μ^c: $\deg \Omega = 2$ and $\deg \rho_{D_\mu^c} = 2\mu + 6$.

Type D_μ^h: $\deg \Omega = 2\mu + 1$ and $\deg \rho_{D_\mu^h} = 4\mu + 5$.

Along with the estimate $\sum_j \mu_j \le k$ (the sum of codimensions) this gives the estimates $\deg \Omega \le 6\mu + 1$ and $\deg \rho \le 14k$, where ρ is the covering function which is direct product of covering functions of all equilibrium points on the polycycle γ (see (64) below). Therefore, the sum of degrees for the forms $\sum_j \deg \Omega_j \le 7k$. This gives required estimates on degrees of the second and third bullets of Theorem 10.

4.4. **Principal functional–Pfaffian system.** We proceed with writing down the principal functional–Pfaffian system explicitly. Slightly abusing notation, we add the subscript or superscript j for objects related to the j-th singularity, while letters without this subscript refer to objects related to the entire polycycle. In this notation we omit the reference to the characteristic size, still keeping in mind that all formulas are explicitly polynomial in r.

Notations: Denote by M_j the domain from Table 2, associated with the j-th singular standard map, by γ_j the corresponding manifold (separating solution) and by $\Omega^{(j)}$ the tuple of Pfaffian forms on M. If the singularity is of the type D_μ or S_0, then $\Omega^{(j)}$ consists of only one form $\omega_j = A_j\,dx_j + B_j\,dy_j$, while in the case S_μ, $\mu > 0$, there are three forms, of which we denote the third one by $A_j\,dx_j + B_j\,dy_j$, (see Table 2). The covering function for M_j is denoted by $\rho_j : M_j \to \mathbb{R}^1_+$.

Construction of the principal system: The phase space for the principal functional–Pfaffian system is the Cartesian product of phase spaces corresponding to all the vertexes of the polycycle and the r-cube in the ε-space:

$$M = M(r) = M_1 \times \cdots \times M_n \times \tilde{B}_r,$$
$$(63) \qquad M_j = M_{j,r} \text{ are taken from the second column of Table 2,}$$
$$\tilde{B}_r = \{|\varepsilon_i| < r,\ i = 1, \dots, k\}.$$

Dimension of the phase space is equal to $2n + 2s + k + m$, where:

- k is the number of the parameters ε (the principal integer index);
- $n(\leq k)$ is the number of vertexes;
- $s(\leq n)$ is the number of resonant saddles on the polycycle (each such a vertex contributes two additional variables z_j, w_j into the list of independent variables);
- $m = n + \sum \mu_j (\leq n + k \leq 2k)$ is the number of additional free parameters $\lambda = (\lambda^1, \dots, \lambda^n)$, $\lambda^j \in \mathbb{R}^{\mu_j+1}$.

The covering function for such a space is the product

$$(64) \qquad \rho = \rho_1 \cdots \rho_n \cdot \rho_\varepsilon : M \to \mathbb{R}^1_+,$$

where the last factor is the covering function for $\tilde{B}_r$. From Table 2 it is clear that ρ is a polynomial of degree at most $\sum_j (6\mu_j + 8) \leq 14k$ in both phase variables and the characteristic size r.

Each form on M_j can be pulled back on M, yielding the form which is independent of all the coordinates except for those related to the j-th vertex. Denote by Ω the union of the tuples $\Omega^{(j)}$: thus Ω is itself the tuple of 1-forms on M, containing $n + 2s$ of them:

$$\Omega = (\Omega^{(1)}, \dots, \Omega^{(n)}) = (\Omega_1, \dots, \Omega_{n+2s}),$$
$$(65) \qquad \Omega^{(j)} = \begin{cases} \{\omega_j\} & \text{if } j \text{ is not a resonant saddle,} \\ \{\omega_{j1}, \omega_{j2}, \omega_j\} & \text{otherwise,} \end{cases}$$
$$\text{where } \omega_{j1} = m_j x_j\, dz_j - z_j\, dx_j, \qquad \omega_{j2} = n_j y_j\, dw_j - w_j\, dy_j,$$
$$\omega_j = A_j\, dx_j + B_j\, dy_j.$$

Each γ_j is a separating solution to the Pfaffian equation or system of equations $\Omega^{(j)} = 0$ on M_j, therefore, the Cartesian product

$$\Gamma = \gamma_1 \times \cdots \times \gamma_n \times \tilde{B}_r$$

is the separating solution to the Pfaffian system $\Omega = 0$ on M. Indeed, one may consider the chain of submanifolds

$$\Gamma_i = \gamma_1 \times \cdots \times \gamma_i \times M_{i+1} \times \cdots \times M_n \times \tilde{B}_r.$$

This chain possesses all the properties required by the definition of a separating solution, see definition 10: there are no singular points of Pfaffian forms on all the manifolds from this chain, and the topological condition of Γ_{i+1} being the boundary of a domain in Γ_i is trivially satisfied, because each γ_{i+1} is the boundary of the corresponding subdomain in M_j. Thus, the Pfaffian part of the principal system is constructed.

In this Pfaffian part we have the following information about the polynomials (recall that $\mathcal{S}_{alg}$ stands for the algebraic part of the specification for the basic system, which is identified by (57) with a tuple of real variables):

$$\begin{aligned} A_j, B_j &\in \mathbb{Z}[x, y, \lambda, \mathcal{S}_{alg}], & \deg A_j, \deg B_j &\le 6\mu + 1, \\ \rho &\in \mathbb{Z}[x, y, \lambda, \varepsilon, r], & \deg \rho &\le 14k. \end{aligned} \tag{66}$$

Now we proceed with description of the loose and rigid functional part of the principal system. It is given by the map

$$\begin{aligned} F &= (F_1, \dots, F_{k+m}) : \mathbb{R}^{k+m} \to \mathbb{R}^{k+m}, \\ F_j &= \begin{cases} \varepsilon_{j-n}, & j = n+1, \dots, n+k, \\ \lambda_{j-n-k}, & j = n+k+1, \dots, n+k+m; \end{cases} \\ \mathcal{F} &= (\mathcal{F}_1, \dots, \mathcal{F}_n) : M \to \mathbb{R}^n \\ \mathcal{F}_j &= x_{j+1} - f_j(y_j, \varepsilon). \end{aligned} \tag{67}$$

Dimension of a generic fiber $(F, \mathcal{F})^{-1}(\cdot)$ is equal to codimension of separating solutions of the Pfaffian system. An essential feature of the above map is the following one: *the coordinate functions of the map F are linear combinations of the coordinates on the source space and generic functions f_j*:

$$F_j \in \mathbb{Z}[\varepsilon, \lambda], \qquad \deg F_j = 1, \tag{68}$$

and all coefficients of those polynomials are ± 1. A more invariant way of formulating the same property is to say that F is a linear map in ε's and λ's. The rigid functional part is defined on the space of 0-jets of vector-functions

$$\mathcal{F}_j \in \mathbb{Z}[x, \varepsilon, \lambda, \mathbf{f}], \qquad \deg \mathcal{F}_j = 1, \tag{69}$$

this makes sense since $\mathbf{f} : M \to \mathbb{R}^n$ is defined on an open set of a Euclidean space.

Definition 17. *The functional–Pfaffian system with the Pfaffian equations (65), the functional equations (67), defined on the domain (63) considered with the covering function (64), will be called the principal functional–Pfaffian system. Given estimates (66), (68), and (69) we say that the principal system is effectively described.*

Later on we will refer to the principal system as simply system (58). Since the principal system is constructed, Theorem 10 is proved. Q.E.D.

4.5. **Reduction of problem of estimating the number of limit cycles to problem of estimating multiplicity of a chain map.** The system (58), whose Khovanski number majorizes the number of solutions to the unspecified basic system (26), satisfies the conditions of Theorem 8. The conclusion of the latter claims that the number $\mathcal{K}\{\Omega, F; \mathcal{F}\}$ is in turn majorized by the combination of Khovanski numbers for some $n + 2s + k + m + 1$ entirely rigid systems (recall that $n+2s$ is the number of Pfaffian equations and $k+m$ is the number of loose functional equations in the principal system, which should be eliminated). The properties of the principal system, listed in the formulation of Theorem 10, yield a complete description of the resulting systems as *chain maps* (the definition is given below).

In what follows we treat the original variables x_j, y_j, the auxiliary variables $z_{j\alpha}$, $w_{j\alpha}$ and the parameters ε, λ in an almost similar way, as it is suggested by the functional equations (67) of the principal system (58). The algebraic part $\mathcal{S}_{alg}$ of the specification, however, plays a different role: the coordinates of the localization points $\mathbf{c_j}$ and the integers $n_{j_\alpha}, m_{j_\alpha}$ determining the hyperbolicity ratios of resonant saddles, would determine the point in the new phase space, around which the resulting chain maps will be considered. Recall that in Sect. 1 we introduced the vectors $\mathbf{c}_j$ and $\mathbf{c}$ as

$$\mathbf{c}_j = (0, \dots, 0, c_j) \in \mathbb{R}^{\mu_j+1}, \qquad c_j \in \mathbb{R}^1,$$
$$\mathbf{c} = (\mathbf{c}_1, \dots, \mathbf{c}_n) \in \mathbb{R}^m, \qquad m = n + \sum \mu_j.$$

For our purposes it would be convenient to consider all (new) variables as taking values around the origin in the corresponding phase space. For this sake we make a parallel translation in the λ-space, which would take the origin into the point $\mathbf{c}$. Clearly, this translation does not affect the algebraic structure of the principal system (58), though changes appearance of the equations.

The characteristic size r retains its original meaning.

Notations: According to what has been said, we introduce the following notations:

$$\mathbf{x} = (x, y, z, w, \varepsilon, \lambda - \mathbf{c}) \in \mathbb{R}^{2n+2s+k+m},$$
$$\mathbf{f} = (f_1, \dots, f_n), \qquad f_j = f_j(y_j, \varepsilon) \iff \mathbf{f} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{f}$ is now considered as a vector-function of the argument $\mathbf{x}$, though each coordinate function f_j of the vector $\mathbf{f}$ depends in fact only on some of the coordinates of the vector $\mathbf{x}$. By $D^{\mathsf{p}}\mathbf{f}$ we denote the collection of all partial derivatives of functions f_j of the order p with respect to variables y_j and ε.

We will also use the same notation $M(r)$ for the domain of the principal system, though in fact it would become a subset of the unit cube $\|\mathbf{x}\| < r$ centered at the origin in the $\mathbf{x}$-space. Now we can formulate the properties of the systems of equations which appear after elimination of Pfaffian equations from the principal system (58) as this was described in §3.4.

Theorem 11. *Let* $\mathbf{m} = 2n + 2s + k + m$. *For any fixed combinatorial type* $\mathcal{T}$ *of the principal functional-Pfaffian system (58), any choice of the algebraic part* $\mathcal{S}_{alg}$ *of the specification, and sufficiently fast decaying to zero sequence of small numbers* $a_1, \dots, a_{\mathbf{m}}$, *i.e.* $1 \gg |a_1| \gg \cdots \gg |a_{\mathbf{m}}| \geq 0$, *the number of regular solutions to the principal system in the domain* $M(r)$ *for any choice of the characteristic size* $r > 0$ *does not exceed the sum of the Khovanski numbers for* $\mathbf{m} - n + 1$ *entirely rigid system of equations in the same domain. Each of these systems has the form*

$$\mathbf{P}^j\big(\mathbf{x}, \mathbf{f}(\mathbf{x}), \dots, D^{\mathbf{m}-n}\mathbf{f}(\mathbf{x}); \mathcal{S}_{alg}, r\big) = (a_1, \dots, a_{\mathbf{m}}), \qquad \mathbf{x} \in M(r) \subseteq \mathbb{R}^{\mathbf{m}},\ j = 0, \dots, \mathbf{m} - n \tag{70}$$

where

- $\mathbf{m} \leq 6k$ *is the total number of variables (dimension of the phase space);*
- $\mathbf{P}_{\mathcal{S}_{alg}}$ *is a vector polynomial,* $\mathbf{P}^j = (P_1^j, \dots, P_{\mathbf{m}}^j)$ *and* $P_i^j \in \mathbb{Z}[\mathbf{x}, \dots; \mathcal{S}_{alg}, r]$; *the degree of each polynomial* P_i^j *is bounded by* $14k2^{i-n-1}$ *for* $i = n + 1, \dots, \mathbf{m}$ *and by* 1 *for* $i = 1, \dots, n$, *where* $j = 0, \dots, \mathbf{m} - n$;
- *the domain* $M(r)$ *belongs to the* r*-cube* U_r *of the space* $\mathbb{R}^{\mathbf{m}}$, *centered at the origin.*

Proof. To prove notice that by Theorem 10 the principal system admits application of the Khovanski reduction method. Since the number of Pfaffian and loose functional equations $\mathbf{m} - n$ is bounded by $5k$, we can apply Theorem 8 and construct $5\mathbf{m} - n + 1 \leq 5k + 1$ additional purely rigid systems. We point out in Sect. 3.4 that the Khovanski method involves only algebraic operations. Theorem 10 provides estimates on degrees of involved polynomials for the principal mixed functional-Pfaffian system (58). These estimates are given by the second and third lines with •'s. Therefore, the principal system (58) admits application of Theorem 9. Straightforward calculations with $D \leq 14k$, $d \leq 7k$, and $n \leq 6k$ prove required in Theorem 11 estimates. Q.E.D.

4.6. **Chain maps and related finiteness theorems.** Now we proceed with a more invariant description of the geometric object corresponding to the system of equations (70).

Definition 18. *Let* $\mathbb{R}^{\mathbf{m}}$ *be a Euclidean space with a fixed coordinate system* $\mathbf{x} = (X_1, \dots, X_{\mathbf{m}})$, *and* $U \subseteq \mathbb{R}^{\mathbf{m}}$ *a domain of the rectangular form,*

$$U = \{\alpha_i < X_i < \beta_i,\ i = 1, \dots, \mathbf{m}\}.$$

Denote by I *the index set* $I = \{1, \dots, \mathbf{m}\}$ *enumerating the coordinates in* $\mathbb{R}^{\mathbf{m}}$, *and let for any* $j = 1, \dots, n$ *and* I_j *be a nonempty subset of* I. *Then we say that a vector-valued function*

$$\mathbf{f} : \mathbb{R}^{\mathbf{m}} \supseteq U \mapsto \mathbb{R}^n, \qquad \mathbf{f} = (f_1, \dots, f_n), \tag{71}$$

is a Cartesian function of the Cartesian type $\mathcal{I} = (I_1, \dots, I_n)$*, if for any* j *the* j*-th component of this function depends only on the coordinates* X_i *with* $i \in I_j$*: in other words,*

$$\forall i \notin I_j \quad \frac{\partial f_j}{\partial X_i} \equiv 0.$$

For any given Cartesian type $\mathcal{I}$ the set of all C^{p}-smooth Cartesian functions (or maps) of this type constitutes a Banach space with the natural C^{p}-norm. We denote this space by $\mathbf{C}^{\mathsf{p}}_{\mathcal{I}}$, sometimes omitting the explicit reference to the type $\mathcal{I}$ when the latter is clear from context. The space $\mathbf{C}^{\mathsf{p}}_{\mathcal{I}}$ will be referred to as the *Cartesian space*. In the same way the *Cartesian spaces of maps* arise. As a consequence, we may say about *genericity* of Cartesian functions (maps) within the given Cartesian type; the notions of openness and density of subsets are also naturally defined.

Definition 19. *Let* $\mathbf{f}$ *be a* C^{p}*-smooth Cartesian map of a given Cartesian type* $\mathcal{I}$*, and* $s \geq 0$ *an nonnegative integer number with* $s \leq \mathsf{p}$*. A Cartesian* s*-jet of the function* $\mathbf{f}$ *at a point* $\mathbf{x_0} \in \mathbf{U}$ *is the equivalence class of all Cartesian functions of the same Cartesian type, which differ from* $\mathbf{f}$ *by a term which is* s*-flat at* $\mathbf{x_0}$*:*

$$\mathbf{j}^s_{\mathcal{I}}\mathbf{f}(\mathbf{x}_0) = \{\mathbf{g} \in \mathbf{C}^{\mathsf{p}}_{\mathcal{I}} : |\mathbf{f} - \mathbf{g}| = o(|\mathbf{x} - \mathbf{x}_0|^s)\}.$$

The space of all s*-jets of functions of the given Cartesian type* $\mathcal{I}$ *at all points* $\mathbf{x}_0 \in U$ *will be denoted by* $\mathbf{J}^s_{\mathcal{I}}(\mathbb{R}^{\mathbf{m}}, \mathbb{R}^{\mathbf{n}})$ *or simply by* $\mathbf{J}^s_{\mathcal{I}}$*, when the environment is unambiguously defined by the context.*

The map

$$\mathbf{x} \mapsto \mathbf{j}^s\mathbf{f}(\mathbf{x})$$

is called the Cartesian s*-jet extension of the Cartesian map* $\mathbf{f}$.

The space of Cartesian jets of any type and any finite order admits a natural coordinate system, in which the Cartesian jet extension of a map $\mathbf{f} = (f_1, \dots, f_{\mathbf{n}})$ takes the form

$$\mathbf{x} = (X_1, \dots, X_{\mathbf{m}}) \mapsto \left(\mathbf{x}, \mathbf{f}(\mathbf{x}), \left\{\frac{\partial \mathbf{f}_j}{\partial X_i},\ i \in I_j\right\}, \dots, \left\{\begin{array}{c}\text{all partial derivatives of functions } f_j \text{ of all orders up to} \\ s \text{ in the variables on which each } f_j \text{ actually depends}\end{array}\right\}\right)$$

The Cartesian jet spaces possess almost all properties of the standard jet spaces. In particular, the natural projections

(72)
$$\mathbb{R}^M \supseteq U \xleftarrow{pr_0} \mathbf{J}^0_{\mathcal{I}} \simeq \mathbb{R}^M \times \mathbb{R}^K \xleftarrow{pr_1} \cdots \xleftarrow{pr_s} \mathbf{J}^s_{\mathcal{I}} \xleftarrow{pr_{s+1}} \cdots$$

are well defined and endow each $\mathbf{J}^s_{\mathcal{I}}$ with the structure of an affine bundle over $\mathbb{R}^{\mathbf{m}}$. Thus it makes sense to say about *polynomial functions* defined on Cartesian bundles.

Definition 20. *A chain map with the exterior part* $\mathbf{P}$ *and the interior part* $\mathbf{f}$ *is a map of the form*

$$\mathbb{R}^{\mathbf{m}} \supseteq U \ni \mathbf{x} \mapsto \mathbf{P}\big(\mathbf{j}^s_{\mathcal{I}}\mathbf{f}(\mathbf{x})\big) \in \mathbb{R}^{\mathbf{m}},$$

where:

- $\mathbf{f}$ *is a Cartesian map from a certain Cartesian space* $\mathbf{C}^{\mathsf{p}}_{\mathcal{I}}(\mathbb{R}^{\mathbf{m}}, \mathbb{R}^{\mathbf{n}})$*, and* $\mathbf{j}^s_{\mathcal{I}}$ *is the corresponding* s*-jet extension of* $\mathbf{f}$*;*
- $\mathbf{P} : \mathbf{J}^s_{\mathcal{I}}(\mathbb{R}^{\mathbf{m}}, \mathbb{R}^n) \to \mathbb{R}^{\mathbf{m}}$ *is a vector polynomial (eventually depending polynomially on some additional parameters),*
- *the composite map is between the spaces of the same dimension:* $\dim \mathbf{x} = \dim \mathbf{P} = \mathbf{m}$.

Once introduced the notions of Cartesian functions, maps, jets etc, we can describe the system (70) as a chain map. Such a chain map is defined on a small cube of some size $r > 0$ with the exterior part $\mathbf{P}$ which is a polynomial with integer coefficients and of a controlled complexity. This polynomial $\mathbf{P}$ depends on r and some additional variables S_{alg} as well, and the interior part $\mathbf{f}$ belongs to some Cartesian space, because the functions f_j depend only on some components of the vector $\mathbf{x} = (x, y, z, w, \varepsilon, \lambda - \mathbf{c})$ (recall that all nonzero coordinates of the vector $\mathbf{c}$ are already included among the variables $\mathcal{S}_{alg}$). Thus our problem of estimating cyclicity of a polycycle takes the following form: describe the Cartesian maps $\mathbf{f}$ for which the chain map admits an upper estimate for the number of preimages of regular values.

Consider chain maps of the form

$$\text{(73)}\qquad \begin{gathered} \mathbf{x} \mapsto \mathbf{G}_r(\mathbf{x}) = \mathbf{P}\big(\mathbf{j}^{\mathbf{m}-n}_{\mathcal{I}}\mathbf{f}(\mathbf{x}), r\big) = (P_1, \dots, P_{\mathbf{m}})\big(\mathbf{j}^{\mathbf{m}-n}_{\mathcal{I}}\mathbf{f}(\mathbf{x}), r\big), \\ \mathbf{x} \in U \subset \mathbb{R}^{\mathbf{m}},\ r > 0, \end{gathered}$$

depending polynomially on an additional variable r, so that

$$\text{(74)}\qquad \mathbf{P} : \mathbf{J}^{\mathbf{m}}_{\mathcal{I}}(\mathbb{R}^{\mathbf{m}}, \mathbb{R}^{\mathbf{n}}) \times \mathbb{R}^1 \to \mathbb{R}^{\mathbf{m}}, \quad \mathbf{f} \in \mathbf{C}^{\mathsf{p}}_{\mathcal{I}}(U, \mathbb{R}^{\mathbf{n}}).$$

We assume that the polynomial $\mathbf{P}$ and the Cartesian type $\mathcal{I}$ are fixed (as before and U denotes a unit cube) and $\mathbf{P}$ is a nontrivial polynomial, i.e. at some point $x \in U$ the linearization matrix $d\mathbf{P}(x)$ has full rank.

Suppose that the smoothness order p is sufficiently high, i.e. $\mathsf{p} > \mathbf{m} + 1$. Now we can formulate Bezout's Theorem for chain maps.

Theorem 12. *Consider a nontrivial polynomial* $\mathbf{P} = (P^1, \dots, P^{\mathbf{m}})$*, defined in (74). Then there is an open dense subset* $\mathsf{F}_{\mathbf{P}} \subset \mathbf{C}^{\mathsf{p}}_{\mathcal{I}}(U, \mathbb{R}^{\mathbf{n}})$ *in the space of Cartesian functions of the given type and an integer* $d = d(P)$ *such that for each* $f \in \mathcal{F}_{\mathbf{P}}$ *there exists a characteristic size* $r_0 > 0$ *and* $\delta > 0$ *with the property if* $0 < |a_1| < \delta$ *and* $0 < |a_{j+1}| < |a_1 \dots a_j|^d$ *for*

$j = 1, \dots, \mathbf{m} - 1$, *then the number of regular preimages of* $(a_1, \dots, a_{\mathbf{m}})$ *(or the Khovanski number) admits the following upper estimate:*

$$\#\{\mathbf{x} \in U_r : \quad \mathbf{G}_r(\mathbf{x}) = (a_1, \dots, a_{\mathbf{m}})\} \le \prod_{i=1}^{\mathbf{m}} \deg P^i, \tag{75}$$

where $0 < r \le r_0$ *and* $U_r \subset \mathbb{R}^{\mathbf{m}}$ *is an r-cube, centered at the origin.*

We consequently reformulate this theorem in form of Theorem 13, then Theorem 15 in Sect. 6 and finally in Sect. 7 we reduce the latter Theorem to Theorem 16 and prove it. Let's apply estimates on degree of P_i's obtained in Theorem 11 to obtain the main result of this paper by modulo of Theorem 12.

Main Corollary. *For any elementary polycycle* γ *occurring in a generic k-parameter family, for any type* $\mathcal{T}_\gamma$ *of an unspecified basic system associated* γ *and any choice of the algebraic part* $\mathcal{S}_{alg,\gamma}$ *corresponding to* γ *one may choose the order of smoothness* $\mathsf{p} > 6k + 1$ *and an open dense subset* $\mathsf{F}_{\mathbf{P}_\gamma} = \mathsf{F}_{\mathcal{T},\mathcal{S}_{\mathsf{alg},,},\mathsf{r}_0}$ *in the space of* C^{p_0}*-smooth functions* $C^{\mathsf{p}_0}(I_{r_0} \times B_{r_0}, \mathbb{R}^n)$ *such that for every* $\mathbf{f}_\gamma = (f_1, \dots, f_n) \in \mathsf{F}_\gamma$ *and a sufficiently small characteristic size* $r_0 = r_0(\mathbf{f})$ *the maximal number of isolated solutions* $\mathcal{B}_k(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}_\gamma; r_0)$ *to the specified basic system* $(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}_\gamma)$ *in the domain* I_{r_0} *is uniformly bounded over all parameter values* $(\lambda, \varepsilon) \in B_{r_0}$*:*

$$\begin{aligned} \mathcal{B}_k(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}_\gamma; r_0) &\le k + (5k+1) \prod_{i=n+1}^{\mathbf{m}} 14k2^{i-n-1} \\ &\le k + (14k)^{5k+1} 2^{5k(5k-1)/2} \le 2^{25k^2} \end{aligned} \tag{76}$$

for any $k \ge 2$.

Remark 5. Recall that $E(2) = 2$ (see the introduction) and by Theorem of Trifonov [Tr2], [IK] which unifies efforts of several mathematicians we know that $E(3) = 3$. Therefore, this is a new estimate for $E(k)$ only for $k \ge 4$.

Proof of the Main Corollary. Recall that the number of isolated solutions $\mathcal{B}_k(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}_\gamma; r_0)$, defined in (26) and definition 9 of the specified basic system $(\mathcal{T}, \mathcal{S}_{alg}, \mathbf{f})$. To prove (76) we apply Theorem 11 and get an estimate on $\mathcal{B}_k(\mathcal{T}_\gamma, \mathcal{S}_{alg,\gamma}, \mathbf{f}_\gamma; r_0)$. Such an estimate is given in terms of the Khovanski numbers for chain maps (73). By Theorem 12 estimates on the Khovanski numbers of these chain maps is given in terms of products of degrees of the involved polynomials. Estimates on degrees of these polynomials are given by Theorem 11. Direct calculations complete the proof of the Main Corollary. Q.E.D.

Cartesian transversality theorem. (Shelkovnikov [IY3]) *The set of Cartesian functions* $f \in C^{\mathsf{p}}_{\mathcal{I}}(U, \mathbb{R}^n)$ *either itself or their jets satisfying a transversality condition form an open and dense set in the space* $C^{\mathsf{p}}_{\mathcal{I}}(U, \mathbb{R}^n)$.

5. Application of stratification theory

Our goal from now on is to prove Theorem 12 below, which is implied by Theorem 13 (Bezout's Theorem for chain maps). The proof of the latter Theorem is product of application of stratified sets theory. First we reformulate Theorem 12 in the form of Theorem 13. Then we reduce Theorem 13 to Theorem 15 and prove the latter one.

5.1. **Reduction to singularity theory applied to chain maps.** To prove Theorem 12 we reformulate this theorem in slightly different way and then prove it.

A bit of terminology: "Replace an $\mathbf{m}$-th jet $j^{\mathbf{m}}F$ by its linear part at a point $a \in \mathbb{R}^{\mathbf{m}}$" means "replace the map $j^{\mathbf{m}}F : \mathbb{R}^{\mathbf{m}} \to J^{\mathbf{m}}_{\mathcal{I}}(\mathbb{R}^{\mathbf{m}}, \mathbb{R}^{\mathbf{n}})$ by its linear part $L_{F,a,\mathbf{m}}$ at the point a".

By the phrase "a map $G : M \to N$ of manifolds satisfies *a transversality condition*" we mean that for some manifold (resp. a collection of manifolds) in the image N the map G is transversal to this manifold (resp. these manifolds).

The present and the last stage of proof of the Main Theorem (2) consists of constructing a stratification of the $\mathbf{m}$-jet space $J^{\mathbf{m}}_{\mathcal{I}}(\mathbb{R}^{\mathbf{m}}, \mathbb{R}^n)$ (a decomposition into a disjoint union of manifolds described below) such that if the $\mathbf{m}$-jet $j^{\mathbf{m}}F$ is transversal to all manifolds of this stratification, then the following theorem is true:

Theorem 13. *Let $\mathcal{I} = (I_1, \dots, I_n)$ be a Cartesian type, $\mathbf{P} = (P_1, \dots, P_{\mathbf{m}})$ be a nontrivial polynomial defined on the Cartesian space of $(\mathbf{m} - n)$-jets $\mathbf{P} : J^{\mathbf{m}-n}_{\mathcal{I}}(\mathbb{R}^{\mathbf{m}}, \mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}^n$ and let $\mathbf{f} : \mathbb{R}^{\mathbf{m}} \to \mathbb{R}^n$ be a C^{p}-smooth map from the space of Cartesian functions $C^{\mathsf{p}}_{\mathcal{I}}(U, \mathbb{R}^n)$ for $\mathsf{p} > (\mathbf{m} - n)$. Suppose the $(\mathbf{m} - n)$-jet $j^{(\mathbf{m}-n)}_{\mathcal{I}}\mathbf{f}$ satisfies a transversality condition $\mathsf{F}_{\mathbf{P}}$ depending only on P. Then for sufficiently small $r > 0$, $\delta > 0$, and $d = d(P) \in \mathbb{Z}_+$ and one can replace in the statement of the previous theorem the $(\mathbf{m} - n)$-jet $j^{\mathbf{m}-n}\mathbf{f}$ at the point a by its linear part $L_{j^{\mathbf{m}-n}\mathbf{f},a}$. Namely,*

$$\begin{aligned}(77)\quad &\#\{x \in B_r(a) : \quad P_1 \circ j^{\mathbf{m}-n}_{\mathcal{I}}\mathbf{f}(x) = a_1, \dots, P_{\mathbf{m}} \circ j^{\mathbf{m}-n}_{\mathcal{I}}\mathbf{f}(x) = a_{\mathbf{m}}\} = \\ &\#\{x \in B_r(a) : \quad P_1 \circ L_{j^{\mathbf{m}-n}\mathbf{f},a}(x) = a_1, \dots, P_{\mathbf{m}} \circ L_{j^{\mathbf{m}-n}\mathbf{f},a}(x) = a_{\mathbf{m}}\},\end{aligned}$$

where $0 < a_1 < \delta$ *and* $0 < |a_{j+1}| < |a_1 \dots a_j|^d$ *for* $j = 1, \dots, \mathbf{m} - 1$.

Remarks. 1. The classical transversality Theorem [AGV] says that for a generic (open dense) map F its $(\mathbf{m}-n)$-jet $j^{\mathbf{m}-n}F$ satisfies any ahead given transversality condition. Prevalent version of this theorem saying that for an "almost every" map F or F "with probability one" a transversality condition for $(\mathbf{m}-n)$-jet $j^{\mathbf{m}-n}F$ satisfied is proved in [Ka4]. For Theorem 13 we need not just the classical transversality Theorem, but the Cartesian Transversality Theorem stated above. For simplicity of notations in what follows we omit Cartesian dependence on $\mathcal{I}$.

2. After a smooth map is replaced by its linear part Bezout's Theorem implies that the number of isolated solutions to the equation in the right-hand side of (77) can be bounded by the product $\prod_{i=1}^{n} \deg P_i$. Therefore, this implies Theorem 12.

5.2. **Stratified manifolds.** Now we recall basic definitions from the theory of stratified sets.

Let M be a smooth manifold, which we call the *ambient manifold*. Consider a singular subset $V \subset M$. Roughly speaking a stratification of V is a decomposition of V into a disjoint union of manifolds (strata) $\{V_\alpha\}_\alpha$ such that *strata of bigger dimension are attached to strata of smaller dimension in a "regular" way*.

"Regular" will obtain a precise meaning in a moment, but the most important property is that *transversality to a smaller stratum implies transversality to an "attached" bigger stratum.* Now we are going to describe the standard language of stratified manifolds and maps of stratified manifolds. This goes back to Whitney and Thom [W], [Th].

Recall the Whitney Conditions (a) and (b). Condition (a) is similar to the notion of a_P-stratification due to Thom [Th] defined in the next subsection. We shall use a_P-stratification to prove condition (15).

Consider a triple (V_β, V_α, x), where $V_\beta,\ V_\alpha$ are C^1 manifolds, x is a point in V_β and $V_\beta \subseteq \bar{V}_\alpha \setminus V_\alpha$.

Definition 21. *A triple (V_β, V_α, x) satisfies the Whitney (a) condition if for any sequence of points $\{x_k\} \subset V_\alpha$ converging to a point $x \in V_\beta$ the sequence of tangent planes $T_k = T_{x_k} V_\alpha$ converges in the corresponding Grassmanian manifold of* $\dim V_\alpha$*-planes in TM and* $\lim T_k = \tau \supset T_x V_\beta$. *Since the Grassmanian manifold of* $\dim V_\alpha$*-planes in* $\dim M$*-space is compact, for any sequence $\{x_k\} \subset V_\alpha$ one can find a subsequence $\{x_{n_k}\} \subset V_\alpha$ such that* $\lim T_{x_{n_k}} V_\alpha$ *exists.*

Remark 6. Since Grassmanian manifold of $\dim V_\alpha$-dimensional planes in $\dim M$-dimensional space is compact, for any converging sequence of points $\{x_k\} \subset V_\alpha$ one can find a subsequence $\{x_{n_k}\} \subset V_\alpha$ such that the limit of tangent planes $T_{x_{n_k}} V_\alpha$ exists.

Definition 22. *A triple (V_β, V_α, x) satisfies the Whitney (b) condition if for any two sequences of points $\{x_k\} \subset V_\alpha$, $\{y_k\} \subset V_\alpha$ converging to a point $x \in V_\beta$ the sequence of "vectors" $\frac{y_k - x_k}{|y_k - x_k|}$ converges to a vector $v \in T_x M$ which belongs to a limiting position of* $\lim T_{x_k} V_\alpha = \tau$*, i.e. $v \in \tau$.* [4]

Since condition (b) is local one can think of M as Euclidean. This explains how to interpret the vector $\frac{y_k - x_k}{|y_k - x_k|}$.

It is easy to show that condition (b) implies condition (a).

[4] Definition of regularity in this form was first formulated by J. Mather [Ma], Whitney's original definition is equivalent to this one provided condition (a) holds

Definition 23. *A locally closed subset V in the ambient manifold M is called a stratified manifold (set, variety) in M, if it is represented as a locally finite disjoint union of smooth submanifolds V_α of M, called strata, of different dimensions in such a way that the closure of each stratum consists of itself and the union of some other strata of strictly smaller dimensions, and Condition (b) of Whitney is satisfied.*

Any union of submanifolds satisfying condition of this definition

$$V = \cup_\alpha V_\alpha \tag{78}$$

is called a *stratification* of V, and the submanifolds V_α are called *strata*. A set V is *stratifiable* if there is a "nice" partition into strata. By *a stratified manifold* we mean a pair $(V, \mathcal{V})$ consisting of a manifold V itself and a partition $\mathcal{V} = \{V_\alpha\}$.

5.3. **Stratified maps and a_P-stratification.** Now we define a smooth map of a stratified manifold $(V, \mathcal{V})$:

Definition 24. *Let $(V, \mathcal{V})$ be a stratified manifold in an ambient manifold M, $V \subseteq M$, then a map $f : V \to N$ is called C^2-smooth if it can be extended to a C^2 smooth map of the ambient manifold M $F : M \to N$ whose restriction to V coincides with f.*

A stratification $V = \cup_\alpha V_\alpha$ stratifies a smooth map $f : V \to \mathbb{R}^k$ if the restriction of f to any stratum V_α has constant rank, i.e., rank $df|_{V_\alpha}(x)$ is independent of $x \in V_\alpha$.

A map $G : L \to M$ is called transversal to a stratified set $(V, \mathcal{V})$ if G is transversal to each strata $V_\alpha \in \mathcal{V}$.

By the Rank Theorem [GG], if a stratification $(V, \mathcal{V})$, $\mathcal{V} = \{V_\alpha\}_{\alpha \in I}$ stratifies a smooth map P, then for each strata V_α the number $d_\alpha(P) = \dim V_\alpha - \operatorname{rank} dP|_{V_\alpha}$ is well defined.

Assume $d_\alpha(P) \geq d_\beta(P)$ for each $V_\beta \subseteq \bar{V}_\alpha \setminus V_\alpha$, i.e. nonempty level sets inside the bigger stratum V_α have dimension $d_\alpha(P)$ greater or equal to dimension of the level sets $d_\beta(P)$ in the smaller stratum V_β. We require that for any sequence of points $\{a_k\} \subset P(V_\alpha)$ converging to a point $a \in P(V_\beta)$, the level sets $\{P^{-1}(a_k) \cap V_\alpha\}$ approach the limiting level set $\{P^{-1}(a) \cap V_\beta\}$ "regularly". In other words, we *require that the level sets in the bigger stratum V_α approach the limit level set in the smaller stratum V_β nicely.*

Definition 25. *Let $P : M \to N$ be a C^2 smooth map of manifolds, and let V_β and V_α be submanifolds of M such that the restrictions $P|_{V_\beta}$ to V_β and $P|_{V_\alpha}$ to V_α have constant ranks $R_{V_\beta}(P)$ and $R_{V_\alpha}(P)$, respectively. Let x be a point in V_β.*

We call the manifold V_α a_P-regular over V_β with respect to the map P at the point x if for any sequence of points $\{x_k\} \subset V_\alpha$ converging to $x \in V_\beta$ such *that the sequence of tangent planes to the level sets* $T_k = ker\ dP|_{V_\alpha}(x_k)$ *converges in the corresponding Grassmanian manifold of* $(\dim V_\alpha - R_{V_\alpha}(P))$*-dimensional planes to some plane* τ*, then we have*

$$\lim ker\ dP|_{V_\alpha}(x_k) = \tau \supseteq ker\ dP|_{V_\beta}(x). \tag{79}$$

Remark 7. Similarly to the case a-regular stratification, the Grassmanian manifold of $(\dim V_\alpha - R_{V_\alpha}(P))$-dimensional planes in $(\dim M - R_{V_\alpha}(P)$-space is compact, so for any sequence $\{x_k\} \subset V_\alpha$ one can find a subsequence $\{x_{n_k}\} \subset V_\alpha$ such that the limit of tangent planes to level sets $\lim ker\ dP|_{V_\alpha}(x_{n_k})$ exists.

Definition 26. *A* C^2 *smooth map* $P : V \to N$ *of a stratifiable manifold* V *to a manifold* N *is called* a_P*-stratifiable if there exist a stratification* $(V, \mathcal{V})$ *such that the following conditions hold:*

a) $(V, \mathcal{V})$ *stratifies the map* P *(see Definition 24);*
b) for all pairs V_β *and* V_α *from* $\mathcal{V}$ *such that* $V_\beta \subseteq \bar{V}_\alpha \setminus V_\alpha$ *the stratum* V_α *is* a_P*-regular over the stratum* V_β *with respect to* P *at point* x *for all* $x \in V_\beta$.

The original definition of a_P-stratification requires an appropriate stratification in the image too [Ma], however for our purposes a stratification in the image is redundant.

5.4. **Relation between existence of a_P-stratification and condition (15).** In Sect. 1.7.1 we showed that the key to the proof of Theorem 12 is condition (15) of Proposition 1. Now we are going to reduce the question whether condition (15) is satisfied to the question whether an a_P-stratification of the polynomial P exists.

Let $P = (P_1, P_2) : \mathbb{R}^N \to \mathbb{R}^2$ be a nontrivial polynomial, $V = P_2^{-1}(0)$ and $V_0 = (P_1, P_2)^{-1}(0)$ be level sets. Assume that there exists a stratification $(V, \mathcal{V})$ that stratifies the map $P|_V$ such that the zero level set V_0 can be represented as a union of strata from $\mathcal{V}$, i.e. $V_0 = \cup_{\alpha \in I_0} V_\alpha$. Denote this stratification of V_0 by $\mathcal{V}_0$. Recall that a map $F : \mathbb{R}^k \to \mathbb{R}^N$ is transversal to a stratification $(V_0, \mathcal{V}_0)$ if it is transversal to each strata $V_\alpha \in \mathcal{V}_0$. Associate to each level set V_a, $a \neq 0$ a natural decomposition $\mathcal{V}_a = \{V_a \cap V_\alpha\}_{\alpha \in I}$.

Proposition 2. *With the above notation if a stratum* $V_\alpha \in \mathcal{V} \setminus \mathcal{V}_0$ *is* a_P*-regular over a stratum* $V_\beta \in \mathcal{V}_0$ *with respect to the polynomial* P*, then any* C^2 *smooth map* $F : \mathbb{R}^n \to \mathbb{R}^2$ *transversal to* $(V_0, \mathcal{V}_0)$ *is also transversal to* $V_a \cap V_\alpha$ *for any small* a*. This implies condition (15).*

Proof. Pick a point x in $V_\beta \subset V_0$ and a point $y \in V_\alpha$. Notice that $ker\ dP|_{V_\beta}(x)$ is the tangent plane to the level set $\{P^{-1}(P(x)) \cap V_\beta\}$ at the point x and $ker\ dP|_{V_\alpha}(y)$ is the tangent plane to the level set $\{P^{-1}(P(y)) \cap V_\alpha\}$.

By condition (79) if a map $F : X \to \mathbb{R}^N$ is transversal to $ker\ dP|_{V_\beta}(x)$ at a point x, then F is transversal to $ker\ dP|_{V_\alpha}(y)$ for any $y \in V_\alpha$ near x.

Therefore, the condition "F is transversal to V_β at a point x" implies the condition "F is transversal to $V_\alpha \cap V_a$ for any small a", see (15). This completes the proof of the Proposition. Q.E.D.

6. Existence of a_P-stratification for polynomial maps

In this section we discuss the question of existence of a_P-stratification. As examples below show existence of a_P-stratification is a nontrivial question. In general, it does not exist.

6.1. **Nonexistence of a_P-stratification because of noncompatibility.** Now we show some clear obstacles for existence of a_P-stratification. For example, let $V \subset \mathbb{R}^n$ be an algebraic variety and let $P : \mathbb{R}^n \to \mathbb{R}^k$ be a polynomial map. Assume that $(V, \mathcal{V})$ stratifies P. If we have two strata V_α and V_β so that V_α lies "over" $V_\beta \subseteq \bar{V}_\alpha \setminus V_\alpha$, then condition (79) can't be satisfied if dimension of the level sets $d_\alpha(P)$ in the upper stratum V_α is strictly less than that of $d_\beta(P)$ in the lower stratum V_β, i.e., $\dim ker\ dP|_{V_\alpha}(y) < \dim ker\ dP|_{V_\beta}$. In this case a plane $ker\ dP|_{V_\beta}(x)$ of the lower stratum V_β should belong to a plane τ of smaller dimension (see condition (79)), which is impossible. Thom constructed the first example when this happens [GWPL].

Thom's example: Consider the "blow-up" vector-polynomial P in the form $P : (x, y) \to (x, xy)$. The line $\{x = 0\}$ is the line of critical points of P. Outside of the line $\{x = 0\}$ P is a diffeomorphism. Therefore, the preimage $P^{-1}(a)$ of any point $a \neq 0$ is the 0-dimensional point. On the other hand, the preimage of 0 is the 1-dimensional line $\{x = 0\}$.

Definition 27. *We call an algebraic set V rank compatible with respect to a polynomial P if there exists a stratification $(V, \mathcal{V})$ which stratifies P and for any pair V_α and V_β from $\mathcal{V}$ such that $V_\beta \subseteq \bar{V}_\alpha \setminus V_\alpha$ the dimension of the nonempty levels sets $d_\beta(P)$ in the lower stratum V_β does not exceed the corresponding dimension $d_\alpha(P)$ in the upper stratum V_α.*

It turns out that even if an algebraic set V is rank compatible with a polynomial P, then a_P-stratification still does not always exist. Let us present an example with this property. The example below belongs to M.Grinberg. It seems that the existence of a counterexample was known before, but we did not find an appropriate reference.

6.2. **Nonexistence of a_P-stratification with compatibility.** Let $V = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 = t^2 y + z\}$ be the three dimensional algebraic variety and $P : V \to \mathbb{R}^2$ be the natural projection to the last two coordinates, i.e. $P : (x, y, z, t) \to (z, t)$.

Lemma 2. *With the above notations the set V is rank compatible with respect to the polynomial map P and does not have a_P-stratification.*

Proof. Consider a rank stratification of V. Such a stratification consists of three stratum: $V_1 = \{x = t = z = 0\}$, $V_2 = \{t = 0,\ x^2 = z,\ x \neq 0\}$, and $V_3 = \{t \neq 0\}$. On each stratum $\text{rank} P|_{V_i} = i - 1$. Level sets $P^{-1}(t, z)$— parabolas for $t \neq 0$ and lines for $t = 0$.

Show that for each point $\mathbf{a} = (0, a, 0, 0) \in V_1$ there exists a family of level sets such that at the point $\mathbf{a}$ the property a_P-regularity of V_3 over V_1 fails.

Consider the preimage of the curve $\{z = -at^2\} \subset \mathbb{R}^2$. This is an algebraic variety of the form $W_a = \{x^2 = t^2(y - a)\}$. One can see that W_a is the Whitney umbrella and the level $x^2 = t_0^2(y - a)$ is the parabola. As $t_0 \to 0$ this parabola tends to semiline $x = t = z = 0$, $y \geq a$. At the point $\mathbf{a} \in V_1$ the property a_P-regularity of V_3 over V_1 clearly fails. This completes the proof of the Lemma. Q.E.D.

Let us mention a positive result on existence of a_P-stratification.

Theorem 14. [Hir1] *If $V \subset \mathbb{R}^n$ is a semialgebraic variety and $P : \mathbb{R}^n \to \mathbb{R}$ is a polynomial function, then there exists an a_P-stratification of $(V, \mathcal{V})$ with respect to P.*

In notes [Ka2] one can find a simple proof of this result.

6.3. **Setting for existence of a_P-stratification.** Let $\mathbb{R}^N$ and $\mathbb{R}^m$ be Euclidean spaces with fixed coordinate systems $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $a = (a_1, \dots, a_m) \in \mathbb{R}^m$ with $N \geq m$ and a nontrivial vector-polynomial $P : \mathbb{R}^N \to \mathbb{R}^m$, i.e. the image $P(\mathbb{R}^N)$ has nonempty interior. In what follows we call *a vector-polynomial* by *a polynomial* for brevity.

Definition 28. *Let $k \in \mathbb{Z}_+$ and $\delta > 0$. We call the (k, δ)-cone $K^m_{k,\delta}$ the following set of points*

$$K^m_{k,\delta} = \{a = (a_1, \dots, a_m) \in \mathbb{R}^m : 0 < a_1 < \delta,\ 0 < a_{j+1} < (a_1 \dots a_j)^k \text{ for } j = 1, \dots, k-1\}. \tag{80}$$

Let $k' \in \mathbb{Z}_+$ and $\delta > 0$. We call the (k', δ')-cone $K^m_{k',\delta'}$ a refinement of the (k, δ)-cone $K^m_{k,\delta}$ if $k' \geq k$ and $\delta' \leq \delta$ and, therefore, $K^m_{k',\delta'} \subseteq K^m_{k,\delta}$.

Define the following sets

$$V_{k,\delta,P} = P^{-1}\big(K^m_{k,\delta}\big), \quad V_{k,0,P} = \cap_{\delta>0} \overline{V_{k,\delta,P}}, \tag{81}$$

where $\bar{V}$ is the closure of V. Then one has

Theorem 15. *For any nontrivial polynomial $P : \mathbb{R}^N \to \mathbb{R}^m$ there exist $k \in \mathbb{Z}_+$ and $\delta > 0$ such that the following conditions hold*

a) the set $V_0^* = V_{k,0,P}$ *(see (81)) is semialgebraic of codimension at least* m.
b) the set $V_{k,\delta,P}$ *consists of regular points of* P*, i.e. if* $b \in V_{k,\delta,P}$*, then the level set* $P^{-1}(b)$ *if nonempty is a manifold of codimension* m.
c) there exists a stratification of V_0^* *by semialgebraic strata* $(V_0^*, \mathcal{V}_0)$ *satisfying the property:* $V_{m,\delta,P}$ *is* a_P*-regular over each strata* $V_\alpha \in \mathcal{V}_0$ *with respect to* P.

Remark 8. 1. Maps from F_P in Theorems 12 and 13 should be transversal to the stratified set $(V_0^*, \mathcal{V}_0)$. Theorem 15 on existence of a_P-stratification along with Cartesian Transversality Theorem stated above implies Theorem 13. To see that one needs to now reproduce arguments from Proposition 2 which is done below.

2. The shape of the (k, δ)-cone $K^m_{k,\delta}$ is important. For example, if we replace definition of $K^m_{k,\delta}$ by $K^m_{k,\delta} = \{a : 0 < a_1 < \delta,\ 0 < a_{j+1} < a_1^{jk}$ for $j = 1, \dots, m-1\}$. Then for the polynomial P, defined by $P_1(x) = x_1,\ P_2(x) = x_2 P_1^k(x), \dots, P_m(x) = x_m P^k_{m-1}(x)$, we have V_0^*, defined by (81), is no longer a set of codimension m. Increase of codimension destroys transversality arguments below.

3. Notice that if for some (k, δ)-cone $K^m_{k,\delta}$ the intersection $P(\mathbb{R}^N) \cap K^m_{k,\delta}$ is empty, there is nothing to prove in the Theorem. An empty set is semialgebraic of the maximal codimension and everything is transversal to an empty set. Therefore, we can assume that for each $\delta > 0$ and $k \in \mathbb{Z}_+$ the intersection $P(\mathbb{R}^N) \cap K^m_{k,\delta}$ is nonempty.

4. If the intersection $P(\mathbb{R}^N) \cap K^m_{k,\delta}$ is nonempty, then it equals $K^m_{k,\delta}$. Indeed, suppose it is not true and there are points $a \in P(\mathbb{R}^N) \cap K^m_{k,\delta}$ and $b \in P(\mathbb{R}^N) \setminus K^m_{k,\delta}$. Connect them by a smooth curve γ which lies inside the cone $K^m_{k,\delta}$. Since $K^m_{k,\delta}$ is a cone of regular values of P, the preimage $P^{-1}(c)$ of each point $c \in K^m_{k,\delta}$ is a smooth manifold of codimension either N or m. If codimensions of $P^{-1}(a)$ and $P^{-1}(c)$ are different, then there is a critical point $c^* \in \gamma \subset K^m_{k,\delta}$. This is a contradiction.

Proof of Theorem 13 based on Theorem 15. Consider the 1-parameter family of maps $F_t = t j^{\mathbf{m}}\mathbf{f} + (1-t)L_{\mathbf{f},a,\mathbf{m}}$ deforming the linear part of $L_{\mathbf{f},a,\mathbf{m}}$ into $j^{\mathbf{m}}\mathbf{f}$. Clearly, $F_1 \equiv j^{\mathbf{m}}\mathbf{f}$ and $F_0 \equiv L_{\mathbf{f},a,\mathbf{m}}$. Fix a small $r > 0$. Since F_0 is transversal to V_0 at 0, and all F_t have the same linear part at 0 we know that all F_t are transversal to V_0 at 0. Condition (15) implies that for any sufficiently small ε and all $t \in [0, 1]$ we have F_t is transversal to V_ε. Since the segment $[0, 1]$ is compact, simple arguments by contradiction show that we have uniformity in t. This completes the proof of Theorem 13. Q.E.D.

7. Proof of Existence of a_P-stratification

In this section we prove existence of a_P-stratification in the special case we are interested in. As examples of Thom and Grinberg (Lemma 2) show, the existence of an a_P-stratification is a nontrivial question. In general, it does not exist. Unfortunately, the existence of a a_P-stratification in our case does not follow from the classical results. Statement of Theorem 15 is the right setting in which a_P-stratification exists and this setting fits to the framework of Theorem 13.

7.1. **C^1-pseudodistance between regular algebraic varieties.** In order to prove Theorem 15 we reformulate it in a convenient for us language. Let $a \in \mathbb{R}^m$. Denote by $L_a = P^{-1}(a)$ the level set of P. Recall that $a \in \mathbb{R}^m$ is called a regular value if for any $x \in L_a$ the rank of linearization of P is maximal, i.e. rank $dP(x) = m$.

Definition 29. *Let $a, b \in \mathbb{R}^m$ be values of $P : \mathbb{R}^N \to \mathbb{R}^m$, $B^N \subset \mathbb{R}^N$ be the unit ball centered at the origin, and*

$$\begin{aligned} &d_0^0 : B^N \times \mathbb{R}^m \to \mathbb{R}, \quad d_0^0(x,a) = \inf_{y \in L_a \cap B^N} \|x-y\|^2 \\ &d_0 : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}, \quad d_0(a,b) = \sup_{x \in L_b \cap B^N} d_0^0(x,a). \end{aligned} \tag{82}$$

Then the C^0-distance between level sets $L_a \cap B^N$ and $L_b \cap B^N$

$$D_P^0(a,b) = dist_{C^0}(L_a \cap B^N, L_b \cap B^N) = \frac{1}{2}\left(d_0(a,b) + d_0(b,a)\right).$$

For any $1 \le m \le N$ denote by $G^{m,N}$ the Grassmanian manifold of m-dimensional planes in the N-dimensional space. Below we introduce algebraic distance in $G^{m,N}$ and then define C^1-distance between regular level sets L_a and L_b to keep all objects inside the category of algebraic sets. The reason we want to stay in the category of (semi)algebraic sets is we want to apply *elimination theory* (e.g. [Mu]). Write the polynomial map P using coordinate functions $P = (P_1, \dots, P_m) : \mathbb{R}^N \to \mathbb{R}^m$. If $x \in \mathbb{R}^N$ is a regular point of P, then gradients $\nabla P_1(x), \dots, \nabla P_m(x)$ are linearly independent and span the space which is the orthogonal complement to the tangent space to the level set $P^{-1}(P(x))$ at x. Define a Gramm-Schmidt orthogonalization operator:

Definition 30. *Let $v_1, \dots, v_m \in \mathbb{R}^N$ be linear independent vectors. Define the Gramm-Schmidt linear operator by*

$$\begin{gathered} * :(v_1, \dots, v_m) \to (v_1^*, \dots, v_m^*), \text{ where} \\ v_1^* = v_1,\ v_2^* = v_2 - \frac{(v_2, v_1^*)}{(v_1^*, v_1^*)} v_1^*, \dots, v_m^* = v_m - \sum_{j<m} \frac{(v_m, v_j^*)}{(v_j^*, v_j^*)} v_j^*. \end{gathered} \tag{83}$$

Remarks. 0. The Gramm-Schmidt linear operator "$*$" has nothing in common with the asterisk operator used for the Khovanski reduction procedure in Sect. 3.

1. Vectors $\{v_j^*\}_j$ are orthogonal, linearly independent and, therefore, span the same m-dimensional space as $\{v_j\}_j$. Denote the space by L. Orthogonality can be proved by induction. First show that v_1^* is orthogonal to v_2^*, v_3^*, and so on. Second suppose that for some $j \le m$ we have v_s^* is orthogonal to $v_{s'}^*$ for each $s \le m$ and $s' < j$. Then show that v_j^* is orthogonal to v_s^* for each $s \le m$. Orthogonality of $\{v_j^*\}_j$ and linear independence of $\{v_j\}_j$ imply linear independence. Linear independence of $\{v_j^*\}_j$ implies that $\{v_j^*\}_j$ spans the same space as $\{v_j\}_j$ does;

2. Vectors $\{v_j^*\}_j$ are not necessarily of unit length;

3. Let $\{L_t\}_{t\in(0,1]}$ be a family of k-dimensional planes in $\mathbb{R}^N$ spanned by a family of vectors $\{v_1(t),\dots,v_m(t)\}_{t\in(0,1]}$ depending continuously on t. Consider $\{v_1^*(t),\dots,v_m^*(t)\}_{t\in(0,1]} = *(v_1(t),\dots,v_m(t))$ as the family of orthogonal basis in $\{L_t\}_{t\in(0,1]}$. Then sufficient condition for $L_t \to L$ as $t \to 0$ in the Grassmanian manifold $G^{m,N}$ is existence of an orthogonal basis $\{v_1^*,\dots,v_m^*\}$ of L such that

$$(84)\qquad \frac{\left(v_j^*(t), v_j^*\right)^2}{(v_j^*(t), v_j^*(t))(v_j^*, v_j^*)} \to 1 \ \text{ as } t \to 0 \text{ for each } j = 1,\dots,m.$$

Define the Gramm-Schmidt operator for the Jacoby matrix $dP(x) = (\nabla P_1(x),\dots,\nabla P_m(x))$ of the polynomial map P

$$(85)\qquad *(\nabla P_1(x),\dots,\nabla P_m(x)) = (*(dP)_1(x),\dots,*(dP)_m(x)).$$

Each vector $*(dP)_j(x)$ is given by the rational function in x.

Let $\Sigma_P \subset \mathbb{R}^N$ be the set of critical points of P. To measure C^1-distance between two regular level sets we introduce the following function: Let $x, y \notin \Sigma_P$. Then

(86)
$$R_P(x,y) = \sum_{j=1}^m \left(1 - \frac{(*(dP)_j(x), *(dP)_j(y))^2}{(*(dP)_j(x), *(dP)_j(x))(*(dP)_j(y), *(dP)_j(y))}\right)$$
$$Q_P(x,y) = \|x-y\|^2 + R_P(x,y).$$

Definition 31. *Let $a, b \in \mathbb{R}^m$ be regular values of $P : \mathbb{R}^N \to \mathbb{R}^m$, $L_a = P^{-1}(a)$ and $L_b = P^{-1}(b)$ be nonempty regular level sets.*

$$d^0_{1,P} : B^N \times \mathbb{R}^m \setminus \Sigma_P \to \mathbb{R}, \quad d^0_{1,P}(x,a) = \inf_{y\in L_a\cap B^N} Q_P(x,y),$$
$$d_{1,P} : \left(\mathbb{R}^m \setminus \Sigma_P\right) \times \left(\mathbb{R}^m \setminus \Sigma_P\right) \to \mathbb{R}, \quad d_{1,P}(a,b) = \sup_{x\in L_b\cap B^N} d^0_{1,P}(x,a).$$

Then the C^1-pseudodistance between regular level sets $L_a \cap B^N$ and $L_b \cap B^N$ is defined by

(87)
$$D^1_P(a,b) = dist_{C^1}\left(L_a \cap B^N, L_b \cap B^N\right) = \frac{1}{2}\left(d_{1,P}(a,b) + d_{1,P}(b,a)\right).$$

Remark 9. The function $D^1_P(a,b)$ is called C^1-pseudodistance, not C^1-distance, because it does not satisfy the triangle inequality. However, it satisfies the following triangle-like inequality

$$2\left(D^1_P(a,b) + D^1_P(b,c)\right) > D^1_P(a,b). \tag{88}$$

The reason we define C^1-pseudodistance $D^1_P(a,b)$ in such a way is because the function $D^1_P(a,b)$ is algebraic in a and b (see Lemma 3 below).

The inequality (88) can be proved as follows. Let $v, w \in \mathbb{R}^N$ be vectors. Denote by $\angle(v,w)$ the angle between v and w. Direct calculation shows that

$$R_P(x,y) = \sum_{j=1}^m \sin^2(\angle(*(dP)_j(x), *(dP)_j(y))).$$

It is easy to check that $2(\sin^2\alpha + \sin^2\beta) \geq \sin^2(\alpha+\beta)$ which is sufficient for the proof of the inequality (88).

Now we can reformulate Theorem 15 in the following way

Theorem 16. *For any nontrivial polynomial $P : \mathbb{R}^N \to \mathbb{R}^m$ there exist $k \in \mathbb{Z}_+$ and $\delta > 0$ such that the following conditions hold*

a) for any two values with the same first coordinate $a = (t, a_2, \dots, a_m)$ and $b = (t, b_2, \dots, b_m)$ in $K^m_{k,\delta}$ we have

$$D^0_P(a,b) < t;$$

b) the same as in Theorem 15;

c) for any two values with the same first coordinate $a = (t, a_2, \dots, a_m)$ and $b = (t, b_2, \dots, b_m)$ in $K^m_{k,\delta}$ we have

$$D^1_P(a,b) < t;$$

Proof of Theorem 15 based on Theorem 16. It is enough to show that parts a) and c) of Theorem 16 imply parts a) and c) of Theorem 15 respectively.

Part a) Theorem 16 $\Longrightarrow$ part a) of Theorem 15. Consider an algebraic curve of the form $\gamma(t) = (t, t^{k+1}, t^{(k+1)^2}, \dots, t^{(k+1)^{m-1}})$. One can check that $\gamma(t) \in K^m_{k,\delta}$ for any $t \in (0,\delta]$. Denote by $V_{t,P} = P^{-1}(\gamma(t))$. The set $\cup_{0<t\leq\delta} V_{t,P}$ is clearly semialgebraic set. By the Tarsky-Seidenberg Theorem the following set

$$\text{closure}\{\cup_{0<t\leq\delta} V_{t,P}\} \setminus \{\cup_{0<t\leq\delta} V_{t,P}\}$$

is semialgebraic. Since for any smooth curve $\tilde{\gamma}(t) = (t, \gamma_2(t), \dots, \gamma_m(t)) \in K^m_{k,\delta}$ for $t \in (0, \delta)$ Hausdorff distance between the level sets $V_{t,P} = P^{-1}(\gamma(t))$ and $\tilde{V}_{t,P} = P^{-1}(\tilde{\gamma}(t))$ is at most t, i.e. $D^0_P(\gamma(t), \tilde{\gamma}(t)) < t$. It implies that Hausdorff distance between any two level sets of the form $V_{t,P}$ and $\tilde{V}_{t,P}$ tends to 0 as $t \to 0$. Therefore,

$$\text{closure}\{\cup_{0<t\le\delta} V_{t,P}\} \setminus \{\cup_{0<t\le\delta} V_{t,P}\} = \text{closure}\big\{P^{-1}\big(K^m_{k,\delta}\big)\big\} \setminus \big\{P^{-1}\big(K^m_{k,\delta}\big)\big\}.$$

This completes the proof of part a) of Theorem 15.

Part c) of Theorem 16 $\Longrightarrow$ part c) of Theorem 15. We use notations of part a). By Theorem from Sect. 6.2 of Hironaka there is a stratification of $V_{0,k,P}$ such that the semialgebraic set $\{\cup_{0<t\le\delta} V_{t,P}\}$ is a a_P-regular over $V_{0,k,P}$. Indeed, let $\pi_1 : \mathbb{R}^m \to \mathbb{R}$ be the natural projection onto the first coordinate. Then a polynomial function $p = \pi_1 \circ P : \mathbb{R}^N \to \mathbb{R}$ is well-defined and $p^{-1}(t) = P^{-1}(\gamma(t))$ for each $0 < t \le \delta$. Application of Theorem of Hironaka to the map

$$p : \text{closure}\{\cup_{0<t\le\delta} V_{t,P}\} \to \mathbb{R}$$

gives existence of the required a_P-stratification.

Since C^1-distance between any two level sets of the form $V_{t,P} = P^{-1}(\gamma(t))$ and $\tilde{V}_{t,P} = P^{-1}(\tilde{\gamma}(t))$ is at most t, i.e. $D^1_P(\gamma(t), \tilde{\gamma}(t)) < t$. It implies that C^1-distance between any two level sets of the form $V_{t,P}$ and $\tilde{V}_{t,P}$ tends to 0 as $t \to 0$. Therefore, a_P-regularity of $P^{-1}(K^m_{k,\delta})$ over $V_{0,k,P}$ follows from a_P-regularity of $\{\cup_{0<t\le\delta} V_{t,P}\}$ over $V_{0,k,P}$. This completes the proof of part c) and Theorem 15. Q.E.D.

Before proving Theorem 16 let us formulate the basic fact from elimination theory [Mu].

7.2. **Elimination theory.** Let $\mathbb{C}^m$ denote the m-dimensional complex space $z = (z_1, \dots, z_m) \in \mathbb{C}^m$, $m \in \mathbb{Z}_+$. A set V in $\mathbb{C}^m$ is called *a closed algebraic set* in $\mathbb{C}^m$ if there is a finite set of polynomials $F_1, \dots F_s$ in $z_1, \dots, z_m$ such that

$$V(F_1, \dots, F_s) = \{(z_1, \dots, z_m) \in \mathbb{C}^m | \; F_j(z_1, \dots, z_m) = 0, \;\; 1 \le j \le s\}.$$

One can define a topology in $\mathbb{C}^m$, called the *Zariski topology*, whose closed sets are closed algebraic sets in $\mathbb{C}^m$. This, indeed, defines a topology, because the set of closed algebraic sets is closed under a finite union and an arbitrary intersection. Sometimes, closed algebraic sets are also called Zariski closed sets.

Definition 32. *A subset S of $\mathbb{C}^m$ is called constructible if it is in the Boolean algebra generated by the closed algebraic sets; or equivalently if S is a disjoint union $T_1 \cup \dots \cup T_m$, where T_i is locally closed, i.e. $T_i = T_i' - T_i''$, T_i' — a closed algebraic set and $T_i'' \subset T_i'$ — a smaller closed algebraic.*

One of the main results of Elimination theory is the following

Theorem 17. *([Mu], Chap. 2.2) Let $V \subset \mathbb{C}^\mu \times \mathbb{C}^N$ be a constructible set and $\pi : \mathbb{C}^\mu \times \mathbb{C}^N \to \mathbb{C}^\mu$ be the natural projection. Then $\pi(V) \subset \mathbb{C}^\mu$ is a constructible set.*

First we shall prove part b) of Theorem 16 and then parts a) and c) of it.

7.3. **Existence of an (k, δ)-cone $K^m_{k,\delta}$ of regular values for P (or the Proof of Part b) of Theorem 16).** The set of critical values $\Sigma_P \subset \mathbb{R}^m$ of a nontrivial polynomial map $P : \mathbb{R}^N \to \mathbb{R}^m$ is an algebraic set of positive codimension. It follows from Sard's lemma for algebraic sets [Mu]. Suppose $d : \mathbb{R}^m \to \mathbb{R}$ is a nonzero polynomial whose zero level set $d^{-1}(0) \supseteq \Sigma$. Fix coordinate systems in $\mathbb{R}^N$. By writing the linearization matrix $dP : \mathbb{R}^N \to \mathbb{R}^m$ and considering $(N - m + 1)$ different $m \times m$ minors one can calculate d explicitly.

Proposition 3. *For a nonzero polynomial function $d : \mathbb{R}^m \to \mathbb{R}$ and $k = \deg d + 1$ there exists $\delta > 0$ such that the function d does not vanish in the (k, δ)-cone $K^m_{k,\delta}$.*

Remark 10. If $\Sigma_P \subseteq d^{-1}(0)$ and $\Sigma_P \cap K^m_{k,\delta} \neq \emptyset$, then there exists $x \in K^m_{k,\delta}$ such that $d(x) = 0$. Contradiction. This shows that part b) of Theorem 16 follows from this proposition.

Proof. Let us prove the statement by induction in dimension m.

For $m = 1$ the zero level set $d_1^{-1}(0) \subset \mathbb{R}$ of any nonzero polynomial $d_1 : \mathbb{R} \to R$ is a finite collection of points and the proposition is obvious. Moreover, it is clear that $\lim_{x_1 \to 0} \dfrac{x_1^k}{d_1(x_1)} = 0$ for any $k > \deg d_1$.

Suppose that $m = s - 1$ we prove that for each nonzero polynomial $d_{s-1} : \mathbb{R}^{s-1} \to \mathbb{R}$ and $k > \deg d_{s-1}$ there is $\delta > 0$ such that d_{s-1} does not vanish on the (k, δ)-cone $K^{s-1}_{k,\delta}$ and

$$\frac{(x_1 \dots x_{s-1})^k}{d_{s-1}(x_1, \dots, x_{s-1})} \to 0 \tag{89}$$

for any $(x_1, \dots, x_{s-1}) \to 0$, provided that $(x_1, \dots, x_{s-1})$ stays inside of the (k, δ)-cone $K^{s-1}_{k,\delta}$.

Consider a nonzero polynomial $d_s : \mathbb{R}^s \to \mathbb{R}$ in s variables. Write in the form

$$d(x_1, \dots, x_s) = \sum_{j=j_0}^{\deg d} a_j(x_1, \dots, x_{s-1}) x_s^j,$$

where j_0 is the first nonzero term in this sum. Notice that $\deg a_{j_0} \le \deg d_s$. Therefore, by the inductive hypothesis we have

$$\frac{(x_1 \dots x_{s-1})^k}{a_{j_0}(x_1, \dots, x_{s-1})} \to 0 \tag{90}$$

for any $(x_1, \dots, x_{s-1}) \to 0$, provided that $(x_1, \dots, x_{s-1})$ stays inside of the (k, δ)-cone $K^{s-1}_{k,\delta}$. This implies that for a sufficiently small $\delta > 0$ we have $x_s = \lambda(x_1 \dots x_{s-1})^k$ for some $|\lambda| < 1$ and

$$(91) \qquad d(x_1, \dots, x_s) = x_s^{j_0} a_{j_0}(x_1, \dots, x_{s-1})(1 + p(x_1, \dots, x_{s-1}, \lambda)),$$

where $p(x_1, \dots, x_{s-1}, \lambda)$ is a rational function which tends to zero while $(x_1, \dots, x_s)$ is inside of the (k, δ)-cone $K^s_{k,\delta}$ and $(x_1, \dots, x_{s-1}) \to 0$. This implies that under these conditions zeroes of $d(x_1, \dots, x_s)$ coincide with zeroes of $a_{j_0}(x_1, \dots, x_{s-1})$. But by inductive hypothesis $a_{j_0}(x_1, \dots, x_{s-1})$ has no zeroes inside the (k, δ)-cone $K^{s-1}_{k,\delta} \subset \mathbb{R}^{s-1} = \{x_s = 0\} \subset \mathbb{R}^s$. Thus the polynomial $d(x_1, \dots, x_s)$ does not vanish inside the (k, δ)-cone $K^s_{k,\delta}$. This completes the proof of Proposition 3. Q.E.D.

7.4. **Reduction to an optimization problem (or Proof of parts a) and c) of Theorem 16).** Let $P = (P_1, \dots, P_m) : \mathbb{R}^N \to \mathbb{R}^m$ be a nontrivial polynomial given by its coordinate functions with $N \geq m$ and let $K_{k,\delta} \subset \operatorname{Im} P(\mathbb{R}^N) \subset \mathbb{R}^m$ be a (m, δ)-cone of regular values of P. Existence of such a cone is proved in the previous section. Recall that $\Sigma \subset \mathbb{R}^N$ denotes the set of critical points of P and $Q_P(x, y)$ is defined in (86). The function $Q_P(x, y)$ is a rational function symmetric with respect to permutation of x and y. It is defined to measure C^1-distance between level sets (see remarks after definition 30). The singular set of Q_P belongs to $(\Sigma_P \times \mathbb{R}^N) \cup (\mathbb{R}^N \times \Sigma_P)$. Recall that $B^N = \{x : \sum_{i=1}^N x_i^2 \leq 1\} \subset \mathbb{R}^N$. Introduce functions $r(x) = 1 - \sum_{i=1}^N x_i^2$.

Assume that the restriction of P to the boundary $S^N = \partial B^N = \{x : \|x\| = 1\}$ has only the regular values in the $(k, \delta)-$cone $K^m_{k,\delta}$. Indeed, regularity of $P|_{S^N} : S^N \to \mathbb{R}^m$ is equivalent to regularity of the polynomial map $(P, r) : \mathbb{R}^N \to \mathbb{R}^m \times \mathbb{R}$ given by $(P, r)(x) = (P(x), 1 - \sum_{i=1}^N x_i^2)$. Existence of an (m, δ)-cone of regular values of the map (P, r) follows from Proposition 3.

Lemma 3. *With the notations above let $a_{\tau_1},\ a_{\tau_2} \in K^m_{k,\delta}$ be two points with the same first $(m - 1)$ coordinates, i.e. $a_{\tau_1} = (a^{m-1}, \tau_1)$ and $a_{\tau_2} = (a^{m-1}, \tau_2)$. Then there exists a polynomial $R^1_P(a^{m-1}, \tau_1, \tau_2, c)$ in variables $a^{m-1} \in \mathbb{R}^{m-1},\ \tau_1,\ \tau_2$, and $c \in \mathbb{R}$ such that*

$$(92) \qquad R^1_P\big(a^{k-1}, \tau_1, \tau_2, d_{1,P}(a_{\tau_1}, a_{\tau_2})\big) = 0.$$

In other words, $d_{1,P}(a_{\tau_1}, a_{\tau_2})$ is algebraic. Moreover, $D_P(a^{m-1}, \tau, \tau, 0) \equiv 0$.

Proof. Recall that $\{*(dP)_j(x)\}_{j=1}^m$ form an orthogonal basis in the orthogonal complement to the tangent plane to the level set $P^{-1}(P(x))$ at the point x (see (85)). Let us make several remarks about the rational function $Q_P(x, y)$ defined by (86).

1. If $a_\tau = (a^{m-1}, \tau) \in K^m_{k,\delta}$ is a regular value for the map $(P_1, \dots, P_m) : \mathbb{R}^N \to \mathbb{R}^m$ for some τ, then $a^{m-1} \in \mathbb{R}^{m-1}$ is a regular value for the map $(P_1, \dots, P_{m-1}) : \mathbb{R}^N \to \mathbb{R}^{m-1}$;

2. If $a^{m-1} \in \mathbb{R}^{m-1}$ is a regular value for the map $(P_1, \dots, P_{m-1}) : B^N \to \mathbb{R}^{m-1}$, then there is a positive $\varepsilon(a^{m-1}) > 0$ such that for each point $x \in B^N$ with the property $(P_1, \dots, P_{m-1})(x) = a^{m-1}$ and each $1 \le j \le m-1$ we have

$$(*(dP)_j(x), *(dP)_j(x)) > \varepsilon(a^{k-1}) > 0. \tag{93}$$

It follows from compactness of B^N and regularity of value a^{m-1} for $(P_1, \dots, P_{m-1})$;

3. Since we consider only those τ that a_τ belongs to the (k, δ)-cone $K^m_{k,\delta}$ of regular values of P there exists a positive function $\varepsilon(a^{m-1}, \tau) > 0$ such that for each $x \in P^{-1}(a_\tau) \cap B^N$

$$(*(dP)_m(x), *(dP)_m(x)) > \varepsilon(a^{m-1}, \tau). \tag{94}$$

It also follows from compactness of B^N and regularity of the value (a^{m-1}, τ) for P;

This shows that $Q_P(x, y)$ restricted to $P^{-1}(K^m_{k,\delta}) \times P^{-1}(K^m_{k,\delta})$ is a smooth function of x and y. We don't say anything about continuation to the boundary of $P^{-1}(K^m_{k,\delta}) \times P^{-1}(K^m_{k,\delta})$.

Consider irreducible representation of the rational function $Q_P(x, y)$ as a ration of two polynomials $Q_P(x, y) = \frac{T(x,y)}{S(x,y)}$. Because of Remarks 2 and 3 we have $S(x, y) \ne 0$ for each pair $(x, y) \in P^{-1}(K^m_{k,\delta}) \times P^{-1}(K^m_{k,\delta})$.

Now notice that we deal with smooth (even analytic) nonsingular objects: smooth level sets $P^{-1}(a_\tau)$ for $a_\tau \in K^m_{k,\delta}$ and the smooth function $Q_P(x, y)$. Notice that $d^0_{1,P}(x, a_{\tau_2})$ is an extremal value of the function $Q_P(x, y)$ provided that $P(y) = a_{\tau_2}$. Similarly, $d_{1,P}(a_{\tau_1}, a_{\tau_2})$ is an extremal value of the function $d^0_{1,P}(x, a_{\tau_2})$ provided that $P(y) = a_{\tau_1}$. To find all extremal values of a smooth function on a smooth manifold one can use the Lagrange multipliers method. We shall prove that functions $d_{1,P}$ and $d^0_{1,P}$ are algebraic functions.

The key point of the Lagrange multipliers method see e.g. [Ti] is that at an extremal point of $Q_P(x, y)$ under the condition $P(y) = a_{\tau_2}$ the gradient $\nabla_y Q_P(x, y)$ can be expressed as a linear combination of gradients $\nabla P_1(y), \dots, \nabla P_m(y)$, and $\nabla r(y)$. The gradient of $Q_P(x, y)$ has the form

$$\begin{aligned}\nabla_y Q_P(x, y) = \nabla_y \left(\frac{T(x, y)}{S(x, y)} \right) = \\ \big(S(x, y)\nabla_y T(x, y) - T(x, y)\nabla_y S(x, y)\big)\, S^{-2}(x, y).\end{aligned} \tag{95}$$

Since $S|_{P^{-1}(K^m_{k,\delta})\times P^{-1}(K^m_{k,\delta})} \neq 0$, we can rewrite the Lagrange system in the following form

$$(96)\qquad \begin{cases} S(x,y)\nabla_y T(x,y) - T(x,y)\nabla_y S(x,y) + \\ +S^2(x)\left[\sum_{j=1}^m \lambda_j \nabla P_j(x) - \lambda_{k+1}\nabla r(y)\right] = 0, \\ P(y) - a_{\tau_2} = 0, \\ T(x,y) - cS(x,y) = 0, \\ \lambda_{k+1} r(y) = 0. \end{cases}$$

Important that all equations are polynomial and we can apply elimination theory! Notice that the last equation is responsible for an extremal point y which might belong to the boundary ∂B^N. If a critical value y belongs to the boundary ∂B^N, then $r(y) = 0$ and $\lambda_{m+1}\nabla r(y)$ is not zero and the gradient $\nabla_y Q_P(x,y)$ should be expressed as a linear combination of $m+1$ vectors $\nabla P_1(y), \dots, \nabla P_m(y)$, and $\nabla r(y)$. If a critical value does not belong to the boundary, i.e. $r(y) \neq 0$, then $\lambda_{m+1} = 0$ and $\lambda_{m+1}\nabla r(y) = 0$.

Complexify the system (96), i.e. consider the system (96) for

$$\left(x, y, \lambda, a^{m-1}, \tau_2, c\right) \in \mathbb{C}^N \times \mathbb{C}^N \times \mathbb{C}^{m+1} \times \mathbb{C}^{m-1} \times \mathbb{C} \times \mathbb{C}.$$

It defines a constructible set, denoted by V, in $\mathbb{C}^{2N+2m+2}$. Let us eliminate variables $\{\lambda_j\}_{j=1}^{k+1}$ and $\{y_i\}_{i=1}^N$ by projecting V along the corresponding $(N+m+1)$-dimensional (λ, y)-plane. The result of projection is a constructible set W in the space $(x, a^{m-1}, \tau_2, c) \in \mathbb{C}^{N+m+1}$. By the construction a point (x, a^{m-1}, τ_2, c) belongs to W if and only if some value of y the following conditions hold: $P(y) = a_{\tau_2}$, $Q_P(x,y) = c$, and y is the critical point of Q_P restricted to $P^{-1}(a_{\tau_2}) \cap B^N$. Counting the number of equations (96) and the number of variables we have that there exists a polynomial map $R'_P : \mathbb{C}^{N+m+1} \to \mathbb{C}$, which is obtained as a result of application of elimination procedure, such that $W \subset (R'_P)^{-1}(0)$.

Now we show that R'_P is not identically zero or the set W has dimension $N+m$. Indeed, consider a polynomial function $\rho_x : P^{-1}(a_{\tau_2}) \cap B^N \to \mathbb{R}$ defined by $\rho_x(y) = Q_P(x,y)$ and is restricted to real.

- The function ρ_x is algebraic and continuous on $P^{-1}(K^m_{k,\delta})$.
- The function ρ_x is not constant. Indeed, if $P(x) = a_{\tau_1} \neq P(y) = a_{\tau_2}$ and both a_{τ_1} and a_{τ_2} belong to $K^m_{k,\delta}$, then $\rho_x(y) = Q_P(x,y)$ must be positive, because C^1-pseudodistance of distinct level sets is positive by definition. On the other side, if $a_{\tau_1} = P(y)$, then $\rho_x \equiv 0$.

Therefore, ρ_x takes its maximal and minimal values, which have to be critical values. Since, ρ_x is algebraic it can have only a finite number of critical values by Sard's lemma for algebraic sets [Mu]. Therefore, over each point (x, a^{m-1}, τ_2) there is a point (x, a^{m-1}, τ_2, c) which belongs to W. Thus, dim W equals dimension of $(x, a^{m-1}, \tau_2,)$-plane, i.e. dim $W = N+m$.

Since R'_P is nonconstant and $\mathbb{C}$ is algebraically closed, there is a polynomial $\tilde{R}_P$ which has the same zero level set and the gradient $\nabla \tilde{R}_P(x, a_{\tau_2}, c) \neq 0$ for all $x \in W$ outside possibly of a set of positive codimension. This $\tilde{R}_P$ exists and generates the radical of the ideal generated by R'_P [Mu]. Therefore, by definition (87) of $d^0_{1,P}(x, a_{\tau_2})$ and by the construction W we have on

$$(97) \qquad \tilde{R}_P\big(x, a, \tau_2, d^0_{1,P}(x, a_{\tau_2})\big) \equiv 0.$$

In order to prove that the function $d_{1,P}(a_{\tau_1}, a_{\tau_2})$ defined by (87) is also algebraic, calculate critical values of $d^0_{1,P}(x, a_{\tau_2})$, provided $P(x) = a_{\tau_1}$. By the implicit function Theorem the gradient $\nabla_x d^0_{1,P}(x, a_{\tau_2})$ can be expressed in terms of partial derivatives of $\tilde{R}_P(x, a_{\tau_2}, c)$ by the following way

$$(98) \qquad \partial_{x_j} d^0_{1,P}(x, a_{\tau_2}) = \partial_c \tilde{R}_P(x, a, \tau_2, c)\left(\partial_{x_j} \tilde{R}_P(x, a, \tau_2, c)\right)^{-1},$$

for $(a, c) = (a_{\tau_2}, d^0_{1,P}(x, a_{\tau_2}))$, provided that $\partial_{x_j} \tilde{R}_P(x, a_{\tau_2}, d^0_{1,P}(x, a_{\tau_2})) \neq 0$ for all $1 \leq j \leq N$.

Fix $a = a_{\tau_2}$ and consider x in $P^{-1}(a_{\tau_1})$ and outside of the union of algebraic sets, where

$$(99) \qquad \mathcal{B}_P(a_{\tau_1}) = \cup_{j=1}^N \left\{x \in P^{-1}(a_{\tau_1}) : \partial_{x_j} \tilde{R}_P(x, a_{\tau_1}, c) = 0\right\}.$$

Each set $\{x : \partial_{x_j} \tilde{R}_P(x, a, c) = 0\}$ has to have a positive codimension, because the gradient $\nabla \tilde{D}_P(\cdot, a_{\tau_2}, c)$ is not identically zero and, therefore, by change of coordinates the same can be said about its coordinate functions.

Then $d^0_{1,P}(x, a)$ is a smooth function in x outside $\mathcal{B}_P(a_{\tau_1})$. Application of the Lagrange multipliers method shows that at an extremal point of the function $d^0_{1,P}(x, a)$, provided $P(x) = a_{\tau_1}$, the gradient $\nabla_x d^0_{1,P}(x, a)$ can be represented as a linear combination $\nabla_x d^0_{1,P}(x, a) = \sum_{j=1}^m \lambda_j \nabla P_j(x) - \lambda_{m+1} \nabla r(x)$. Plugging in the expression for $\nabla_x d^0_{1,P}(x, a)$ in terms of $\partial_c \tilde{R}_P(x, a_{\tau_2}, c)$ and $\partial_{x_j} \tilde{R}_P(x, a_{\tau_2}, c)$ for $j = 1, \dots, N$ we can present a Lagrange multiplier system in the following form

$$(100) \qquad \begin{cases} \partial_{x_j} \tilde{R}_P(x, a_{\tau_2}, c) \partial_c \tilde{R}_P(x, a_{\tau_2}, c) = \\ = \left[\sum_{j=1}^m \lambda_j \nabla P_j(x) - \lambda_{m+1} \nabla r_1(x)\right] \left(\partial_{x_j} \tilde{R}_P(x, a_{\tau_2}, c)\right)^2 \\ \tilde{R}_P(x, a_{\tau_2}, c) = 0, \\ P(x) - a_{\tau_1} = 0, \\ \lambda_{k+1} r(x) = 0. \end{cases}$$

Again the system (100) consists of only polynomial equations and we can apply elimination theory. Consider this system for

$$\left(x, \lambda, a^{m-1}, \tau_1, \tau_2, c\right) \in \mathbb{C}^N \times \mathbb{C}^{m+1} \times \mathbb{C}^{m-1} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}.$$

It defines a constructible set, denoted by V_1, in $\mathbb{C}^{N+2m+3}$. Let us eliminate variables $\{\lambda_j\}_{j=1}^{m+1}$, $\{x_i\}_{i=1}^{N}$ by projecting V_1 along the corresponding $(N+m+1)$-dimensional (λ, x)-plane. The result of projection is a constructible set W_1 in the space $(a^{m-1}, \tau_1, \tau_2, c) \in \mathbb{C}^{m+2}$.

Similarly to the arguments for the constructible set W one can show that W_1 has dimension k. Since W_1 is constructible and has codimension 1, there is a nonconstant polynomial $R_P^1(a^{m-1}, \tau_1, \tau_2, c)$ such that $W_1 \subseteq (R_P^1)^{-1}(0)$. By the definition (87) of $d_{1,P}(a_{\tau_1}, a_{\tau_2})$ and by the construction

$$R_P^1(a_{\tau_1}, a_{\tau_2}, d_{1,P}(a_{\tau_1}, a_{\tau_2})) \equiv 0. \tag{101}$$

By the construction if $a_{\tau_1} = a_{\tau_2}$, then $R_P^1(a_{\tau_1}, a_{\tau_1}, 0) \equiv 0$, because in this case both level sets are the same and C^1-distance between them must equal zero. This completes the proof of Lemma 3. Q.E.D.

Lemma 4. *With the notations above there exists a (k', δ')-cone $K^m_{k',\delta'}$, which is a refinement of $K^m_{k,\delta}$ such that for any pair of points $a_{\tau_1} = (a^{k-1}, \tau_1)$ and $a_{\tau_2} = (a^{m-1}, \tau_2)$ from $K^m_{k',\delta'}$ we have C^1-distance $R_P^1(a_{\tau_1}, a_{\tau_2})$ between level sets $P^{-1}(a_{\tau_1})$ and $P^{-1}(a_{\tau_2})$ bounded by*

$$D_P^1(a_{\tau_1}, a_{\tau_2}) \le (a_1 \dots a_{m-1})^2, \text{ where } a^{m-1} = (a_1, \dots, a_{m-1}) \in \mathbb{R}^{m-1}.$$

Proof. It follows from Lemma 3 that there is a polynomial $R_P^1(a_{\tau_1}, a_{\tau_2})$ such that $R_P^1(a_{\tau_1}, a_{\tau_2}, d_{1,P}(a_{\tau_1}, a_{\tau_2})) \equiv 0$ and $R_P^1(a, \tau, \tau, 0) \equiv 0$.

For a_{τ_1} and a_{τ_2} belonging to the (k, δ)-cone $K^m_{k,\delta}$ of regular values of P the function $d_{1,P}(a_{\tau_1}, a_{\tau_2})$ depends continuously on τ_1 and τ_2. Let us rewrite $R(a_{\tau_1}, a_{\tau_2}, c)$ in the form $R_P^1(a^{m-1}, \tau_1, \tau_2, c)$. Recall that in our notations $a_\tau = (a^{m-1}, \tau)$.

Suppose that $\tau_1 > \tau_2$ otherwise renumerate. Notice that each sufficiently small positive root $c_j(a^{m-1}, \tau_1, \tau_2)$ of $R_P^1(a^{m-1}, \tau_1, \tau_2, \cdot)$ is increasing in τ_1 and decreasing in τ_2 in a neighborhood of $(a^{k-1}, \tau_1, \tau_2) = 0$. Therefore, $c_j(a^{m-1}, \tau_1, 0) > c_j(a^{m-1}, \tau_1, \tau_2)$ We can assume this because each $c_j(a^{m-1}, \tau_1, \tau_2)$ is algebraic and its derivative algebraic too. Therefore, refining (k, δ)-cone $K^m_{k,\delta}$ if necessary we can assume without loss of generality as in Lemma 3 that $c_j(a^{m-1}, \tau_1, \tau_2)$ is monotonic in τ_1 and τ_2.

By Proposition 3 there are $\tilde{k} \in \mathbb{Z}_+$ and $\tilde{\delta} > 0$ such that if $(a_1, \dots, a_{m-1}, c, \tau) \in K^{m+1}_{\tilde{k},\tilde{\delta}}$, then $\tilde{R}_P^1(a^{m-1}, \tau, c) \neq 0$, i.e. $(\tilde{k}, \tilde{\delta})$-cone $K^{m+1}_{\tilde{k},\tilde{\delta}}$ is free from zeroes of R_P^1. Therefore, each root $c_j(a^{k-1}, \tau)$ of R_P^1 with $c_j(0) = 0$ should satisfy

$$|\tau| > |a_1 \dots a_{m-1} c|^{\tilde{k}} > 0. \tag{102}$$

Restrict now $(a_1, \dots, a_{m-1}, \tau)$ to $(2\tilde{k}, \delta)$-cone $K^m_{2\tilde{k},\tilde{d}t}$, i.e. in particular $|\tau| < |a_1 \dots a_{m-1}|^{2\tilde{k}}$. This gives that for any point $(a_1, \dots, a_{m-1}, \tau)$ from $K^m_{2\tilde{k},\tilde{d}t}$

a root $c_j(a^{m-1}, \tau)$ we have

$$0 < |c| < |a_1 \dots a_{m-1}|^{\tilde{k}}. \tag{103}$$

This completes the proof of the lemma. Q.E.D.

Let m and $\delta > 0$. Define a sequence of polynomials $P = (P_1, \dots, P_m)$ and numbers $m \in \mathbb{Z}_+$ and $\delta > 0$ defined by their coordinate functions:

$$\begin{gathered} P_j^0 = P_j - (P_1 \dots P_{j-1})^k, \ \ j = 2, \dots, m \\ P^s = \left(P_1^0, \dots, P_s^0, P_{s+1}, \dots, P_{m-s}\right), \ \ s = 2, \dots, m. \end{gathered} \tag{104}$$

Define the restriction of the (k, δ)-cone $K^m_{k,\delta}$ to the s-dimensional plane, denoted by $K^s_{k,\delta}$, generated by the first k-coordinates by the following way:

$$\begin{gathered} K^s_{k,\delta} = \{a^s = (a_1, \dots, a_s) \in \mathbb{R}^s : \ 0 < a_1 < \delta, \\ 0 < |a_{j+1}| < |a_1 \dots a_j|^{m_{j+1}} \text{ for } j = 1, \dots, s-1\}. \end{gathered} \tag{105}$$

By the construction of an (k, δ)-cone $K^m_{k,\delta}$. This cone consists of regular values of P (see Sect. 7.3), i.e. any point $(0, a^{m-1}) \in \mathbb{R} \times K^{m-1}_{k,\delta}$ is a regular point for the polynomial P^{m-1}.

By Lemma 92 there is refinement of $K^{m-1}_{k,\delta}$, denoted the same, such that for any two points $a^{m-1}_{\tau_1} = (0, a^{m-2}, \tau_1)$ and $a^{m-1}_{\tau_1}(0, a^{m-2}, \tau_2)$ from $\mathbb{R} \times K^{m-1}_{k,\delta}$

$$D^1_{P^1}\left(a^{m-1}_{\tau_1}, a^{m-1}_{\tau_2}\right) \leq (a_1 \dots a_{m-2})^2. \tag{106}$$

By induction one can show that there is a refinement an (m, δ)-cone $K^m_{k,\delta}$ such that for any two points $a^{m-s}_{\tau_1} = (0, a^{m-s-1}, \tau_1)$ and $a^{m-s}_{\tau_1} = (0, a^{m-s-1}, \tau_2)$ from the restriction cone $K^{m-s}_{k,\delta}$ such that

$$D^1_{P^s}\left(a^{m-s}_{\tau_1}, a^{m-s}_{\tau_2}\right) \leq (a_1 \dots a_{m-s-1})^2. \tag{107}$$

Notice that for any $1 \leq s \leq m$ level sets of the polynomial P^s correspond to level sets of the initial polynomial P. Combining this with all estimates for $D^1_{P^s}(a^{m-s}_{\tau_1}, a^{m-s}_{\tau_2})$ and the triangle-like inequality (88) one can show that part c) of Theorem 16 holds true. Part a) of Theorem 16 follows from part c) because $Q_P(x, y) \geq \|x - y\|^2$, which implies that $D^1_P(a, b) \geq D^0_P(a, b)$ for any pair $a, b \in K^m_{k,\delta}$. This completes the proof of Theorem 16. Q.E.D.

Theorem 16 implies Theorem 15. The latter one implies Theorem 13, which in turn, gives the Main Theorem. So, the proof of the Main Theorem is completed. Q.E.D.

8. Geometric multiplicity of generic germs

In this section we describe how using developed in the paper technique we can prove Theorems 1 and 2. Notice first that Theorem 2 implies Theorem 1 for $N = n$ and $P \equiv Id$.

Consider now a chain map $P \circ F : \mathbb{R}^n \to \mathbb{R}^n$, where $F : \mathbb{R}^n \to \mathbb{R}^N$ is a generic C^m-smooth map with $k > n$ and $P : \mathbb{R}^N \to \mathbb{R}^n$ is a polynomial map of degree d. Fix a coordinate system $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$. Then the chain map $P \circ F$ can be given by its coordinate functions

$$P \circ F(x) = (P_1 \circ F(x), \dots, P_n \circ F(x)). \tag{108}$$

W.l.o.g. one can assume that we need to estimate geometric multiplicity of 0 and $P \circ F(0) = 0$. Fix a sufficiently small positive r and the open r-ball B_r centered at 0 in $\mathbb{R}^n$. Suppose for some $b \in \mathbb{R}^n$ the number of regular preimages $\#\{F^{-1}(b) \cap B_r\}$ is maximal. By the Implicit Function Theorem $\#\{F^{-1}(\tilde{b}) \cap B_r\}$ is locally constant, i.e. it coincides for all $\tilde{b}$ sufficiently close to b. Thus, by Sard's lemma we can assume that b is regular nonzero value of the chain map $P \circ F$. Moreover, w.l.o.g. the first coordinate $b_1 = a_1$ is nonzero.

Consider now the problem of estimating geometric multiplicity as the problem of estimating the Khovanski number of the following system:

$$\begin{cases} P_1 \circ F(x) = a_1, \\ P_2 \circ F(x) = b_2, \\ \cdots & x \in B_r \subset \mathbb{R}^n \\ P_n \circ F(x) = b_n, \end{cases} \tag{109}$$

where a_1 is fixed and the first equation is rigid and all the other equations are loose and $b_2, \dots, b_n$ can vary. Notice B_r admits algebraic covering function. So Theorems 8 and 9 can be applied to (109). This gives that the Khovanski number of (109) is bounded by the sum of the Khovanski numbers of additional n systems, where n determined by #{loose equations} + 1. Notice that application of the Khovanski method to (109) also cooks up chain maps of the form $\{P^i \circ j^n F\}_{i=1}^n$ and allows to consider for additional systems only points in the image $a = (a_1, \dots, a_n)$ with sufficiently fast decay $1 \gg |a_1| \gg \cdots \gg |a_n|$. This makes it possible to apply Theorem 13 with no Cartesian restrictions. Therefore, estimates for the Khovanski numbers of the additional n systems are given by product of degrees of the corresponding coordinate polynomials of $\{P^i\}$'s. Direct computation completes the proof of Theorem 2. Q.E.D.

Acknowledgements. I would like to express my sincere gratitude to my teacher Yulij Sergeevich Ilyashenko for teaching me bifurcation theory, for his guidance and encouragement. Numerous discussions with him had been fruitful and indispensable for me to get into the subject. Without his enthusiasm and curiosity this work would have never been finished.

Several discussions with Askold Georgievich Khovanski about his method were very important to simplify presentation of the method in the paper. I would also like to thank Christine Rousseau for inviting me to the workshop in CRM, University of Montreal in January, 1999 and Robert Roussarie for inviting me to Dijon University in September, 1999 to give series of lectures about the present work. Professors Rousseau and Roussarie along with active participants of those seminar F. Dumortier, J.-P. Francoise, P. Mardesic, A. Mourtada, R. Moussu, D. Schlomiuk, and others have made many important comments on the subject to the paper. Countless discussions with J. Mather, and several discussions with D. Novikov and S. Yakovenko on the subject of the paper helped ameliorating the text significantly. I would like to made the last, but far not the least thank to Dana Schlomiuk for inviting me to give a course of lectures on Hilbert-Arnold problem and the present work in the workshop "Asymptotic series, differential algebra and finiteness problems in non-linear dynamical systems" in CRM, University of Montreal, in June 2000, which will be published in [Ka2]. It is my pleasure to thank all the above place and people for hospitality and exciting atmosphere to work.

References

[AAIS] Aframovich, V., Arnold, V., Ilyashenko, Yu., Shilnikov, L.: Encyclopedia Math. Sci., Vol. 5, Bifurcation Theory. Dynamical Systems VINITI (Russian) & Springer-Verlag (English) 1986 & 1994

[A] Arnold, V.: Problems for Arnold's seminar, 1989

[AGV] Arnold, V., Gusein-Zade, S., Varchenko, A.: Singularities of differentiable maps, Vol. I. Monographs in Mathematics 82. Birkhäuser, Boston 1985

[BCR] Bochnak, J., Coste, M., Roy, M.-F.: Real Algebraic Geometry. Springer-Verlag, Berlin 1998

[DeR] Denkowska, Z., Roussarie, R.: A Method of Desingularization for Analytic two-dimensional vector field families. Bol. Soc. Bras. Mat. **22**, 93–126 (1991)

[Du] Dulac, H.: Sur les cycles limite. Bull. Soc. Math. Fr. **51**, 45–188 (1923)

[D] Dumortier, F.: Singularities of vector fields on the plane. J. Differ. Equations **23**, 53–106 (1977)

[DIR] Dumortier, F., Ilyashenko, Yu., Rousseau, C.: Normal forms near a saddle-node and applications to finite cyclicity of graphics. Preprint 2000

[DMR] Dumortier, F., El Morsalani, M., Rousseau, C.: Hilbert's 16th problem for quadratic systems and cyclicity of elementary graphics. Nonlinearity **9**, 1209–1251 (1996)

[DRR] Dumortier, F., Roussarie, R., Rousseau, C.: Hilbert's 16th problem for quadratic vector fields. Differ. Equations **110**, 86–133 (1994)

[E] Écalle, J.: Introduction aux fonctions analysables et preuve consrustive de la conjecture de Dulac. Herman, Paris 1992

[FP] Francoise, J.-P., Pugh, C.: Keeping track of limit cycles. J. Differ. Equations **65**, 139–157 (1986)

[GK] Gabrielov, A., Khovanski, A.: Multiplicity of a Noetherian intersection. Geometry of differential equations, 119–130. AMS Transl. Ser. 2, 186. AMS, Providence, RI 1998

[GWPL] Gibson, C., Wirthmuller, K., du Plessis, A., Loojenga, E.: Topological Stability of Smooth Mappings. Lectures Notes in Mathematics **552**. Springer 1976

[GG] Golubitsky, M., Guillemin, V.: Stable Mappings and Their Singularities. Graduate Texts in Mathematics **14**. Springer-Verlag 1973

[GM] Goresky, M., MacPherson, R.: Stratified Morse Theory. Springer-Verlag 1987

[G] Grozovskii, T.: Bifurcations of polycycles an "apple" and a "half-apple" in generic two-parameter families (in Russian). Differ. Equations **32**, 458–469 (1996)

[GR] Guzman, A., Rousseau, C.: Genericity condition for cyclicity of elementary graphics. Differ. Equations **155**, 44–72 (1999)
[Hir1] Hironaka, H.: Stratification and flatness Real and Complex singularities. Nordic Summer School (Oslo, 1976). Sijthoff-Noordhoff, Growhgen 1977
[Hir2] Hironaka, H.: Introduction to real-algebraic sets and real-analytic maps. Instituto Mathematico L. Tonelli, Dell' Universita'di Piza 1973
[I1] Ilyashenko, Yu.: Finiteness Theorem for limit cycles. Am. Math. Soc., Providence, RI 1991
[I2] Ilyashenko, Yu.: Normal forms for local families and nonlocal bifurcations. Astérisque **222**, 233–258 (1994)
[I3] Ilyashenko, Yu.: Centennial history of the Hilbert 16th problem. Bull. Am. Math. Soc. **39**, 301–354 (2002)
[IK] Ilyashenko, Yu., Kaloshin, V.: Bifurcations of planar and spatial polycycles: Arnold's program and its development. Fields Inst. Commun. **24**, 241–271 (1999)
[IY1] Ilyashenko, Yu., Yakovenko, S.: Finitely smooth normal forms of local families of diffeomorphisms and vector fields. Russ. Math. Surv. **46**, 1–43 (1991)
[IY2] Ilyashenko, Yu., Yakovenko, S.: Concerning Hilbert sixteenth problem. Am. Math. Soc. Transl., Ser. 2 **165**, 1–20 (1995)
[IY3] Ilyashenko, Yu., Yakovenko, S.: Finite Cyclicity of Elementary Polycycles in Generic Families. Am. Math. Soc. Transl. **165**, 21–95 (1995)
[J] Jacobson, N.: Basic Algebra, Vol. 1, 1974
[Ka1] Kaloshin, V.: A Geometric Proof of Existence of Whitney's stratifications. Preprint, http://www.aimath.org/preprints.html, 2000-11
[Ka2] Kaloshin, V.: Around Hilbert-Arnold Problem. CRM, Montreal, to appear
[Ka3] Kaloshin, V.: Hilbert-Arnold Problem and an estimate for cyclicity of polycycles on the plane and in the space. Funct. Anal. Appl. **35**, 78–81 (2001)
[Ka4] Kaloshin, V.: Some prevalent properties of smooth dynamical systems. Proc. Steklov Inst. Math. **213**, 115–140 (1996)
[Kh] Khovanski, A.: Fewnomials. Am. Math. Soc., Providence, RI 1991
[Kl] Kleban, O.: Order of the topologically sufficient jet of a smooth vector field on the real plane at a singular point of finite multiplicity. Am. Math. Soc. Transl., Ser. 2 **165**, 131–153 (1995)
[KS] Kotova, A., Stanzo, V.: Few-Parameter Generic Families on the Sphere. Am. Math. Soc. Transl., Ser. 2, Providence, RI **213**, 155–202 (1996)
[Ma] Mather, J.: Notes on topological stability. Harvard University 1970
[Mi] Milnor, J.: Singular points of complex hypersurfaces. Princeton University Press, Princeton, NJ **61**, 1968
[MR] Moussu, R., Roche, C.: Khovanski's theory and the Dulac Problem. Invent. Math. **105**, 431–441 (1991)
[Mu] Mumford, D.: Algebraic Geometry I, Complex Projective Algebraic Varieties. Springer-Verlag 1976
[P] Perko, L.: Differential Equations and Dynamical Systems. Appl. Math. 7. Springer-Verlag 1993
[PW] du Plessis, A., Wall, T.: The Geometry of Topological Stability. Oxford 1995
[R1] Roussarie, R.: Cyclicite finie et le 16 problem d'Hilbert, Dynamical systems (Volparaiso 1986), R. Bauon, R. Lavarca, J. Palis (eds.). Lecture Notes in Mathematics **1331**. Springer-Verlag, Berlin, New York 1988, pp. 161–188
[R2] Roussarie, R.: Bifurcation of planar vector fields and Hilbert's sixteenth problem. Progr. Math. **164**. Birkhäuser Verlag, Basel 1998
[RSZ] Rousseau, C., Świrszcz, G., Zoladek, H.: Cyclicity of graphics with semi-hyperbolic points inside quadratic systems. Dyn. Control Syst. **4**, 149–189 (1988)
[S] Schlomiuk, D. (ed.): Bifurcations and periodic orbits of vector fields. NATO AS1, Series C (Math. and Phys. Sciences), Vol. 408. Kluwer, Dordrecht, Boston, London 1993
[Ta] Takens, F.: Unfoldings of certain singularities of vector fields: generalized Hopf bifurcations. J. Differ. Equations **14**, 476–493 (1973)

[Th] Thom, R.: Ensembles et morphismes stratifiés. Bull. Am. Math. Soc. **75**, 240–284 (1969)

[Ti] Tikhomirov, V.: Fundamental principles of the theory of extremal problems. Transl. by Bernd Luderer. John Wiley & Sons 1986

[Tr1] Trifonov, S.: Desingularization in Families of Analytic Differential Equations. Am. Math. Soc. Transl., Ser. 2 **165**, 97–129 (1995)

[Tr2] Trifonov, S.: Cyclicity of elementary polycycles in generic smooth vector fields, Differential equations with real and complex time, Yu. Ilyashenko (ed.). Proc. Steklov Inst. Math. **213**, 141–199 (1996)

[Wa] Wall, T.: Regular Stratifications. Lecture Notes in Mathematics **468**, 332–344 (1975)

[W] Whitney, H.: Elementary structure of real algebraic varieties. Ann. Math. **66**, 545–556 (1957)