Non-symmetric Simple Random Walks along Orbits of Ergodic Automorphisms.

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Consider an ergodic measure-preserving automorphism $T$ of probability space $(M, \mathcal{M}, \mu)$. Having a measurable function $p, 0 < p(x) < 1$ a.e., we can construct Markov chain whose phase space is $M$ and a moving point $x \in M$ jumps from $x$ to $Tx$ with probability $p(x)$ and to $T^{-1}x$ with probability $1 - p(x)$. Such Markov chains are called simple random walks along orbits of $T$. If $x_0$ is an initial position then the position $x_n$ at any moment of time $n$ can be written as $x_n = T^{b_n}x_0$. For the sequence $b_n$ we have $b_0 = 0$, $b_{n+1} - b_n = \pm 1$, i.e. $b_n$ is a simple random walk on $\mathbb{Z}^1$, for which $P\{b_{n+1} = k + 1|b_n = k\} = p(T^kx)$, $P\{b_{n+1} = k - 1|b_n = k\} = 1 - p(T^kx)$. The first question which arises in this situation is whether this Markov chain has a stationary measure absolutely continuous wrt to $\mu$. The density $r$ of this measure must satisfy the equation

$$r(x) = r(T^{-1}x)p(T^{-1}x) + r(Tx)(1 - p(Tx)) \tag{1}$$

Assume that $0 < c_1 = const < p(x) < c_2 = const < 1$ a.e. and put $q(x) = r(x)p(x)$. Then

$$\frac{q(x)}{p(x)} = q(T^{-1}x) + q(Tx)\frac{1 - p(Tx)}{p(Tx)}$$

or

$$q(x) - q(Tx)\frac{1 - p(Tx)}{p(Tx)} = q(T^{-1}x) - q(x)\frac{1 - p(x)}{p(x)}.$$

This shows that $q(x) - q(Tx)\frac{1 - p(Tx)}{p(Tx)}$ is invariant under $T$ and must be a constant a.e. since $T$ is ergodic:

$$q(x) - q(Tx)\frac{1 - p(Tx)}{p(Tx)} = C$$

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or

\[ q(x) = q(Tx) \frac{1 - p(Tx)}{p(Tx)} + C, \quad C = \text{const} \]  \quad (2)

**Definition 1.** Simple random walks is called symmetric if

\[ \int \ln p(x) d\mu(x) = \int \ln(1 - p(x)) d\mu(x). \]

Otherwise it is called non-symmetric.

Let us show that in the non-symmetric case (2) and therefore (1) always have a solution. Assume for the definiteness that

\[ \int \ln(1 - p(x)) d\mu(x) < \int \ln p(x) d\mu(x). \]  \quad (3)

It is easy to see that the series below converges for a.e. \( x \)

\[ q(x) = C + C \frac{1 - p(Tx)}{p(Tx)} + C \frac{1 - p(Tx)}{p(Tx)} \cdot \frac{1 - p(T^2x)}{p(T^2x)} + \ldots \]  \quad (4)

Indeed, the product of \( k \) terms can be rewritten as

\[ \prod_{i=1}^{k} \frac{1 - p(T^i x)}{p(T^i x)} = \exp \left[ \sum_{i=1}^{k} \ln \frac{1 - p(T^i x)}{p(T^i x)} \right]. \]

The last sum behaves a.e. as \(-ck\) as \( k \to \infty\) in a view of ergodic theorem where \( c = \int \ln \frac{1}{1 - p(x)} d\mu(x) > 0 \). The case \( \int \ln(1 - p(x)) d\mu(x) > \int \ln p(x) d\mu(x) \) can be considered in the same way.

If the random walk is symmetric then \( C = 0 \). Indeed, we must have

\[ \int q(x) d\mu(x) = \int \ln \left( q(Tx) \cdot \frac{1 - p(Tx)}{p(Tx)} + C \right) d\mu(x) \]  \quad (5)

The right-hand side is a monotone function of \( C \). Since (5) holds for \( C = 0 \) we get the result. In this case (2) is reduced to

\[ \frac{q(x)}{q(Tx)} = \frac{1 - p(Tx)}{p(Tx)} \]

or

\[ \ln q(x) - \ln q(Tx) = \ln(1 - p(Tx)) - \ln p(Tx) \]  \quad (6)
The last equation has a solution for generic \( p \) only if \( T \) is a shift on a compact abelian group i.e. it is an automorphism with pure point spectrum [S2] Lect.4. This follows easily from the spectral theory of dynamical systems. This case with \( M = Tor^d \) was considered in [S1]. For other transformations having a continuous component in the spectrum the equation (6) has no solution for generic \( p \).

Fix \( x \in M \) and consider the probabilities \( p^{(+)}_{2n}(x) \) of random walks \( \{b(m) \ 0 \leq m \leq 2n\} \) (see above) for which \( b(m) > 0 \) for \( 0 < m < 2n \), \( b(2n) = 0 \). Following P.Levy we call such walks positive excursions. One can consider also negative excursions and corresponding probabilities \( p^{(-)}_{2n}(x) \). It is easy to see that

\[
p^{(+)}_{2}(x) = p(x)(1 - p(Tx)),
\]

\[
p^{(+)}_{2n}(x) = p(x) \left[ \sum_{s \geq 1} \sum_{n_1 + n_2 + \cdots + n_s = n - 1 \atop n_i \geq 0, 1 \leq i \leq s} p^{(+)}_{2n_1}(Tx) \cdots p^{(+)}_{2n_s}(Tx) \right] (1 - p(Tx))
\]

For the generating \( \varphi^{(+)}(x, \Theta) = \sum_{n \geq 1} p^{(+)}_{2n}(x) \Theta^{2n}, \ |\Theta| \leq 1 \), we derive from (7) and (8) the equation (see also [S1])

\[
\varphi^{(+)}(x, \Theta) = \frac{p(x)(1 - p(Tx))\Theta^2}{1 - \varphi^{(+)}(Tx, \Theta)}.
\]  

**Lemma 1.** If (3) is valid then \( \varphi^{(+)}(x, 1) = p(x) \frac{\Lambda(x)}{\Lambda(x) + 1} \) where \( \Lambda(x) = \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{1 - p(T^i x)}{p(T^i x)} \).

The last series converges a.e.

**Proof.** For \( \varphi^{(+)}(x, 1) \) we have from (9)

\[
\varphi^{(+)}(x, 1) = \frac{p(x)(1 - p(Tx))}{1 - \varphi^{(+)}(Tx, 1)}.
\]  

Then for \( \psi^{(+)}(x, 1) = \varphi^{(+)}(x, 1)/p(x) \leq 1 \) we have

\[
\psi^{(+)}(x, 1) = \frac{\frac{1 - p(Tx)}{p(Tx)}}{1 - \psi^{(+)}(Tx, 1)}.
\]  

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Since $\psi^+(y, 1) > 0$ one can write $\psi^+(T^k x, 1) = \frac{\Lambda(T^k x)}{\Lambda(T^k x) + 1}$. Then (11) gives

$$\frac{\Lambda(x)}{\Lambda(x) + 1} = \frac{1 - p(Tx)}{p(Tx)} \Lambda(Tx) + \frac{1}{\varphi(Tx)}.$$ 

or

$$\Lambda(x) = \frac{1 - p(Tx)}{p(Tx)} + \frac{1 - p(Tx)}{p(Tx)} \Lambda(Tx)$$

(12)

This shows that

$$\Lambda(x) = \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{1 - p(T^i x)}{p(T^i x)}$$

The last series converges a.e. in view of (3) and gives the result, QED. Notice that $\Lambda(x) = C^{-1} q(x)$, where $q(x)$ is defined by formula (4).

If we calculate in the same way the distribution of negative Levy excursions we shall easily get for the corresponding generating functions that $\varphi^-(x, 1) = 1 - p(x)$.

Thus in the non-symmetric case (3) the probability of the negative excursion is $1 - p(x)$, the probability of the positive excursion is $p(x) \frac{\Lambda(x)}{\Lambda(x) + 1}$ and the probability that the moving point jumps from $x$ to $Tx$ and does not return back to $x$ is $p(x) \frac{\Lambda(x)}{\Lambda(x) + 1}$.

Let us calculate the expectation of the length of the positive excursion $m^+_1(x) = \sum_{n \geq 1} 2n p^+_n(x)$. We have by differentiating (9)

$$m^+_1(x) = \frac{\partial \varphi^+(x, 1)}{\partial \Theta} = \frac{2p(x)(1 - p(Tx))}{1 - \varphi^+(Tx, 1)} + \frac{p(x)(1 - p(Tx))}{(1 - \varphi^+(Tx, 1))^2} m^+_1(Tx)$$

(13)

The ratio $\frac{p(x)(1 - p(Tx))}{(1 - \varphi^+(Tx, 1))^2}$ equals

$$\frac{p(x)(1 - p(Tx))(\Lambda(Tx) + 1)^2}{((1 - p(Tx))\Lambda(Tx) + 1)^2} = \frac{p(x)}{p(Tx)} \cdot \frac{1 - p(Tx)}{p(Tx)} \cdot \frac{(\Lambda(Tx) + 1)^2}{\left(1 - \frac{p(Tx)}{p(Tx)} \cdot \Lambda(Tx) + \frac{1 - p(Tx)}{p(Tx)} + 1\right)^2}$$

$$= \frac{p(x)}{p(Tx)} \cdot \frac{(\Lambda(Tx) + 1)^2}{(\Lambda(x) + 1)^2} \cdot \frac{1 - p(Tx)}{p(Tx)}.$$
The last step was based on (12). If we put \( h(x) = \frac{2p(x)(1-p(Tx))}{1-\varphi^{(+)}(Tx,1)} \) then we can write \( m_1^{(+)}(x) \) in the form

\[
m_1^{(+)}(x) = h(x) + \frac{p(x)}{(\Lambda(x) + 1)^2} \cdot \sum_{k=1}^{\infty} h(T^k x) \prod_{i=1}^{k} \frac{1 - p(T^i x)}{p(T^i x)} \cdot \frac{(\Lambda(T^k x) + 1)^2}{p(T^{k+1} x)} \quad (14)
\]

In view of our assumption concerning \( p \) the function \( h \) is bounded from above and below. We need

**Assumption 1.** The function \( \Lambda \in L^1 (M, \mathcal{M}, \mu) \).

Under this assumption the density \( r(x) \in L^1 (M, \mathcal{M}, \mu) \) (see (1) and (4)).

In a similar way one can write down the expressions for the derivatives \( m_i^{(+)}(x) = \frac{d^i \varphi^{(+)}(x,1)}{dx^i} \) which are finite a.e. if the Assumption 1 holds.

The same analysis can be applied to \( \varphi^{(-)}(x,1) \) and its derivatives. One can show that if (3) holds then the probabilities \( p_n^{(-)}(x) \) decay exponentially. The sum \( \varphi(x, \Theta) = \varphi^{(+)}(x, \Theta) + \varphi^{(-)}(x, \Theta) \) is generating function for excursions, both negative and positive.

Consider the probability distribution \( P_{x}^{(n)} \) concentrated on the set \( \bigcup_{|k| \leq n} T^k x \) where \( P_{x}^{(n)}(k) \) is the probability of random walks starting from \( x = x_0 \) for which \( x_n = T^k x_0 \) or \( b(n) = k \). For \( k > 0 \) such random walks have the following structure. First it makes some numbers of positive or negative excursions coming back to \( x_0 \). Then it jumps to \( T x_0 \) and makes \( \nu_1 \) positive excursions. After this it jumps to \( T^2 x_1 \), makes there \( \nu_2 \) positive excursions and so on. We denote \( \nu_0, \nu_1, \cdots, \nu_k \) the numbers of these excursions and \( \xi_{i1}, \xi_{i2}, \cdots, \xi_{i\nu_i} \) are their lengths, \( i = 0, 1, \cdots, k \) \( \zeta_i = \xi_{i1} + \cdots + \xi_{i\nu_i} \). Then \( P\{\nu_i = r\} = \left( \frac{p(T^i x) \Lambda(T^i x)}{\Lambda(T^i x) + 1} \right)^r \), \( r = 1, 2, \cdots \) and we can write

\[
P_{x}^{(n)}(k) = \frac{1}{p(T^k x_0)} \cdot \frac{1}{1 - \frac{p(T^k x_0) \Lambda(T^k x_0)}{\Lambda(T^k x_0) + 1}} \cdot \sum_{r=0}^{\infty} \left( \frac{p(T^r x) \Lambda(T^r x)}{\Lambda(T^r x) + 1} \right)^r \quad (15)
\]

The factor \( \sum_{r=0}^{\infty} \left( \frac{p(T^r x) \Lambda(T^r x)}{\Lambda(T^r x) + 1} \right)^r \) is the sum over the values of \( \nu_i \) of probabilities to have \( r \) positive excursions. From (12)

\[
\frac{p(T^i x)}{1 - \frac{p(T^i x) \Lambda(T^i x)}{\Lambda(T^i x) + 1}} = \frac{\Lambda(T^i x) + 1}{(1-p(T^i x)) (\Lambda(T^i x) + 1)} = \frac{\Lambda(T^i x) + 1}{\Lambda(T^{i-1} x) + 1}
\]

Therefore
\[ P_x^{(n)}(k) = \frac{1}{p(T^k x)} \frac{\Lambda(T^k x)}{\Lambda(x)} + 1 \hat{P}\{\zeta_0 + \zeta_1 + \cdots + \zeta_k = n - k\} = \frac{r(T^k x)}{\Lambda(x)} + 1 \hat{P}\{\zeta_0 + \zeta_1 + \cdots + \zeta_k = n - k\} \quad (16) \]

Here \( \hat{P} \) is the probability distribution corresponding to the distribution of the sum of \( k+1 \) independent random variables \( \zeta_i \) for which
\[
\hat{P}\{\zeta_i = 2m\} = \left(1 - \frac{p(T^i x)\Lambda(T^i x)}{\Lambda(T^i x) + 1}\right) P\{\zeta_i = 2m\}.
\]

Now we can formulate the main result of this paper.

**Main theorem.** For a non-symmetric random walks along orbits of \( T \) for which inequality (3), Assumption 1 hold and \( m_1^{(+)}(x), m_2^{(+)}(x), m_3^{(+)}(x) \in L^1(M, \mathcal{M}, \mu) \) the distribution \( P_x^{(n)} \) converges to \( r(z)d\mu(z) \) in the following sense: for any bounded measurable \( \varphi \)
\[
\lim_{n \to \infty} \int M \varphi(z) dP_x^{(n)}(z) = \lim_{n \to \infty} \sum_k P_x^{(n)}(k) \varphi(T^k x) = \int M \varphi(z) r(z) d\mu(z)
\]

for \( \mu \)-a.e. \( x \).

**Proof.** Using (16) we can write
\[
\sum_k P_x^{(n)}(k) \varphi(T^k x) = \frac{1}{\Lambda(x)} + 1 \sum_{i=1}^k \frac{r(T^i x) \varphi(T^i x) \hat{P}\{\zeta_0 + \zeta_1 + \cdots + \zeta_k = n - k\}}{\Lambda(x) + 1}
\]

In view of our assumptions the sequence of random variables \( \zeta_0, \zeta_1, \cdots, \zeta_k \) satisfies the local central limit theorem of probability theory. Therefore if \( a(T^i x) \) is the expectation of \( \zeta_i \) (with respect to \( \mathcal{P} \)), \( \sigma(T^i x) \) is its variance we can write
\[
\hat{P}\{\zeta_0 + \zeta_1 + \cdots + \zeta_k = n - k\} \sim \frac{1}{\sqrt{2\pi k \sigma}} \cdot \exp \left\{ -\frac{1}{2} \frac{(n - k - \sum_{i=1}^k a(T^i x))^2}{\sum_{i=1}^k \sigma(T^i x)} \right\}
\]

\[ \sigma = \frac{1}{k} \sum_{i=1}^k \sigma(T^i x). \] The last formula shows that \( \hat{P} \) is a slowly varying function of \( n - k \). Therefore the average over \( k \) of \( r(T^k x) \varphi(T^k x) \) with respect to this probability distribution is equivalent to the usual average with respect to the uniform distribution which gives in the limit \( \int M \varphi(z) d\mu(z) \), QED.
Thus, the Proposition. QED.

Let us rewrite \( \Delta(\mathbf{x}) \) and any \( \mathbf{y} \). By formula (17), \( \Delta(\mathbf{x}) \) converges to \( r(\mathbf{y})d\mu(\mathbf{z}) \) in the following sense

\[
\lim_{n \to \infty} \int \phi(z) dP^{(n)}_x(z) = \lim_{n \to \infty} \sum_{k=1}^n P^{(n)}_x(k) \phi(T^k \mathbf{x}) = \int \phi(z) r(\mathbf{y}) d\mu(\mathbf{y})
\]

for all \( \mathbf{x} \in M \).

This theorem was proved in [S1] for any diophantine \( \omega \).

**Proof** Assume for definiteness that \( \int \ln(1 - p(x)) d\mu(x) \neq \int \ln p(x) d\mu(x) \).

**Proposition** \( \Lambda(\mathbf{x}) = \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{1-p(T^i x)}{p(T^i x)} \) is a continuous function.

**Proof of the Proposition.** Put \( \eta(x) = \ln(1 - p(x))/p(x) \) and \( -C = \int \eta(x) d\mu(x) < 0 \). Rewrite \( \Lambda(\mathbf{x}) = \sum_{k=1}^{n} \exp(\sum_{k=1}^{n} \eta(T^k \mathbf{x})) \). Show that for any \( \epsilon > 0 \) there exists a constant \( n_0 = n_0(\epsilon) \) such that for any \( x \in M \) and \( n > n_0 \) the following holds:

\[
\sum_{k=1}^{n} \eta(T^k \mathbf{x}) < -(C - \epsilon)n < -Cn/2.
\]  

(17)

Since, \( \eta(x) \) is a continuous function for any \( \epsilon > 0 \) there exists \( \delta = \delta(\epsilon) \) such that \( |\eta(x) - \eta(y)| < \epsilon/2 \) for any \( |x - y| < d\delta \). Fix a \( \delta \)-grid \( \{x_j\}_{i=1}^{N} \) on \( M = \text{Torus}^d \). By ergodic theorem there exists a universal constant \( n_0 \) such that for any \( j = 1, \ldots, N \) and any \( n > n_0 \)

\[
\left| \frac{1}{n} \sum_{i=1}^{n} \eta(T^i x_j) - \int \eta(y) d\mu(y) \right| < \epsilon/2.
\]

Then for any point \( x \in M \) there exists a point of the grid \( x_j \) such that \( |x - x_j| < \delta \). Thus,

\[
\left| \frac{1}{n} \sum_{i=1}^{n} \eta(T^i x) - \int \eta(y) d\mu(y) \right| < \epsilon.
\]

Formula (17) implies continuity of \( \Lambda(\mathbf{x}) \), because \( a_n(x) = \exp(\sum_{k=1}^{n} \eta(T^k \mathbf{x})) \) exponentially small and the sum over \( n > n_0 \) is uniformly small. This completes the proof of the Proposition. QED.

Let us rewrite

\[
m_1^{(+)}(x) = \frac{\partial \phi^+(x, \theta)}{\partial \theta} |_{\theta=1}
\]
in the form
\[ m_1^{(+)}(x) = h(x) + h(x) \sum_{k=1}^{\infty} \prod_{j=0}^{k} \frac{\Lambda(T^{j+1}x)}{\Lambda(T^jx) + 1}. \]

\( \Lambda(x) \) is continuous and, therefore, bounded. This implies that \( m^{(+)}(x) \) is a bounded function of \( x \). One can check that
\[ m_2^{(+)}(x) = \frac{\partial^2 \phi^{(+)}(x, \theta)}{\partial \theta^2} \bigg|_{\theta=1} \]
can be presented in the following form: Put \( \Phi(x) = 1 + \sum_{k=1}^{\infty} \prod_{j=0}^{k} \frac{\Lambda(T^{j+1}x)}{\Lambda(T^jx) + 1}. \) Then
\[ m_2^{(+)}(x) = \left( h(x) + 2h(x)h(Tx) \frac{\Lambda(Tx) + 1}{p(Tx)(\Lambda(x) + 1)} \Phi(Tx) \right) \Phi(x). \]

It is easy to see that \( \Phi(x) \) is continuous and bounded. Therefore, \( m_2^{(+)}(x) \) is bounded. In a similar fashion one can check the third moment. Thus, we can apply the Main theorem. This completes the proof of the theorem. QED.

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Reference.