# Non-symmetric Simple Random Walks along Orbits of Ergodic Automorphisms.

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Consider an ergodic measure-preserving automorphism $T$ of probability space $(M, \mathcal{M}, \mu)$. Having a measurable function $p, 0 < p(x) < 1$ a.e., we can construct Markov chain whose phase space is $M$ and a moving point $x \in M$ jumps from $x$ to $Tx$ with probability $p(x)$ and to $T^{-1}x$ with probability $1 - p(x)$. Such Markov chains are called simple random walks along orbits of $T$. If $x_0$ is an initial position then the position $x_n$ at any moment of time $n$ can be written as $x_n = T^{b_n} x_0$. For the sequence $b_n$ we have $b_0 = 0$, $b_{n+1} - b_n = \pm 1$, i.e. $b_n$ is a simple random walk on $\mathbb{Z}^1$, for which $P\{b_{n+1} = k+1 | b_n = k\} = p(T^k x)$, $P\{b_{n+1} = k-1 | b_n = k\} = 1 - p(T^k x)$. The first question which arises in this situation is whether this Markov chain has a stationary measure absolutely continuous wrt to $\mu$. The density $r$ of this measure must satisfy the equation

$$r(x) = r(T^{-1}x)p(T^{-1}x) + r(Tx)(1 - p(Tx)) \tag{1}$$

Assume that $0 < c_1 = const < p(x) < c_2 = const < 1$ a.e. and put $q(x) = r(x)p(x)$. Then

$$\frac{q(x)}{p(x)} = q(T^{-1}x) + q(Tx)\frac{1 - p(Tx)}{p(Tx)}$$

or

$$q(x) - q(Tx)\frac{1 - p(Tx)}{p(Tx)} = q(T^{-1}x) - q(x)\frac{1 - p(x)}{p(x)}.$$

This shows that $q(x) - q(Tx)\frac{1-p(Tx)}{p(Tx)}$ is invariant under $T$ and must be a constant a.e. since $T$ is ergodic:

$$q(x) - q(Tx)\frac{1 - p(Tx)}{p(Tx)} = C$$

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or

$$q(x) = q(Tx)\frac{1-p(Tx)}{p(Tx)} + C, \quad C = const \tag{2}$$

**Definition 1.** Simple random walks is called symmetric if $\int \ln p(x)d\mu(x) = \int \ln(1-p(x))d\mu(x)$. Otherwise it is called non-symmetric.

Let us show that in the non-symmetric case (2) and therefore (1) always have a solution. Assume for the definiteness that

$$\int \ln(1-p(x))d\mu(x) < \int \ln p(x)d\mu(x). \tag{3}$$

It is easy to see that the series below converges for a.e. $x$

$$q(x) = C + C\frac{1-p(Tx)}{p(Tx)} + C\frac{1-p(Tx)}{p(Tx)} \cdot \frac{1-p(T^2x)}{p(T^2x)} + \cdots \tag{4}$$

Indeed, the product of $k$ terms can be rewritten as

$$\prod_{i=1}^{k} \frac{(1-p(T^ix))}{p(T^ix)} = \exp\left[\sum_{i=1}^{k} \ln\frac{(1-p(T^ix))}{p(T^ix)}\right].$$

The last sum behaves a.e. as $-ck$ as $k \to \infty$ in a view of ergodic theorem where $c = \int \ln \frac{p(x)}{1-p(x)} d\mu(x) > 0$. The case $\int \ln(1-p(x))d\mu > \int \ln p(x)d\mu(x)$ can be considered in the same way.

If the random walk is symmetric then $C = 0$. Indeed, we must have

$$\int \ln q(x)d\mu(x) = \int \ln\left(q(Tx) \cdot \frac{1-p(Tx)}{p(Tx)} + C\right) d\mu(x) \tag{5}$$

The right-hand side is a monotone function of $C$. Since (5) holds for $C = 0$ we get the result. In this case (2) is reduced to

$$\frac{q(x)}{q(Tx)} = \frac{1-p(Tx)}{p(Tx)}$$

or

$$\ln q(x) - \ln q(Tx) = \ln(1-p(Tx)) - \ln p(Tx) \tag{6}$$

The last equation has a solution for generic $p$ only if $T$ is a shift on a compact abelian group i.e. it is an automorphism with pure point spectrum [S2] Lect.4. This follows easily from the spectral theory of dynamical systems. This case with $M = Tor^d$ was considered in [S1]. For other transformations having a continuous component in the spectrum the equation (6) has no solution for generic $p$.

Fix $x \in M$ and consider the probabilities $p_{2n}^{(+)}(x)$ of random walks $\{b(m)\ 0 \le m \le 2n\}$ (see above) for which $b(m) > 0$ for $0 < m < 2n, b(2n) = 0$. Following P.Levy we call such walks positive excursions. One can consider also negative excursions and corresponding probabilities $p_{2n}^{(-)}(x)$. It is easy to see that

$$p_2^{(+)}(x) = p(x)(1 - p(Tx)), \tag{7}$$

$$p_{2n}^{(+)}(x) = p(x) \left[ \sum_{s \ge 1} \sum_{\substack{n_1+n_2+\cdots+n_s=n-1 \\ n_i>0, 1\le i\le s}} p_{2n_1}^{(+)}(Tx) \cdot \cdots \cdot p_{2n_s}^{(+)}(Tx) \right] (1 - p(Tx)) \tag{8}$$

For the generating $\varphi^{(+)}(x, \Theta) = \sum_{n \ge 1} p_{2n}^{(+)}(x)\Theta^{2n}$, $|\Theta| \le 1$, we derive from (7) and (8) the equation (see also [S1])

$$\varphi^{(+)}(x, \Theta) = \frac{p(x)(1 - p(Tx))\Theta^2}{1 - \varphi^{(+)}(Tx, \Theta)}. \tag{9}$$

**Lemma 1.** *If (3) is valid then* $\varphi^{(+)}(x, 1) = p(x)\frac{\Lambda(x)}{\Lambda(x)+1}$ *where* $\Lambda(x) = \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{1-p(T^i x)}{p(T^i x)}$. *The last series converges a.e.*

**Proof.** For $\varphi^{(+)}(x, 1)$ we have from (9)

$$\varphi^{(+)}(x, 1) = \frac{p(x)(1 - p(Tx))}{1 - \varphi^{(+)}(Tx, 1)}. \tag{10}$$

Then for $\psi^{(+)}(x, 1) = \varphi^{(+)}(x, 1)/p(x) \le 1$ we have

$$\psi^{(+)}(x, 1) = \frac{\frac{1-p(Tx)}{p(Tx)}}{\frac{1}{p(Tx)} - \psi^{(+)}(Tx, 1)} \tag{11}$$

Since $\psi^{(+)}(y,1) > 0$ one can write $\psi^{(+)}\left(T^k x, 1\right) = \frac{\Lambda(T^k x)}{\Lambda(T^k x)+1}$. Then (11) gives

$$\frac{\Lambda(x)}{\Lambda(x)+1} = \frac{\frac{1-p(Tx)}{p(Tx)}\left(\Lambda(Tx)+1\right)}{\frac{1-p(Tx)}{p(Tx)}\Lambda(Tx) + \frac{1}{p(Tx)}},$$

or

$$\Lambda(x) = \frac{1-p(Tx)}{p(Tx)} + \frac{1-p(Tx)}{p(Tx)}\Lambda(Tx) \tag{12}$$

This shows that

$$\Lambda(x) = \sum_{k=1}^{\infty}\prod_{i=1}^{k}\frac{1-p(T^i x)}{p(T^i x)}$$

The last series converges a.e. in view of (3) and gives the result, QED. Notice that $\Lambda(x) = C^{-1}q(x)$, where $q(x)$ is defined by formula (4).

If we calculate in the same way the distribution of negative Levy excursions we shall easily get for the corresponding generating functions that $\varphi^{(-)}(x,1) = 1 - p(x)$.

Thus in the non-symmetric case (3) the probability of the negative excursion is $1-p(x)$, the probability of the positive excursion is $p(x)\frac{\Lambda(x)}{\Lambda(x)+1}$ and the probability that the moving point jumps from $x$ to $Tx$ and does not return back to $x$ is $p(x)\frac{1}{\Lambda(x)+1}$. Let us calculate the expectation of the length of the positive excursion $m_1^{(+)}(x) = \sum\limits_{n\geq 1} 2np_{2n}^{(+)}(x)$. We have by differentiating (9)

$$m_1^{(+)}(x) = \frac{\partial\varphi^{(+)}(x,1)}{\partial\Theta} = \frac{2p(x)(1-p(Tx))}{1-\varphi^{(+)}(Tx,1)} + \frac{p(x)(1-p(Tx))}{(1-\varphi^{(+)}(Tx,1))^2}m_1^{(+)}(Tx) \tag{13}$$

The ratio $\frac{p(x)(1-p(Tx))}{(1-\varphi^{(+)}(Tx,1))^2}$ equals

$$\begin{aligned}\frac{p(x)(1-p(Tx))(\Lambda(Tx)+1)^2}{((1-p(Tx))\Lambda(Tx)+1)^2} &= \frac{p(x)}{p(Tx)}\cdot\frac{1-p(Tx)}{p(Tx)}\cdot\frac{(\Lambda(Tx)+1)^2}{\left(\frac{1-p(Tx)}{p(Tx)}\cdot\Lambda(Tx)+\frac{1-p(Tx)}{p(Tx)}+1\right)^2} = \\ &= \frac{p(x)}{p(Tx)}\cdot\frac{(\Lambda(Tx)+1)^2}{(\Lambda(x)+1)^2}\cdot\frac{1-p(Tx)}{p(Tx)}.\end{aligned}$$

The last step was based on (12). If we put $h(x) = \frac{2p(x)(1-p(Tx))}{1-\varphi^{(+)}(Tx,1)}$ then we can write $m_1^{(+)}(x)$ in the form

$$m_1^{(+)}(x) = h(x) + \frac{p(x)}{(\Lambda(x)+1)^2} \cdot \sum_{k=1}^{\infty} h(T^k x) \prod_{i=1}^{k} \frac{1-p(T^i x)}{p(T^i x)} \cdot \frac{(\Lambda(T^k x)+1)^2}{p(T^{k+1}x)} \tag{14}$$

In view of our assumption concerning $p$ the function $h$ is bounded from above and below. We need

**Assumption 1.** *The function* $\Lambda \in L^1(M, \mathcal{M}, \mu)$.

Under this assumption the density $r(x) \in L^1(M, \mathcal{M}, \mu)$ (see (1) and (4)).

In a similar way one can write down the expressions for the derivatives $m_r^{(+)}(x) = \frac{d^r \varphi^{(+)}(x,1)}{d\Theta^r}$ which are finite a.e. if the Assumption 1 holds.

The same analysis can be applied to $\varphi^{(-)}(x,1)$ and its derivatives. One can show that if (3) holds then the probabilities $p_{2n}^{(-)}(x)$ decay exponentially. The sum $\varphi(x,\Theta) = \varphi^{(+)}(x,\Theta) + \varphi^{(-)}(x,\Theta)$ is generating function for excursions, both negative and positive.

Consider the probability distribution $P_x^{(n)}$ concentrated on the set $\bigcup_{|k|\le n} T^k x$ where $P_x^{(n)}(k)$ is the probability of random walks starting from $x = x_0$ for which $x_n = T^k x_0$ or $b(n) = k$. For $k > 0$ such random walks have the following structure. First it makes some numbers $\nu_0$ of positive or negative excursions coming back to $x_0$. Then it jumps to $Tx_0$ and makes $\nu_1$ positive excursions. After this it jumps to $T^2 x_1$, makes there $\nu_2$ positive excursions and so on. We denote $\nu_0, \nu_1, \cdots, \nu_k$ the numbers of these excursions and $\xi_{i1}, \xi_{i2}, \cdots, \xi_{i\nu_i}$ are their lengths, $i = 0, 1, \cdots, k$ $\zeta_i = \xi_{i1} + \cdots + \xi_{i\nu_i}$. Then $P\{\nu_i = r\} = \left(\frac{p(T^i x)\Lambda(T^i x)}{\Lambda(T^i x)+1}\right)^r$, $r = 1, 2, \cdots$ and we can write

$$P_x^{(n)}(k) = \frac{1}{p(T^k x_0)} \cdot \prod_{i=1}^{k} p(T^i x) \cdot \frac{1}{1 - \frac{p(T^i x)\Lambda(T^i x)}{\Lambda(T^i x)+1}} \cdot \tilde{\mathcal{P}}\{\zeta_0 + \zeta_1 + \cdots + \zeta_k = n-k\} \tag{15}$$

The factor $\frac{1}{1-\frac{p(T^i x)\Lambda(T^i x)}{\Lambda(T^i x)+1}} = \sum_{r=0}^{\infty} \left(\frac{p(T^i x)\Lambda(T^i x)}{\Lambda(T^i x)+1}\right)^r$ is the sum over the values of $\nu_i$ of probabilities to have $r$ positive excursions. From (12)

$$\frac{p(T^i x)}{1 - \frac{p(T^i x)\Lambda(T^i x)}{\Lambda(T^i x)+1}} = \frac{\Lambda(T^i x)+1}{\frac{(1-p(T^i x))}{p(T^i x)}(\Lambda(T^i x)+1)} = \frac{\Lambda(T^i x)+1}{\Lambda(T^{i-1}x)+1}$$

Therefore

$$
\begin{aligned}
P_x^{(n)}(k) = \frac{1}{p(T^k x)} \frac{\Lambda(T^k x)+1}{\Lambda(x)+1} \tilde{\mathcal{P}}\{\zeta_0+\zeta_1+\cdots+\zeta_k = n-k\} = \\
\frac{r(T^k x)}{\Lambda(x)+1} \tilde{\mathcal{P}}\{\zeta_0+\zeta_1+\cdots+\zeta_k = n-k\}
\end{aligned}
\tag{16}
$$

Here $\tilde{\mathcal{P}}$ is the probability distribution corresponding to the distribution of the sum of $k+1$ independent random variables $\zeta_i$ for which $\tilde{\mathcal{P}}\{\zeta_i = 2m\} = \left(1 - \frac{p(T^i x)\Lambda(T^i x)}{\Lambda(T^i x)+1}\right) \mathcal{P}\{\zeta_i = 2m\}$.

Now we can formulate the main result of this paper.

**Main theorem.** *For a non-symmetric random walks along orbits of $T$ for which inequality (3), Assumption 1 hold and $m_1^{(+)}(x), m_2^{(+)}(x), m_3^{(+)}(x) \in L^1(M, \mathcal{M}, \mu)$ the distribution $P_x^{(n)}$ converges to $r(z)d\mu(z)$ in the following sense: for any bounded measurable $\varphi$*

$$
\lim_{n\to\infty} \int \varphi(z) dP_x^{(n)}(z) = \lim_{n\to\infty} \sum P_x^{(n)}(k)\varphi(T^k x) = \int_M \varphi(z) r(z) d\mu(z)
$$

*for $\mu$ - a.e. $x$.*

**Proof.** Using (16) we can write

$$
\sum_k P_x^{(n)}(k)\varphi(T^k x) = \frac{1}{\Lambda(x)+1} \sum r(T^k x)\varphi(T^k x) \tilde{\mathcal{P}}\{\zeta_0+\zeta_1+\cdots+\zeta_k = n-k\}
$$

In view of our assumptions the sequence of random variables $\zeta_0, \zeta_1, \cdots \zeta_k$ satisfies the local central limit theorem of probability theory. Therefore if $a(T^i x)$ is the expectation of $\zeta_i$ (with respect to $\tilde{\mathcal{P}}$), $\sigma(T^i x)$ is its variance we can write

$$
\tilde{\mathcal{P}}\{\zeta_0+\zeta_1+\cdots+\zeta_k = n-k\} \sim \frac{1}{\sqrt{2\pi\sigma k}} \cdot \exp\left\{ -\frac{1}{2} \frac{(n-k-\sum_{i=1}^{k} a_i(T^i x))^2}{\sum_{i=1}^{k} \sigma(T^i x)} \right\},
$$

$\sigma = \frac{1}{k}\sum_{i=1}^{k} \sigma(T^i x)$. The last formula shows that $\tilde{\mathcal{P}}$ is a slowly varying function of $n-k$. Therefore the average over $k$ of $r(T^k x)\varphi(T^k x)$ with respect to this probability distribution is equivalent to the usual average with respect to the uniform distribution which gives in the limit $\int r(z)\varphi(z)d\mu(z)$, QED.

**Theorem** *(Corollary from the Main Theorem) Let the probability space* $(M, \mathcal{M}, \mu)$ *be the d-dimensional torus* $Tor^d$ *with the Lebesgue measure* $\mu = Leb_d$ *and let* $T : Tor^d \to Tor^d$ *be a rotation by an irrational angle* $\omega$*, i.e.* $T : x \mapsto x + \omega$*. If* $p : M \to (0,1)$ *be a continuous function such that* $\int \ln(1-p(x))\ d\mu(x) \neq \int \ln p(x) d\mu(x)$*, then the distribution* $P_x^{(n)}$ *converges to* $r(z)d\mu(z)$ *in the following sense*

$$\lim_{n\to\infty} \int \phi(z) dP_x^{(n)}(z) = \lim_{n\to\infty} \sum P_x^{(n)}(k)\phi(T^k x) = \int \phi(z) r(z)\ d\mu(x)$$

*for all* $x \in M$*.*

This theorem was proved in [S1] for any *diophantine* $\omega$.

*Proof* Assume for definiteness that $\int \ln(1-p(x))\ d\mu(x) < \int \ln p(x) d\mu(x)$.

**Proposition** $\Lambda(x) = \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{1-p(T^i x)}{p(T^i x)}$ *is a continuous function.*

*Proof of the Proposition.* Put $\eta(x) = \ln(1-p(x))/p(x)$ and $-C = \int \eta(x)\ d\mu(x) < 0$. Rewrite $\Lambda(x) = \sum_{n=1}^{\infty} \exp(\sum_{k=1}^{n} \eta(T^k x))$. Show that for any $\epsilon > 0$ there exists a constant $n_0 = n_0(\epsilon)$ such that for any $x \in M$ and $n > n_0$ the following holds:

$$\sum_{k=1}^{n} \eta(T^k x) < -(C-\epsilon)n < -Cn/2. \tag{17}$$

Since, $\eta(x)$ is a continuous function for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that $|\eta(x) - \eta(y)| < \epsilon/2$ for any $|x - y| < d\delta$. Fix a $\delta$-grid $\{x_j\}_{j=1}^{N}$ on $M = Tor^d$. By ergodic theorem there exists a universal constant $n_0$ such that for any $j = 1, \ldots, N$ and any $n > n_0$

$$\left| \frac{1}{n} \sum_{i=1}^{n} \eta(T^i x_j) - \int \eta(y)\ d\mu(y) \right| < \epsilon/2.$$

Then for any point $x \in M$ there exists a point of the grid $x_j$ such that $|x - x_j| \leq \delta$. Thus,

$$\left| \frac{1}{n} \sum_{i=1}^{n} \eta(T^i x) - \int \eta(y)\ d\mu(y) \right| <$$

$$\left| \frac{1}{n} \left( \sum_{i=1}^{n} (\eta(T^i x) - \eta(T^i x_j)) \right) \right| + \left| \frac{1}{n} \sum_{i=1}^{n} \eta(T^i x_j) - \int \eta(y)\ d\mu(y) \right| < \epsilon.$$

Formula (17) implies continuity of $\Lambda(x)$, because $a_n(x) = \exp(\sum_{k=1}^{n} \eta(T^k x)$ exponentially small and the sum over $n > n_0$ is uniformly small. This completes the proof of the Proposition. QED.

Let us rewrite

$$m_1^{(+)}(x) = \frac{\partial \phi^+(x, \theta)}{\partial \theta}|_{\theta=1}$$

in the form

$$m_1^{(+)}(x) = h(x) + h(x)\sum_{k=1}^{\infty}\prod_{j=0}^{k}\frac{\Lambda(T^{j+1}x)}{\Lambda(T^j x)+1}.$$

$\Lambda(x)$ is continuous and, therefore, bounded. This implies that $m^{(+)}(x)$ is a bounded function of $x$. One can check that

$$m_2^{(+)}(x) = \frac{\partial^2\phi^+(x,\theta)}{\partial\theta^2}|_{\theta=1}$$

can be presented in the follwing form: Put $\Phi(x) = 1 + \sum_{k=1}^{\infty}\prod_{j=0}^{k}\frac{\Lambda(T^{j+1}x)}{\Lambda(T^j x)+1}$. Then

$$m_2^{(+)}(x) = \left(h(x) + 2h(x)h(Tx)\frac{\Lambda(Tx)+1}{p(Tx)(\Lambda(x)+1)}\Phi(Tx)\right)\Phi(x).$$

It is easy to see that $\Phi(x)$ is continuous and bounded. Therefore, $m_2^{(+)}(x)$ is bounded. In a similar fashion one can check the third moment. Thus, we can apply the Main theorem. This completes the proof of the theorem. QED.

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Reference.

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