

Stretched exponential estimate on growth of the number of
periodic points for prevalent diffeomorphisms

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Abstract

For diffeomorphisms of smooth compact finite-dimensional manifolds, we consider the problem of how fast the number of periodic points with period n grows as a function of n . In many familiar cases (*e.g.*, Anosov systems) the growth is exponential, but arbitrarily fast growth is possible; in fact, the first author has shown that arbitrarily fast growth is topologically (Baire) generic for C^2 or smoother diffeomorphisms. In the present work we show that, by contrast, for a measure-theoretic notion of genericity we call “prevalence”, the growth is not much faster than exponential. Specifically, we show that for each $\rho, \delta > 0$, there is a prevalent set of $C^{1+\rho}$ (or smoother) diffeomorphisms for which the number of period n points is bounded above by $\exp(Cn^{1+\delta})$ for some C independent of n . We also obtain a related bound on the decay of hyperbolicity of the periodic points as a function of n , and obtain the same results for 1-dimensional endomorphisms. The contrast between topologically generic and measure-theoretically generic behavior for the growth of the number of periodic points and the decay of their hyperbolicity shows this to be a subtle and complex phenomenon, reminiscent of KAM theory. Here in Part I we state our results and describe the methods we use. We complete most of the proof in the 1-dimensional C^2 -smooth case and outline the remaining steps, deferred to Part II, that are needed to establish the general case.

The novel feature of the approach we develop in this paper is an introduction of Newton Interpolation Polynomials as a tool of perturbing trajectories of iterated maps.

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Chapter 1

A Problem of the Growth of the Number of Periodic Points and Decay of Hyperbolicity for Generic Diffeomorphisms.

1.1 Introduction

Let $\text{Diff}^r(M)$ be the space of C^r diffeomorphisms of a finite-dimensional smooth compact manifold M with the uniform C^r -topology, where $\dim M \geq 2$, and let $f \in \text{Diff}^r(M)$. Consider the number of periodic points of period n

$$P_n(f) = \#\{x \in M : x = f^n(x)\}. \quad (1.1)$$

The main question of this paper is:

Question 1.1.1. *How quickly can $P_n(f)$ grow with n for a “generic” C^r diffeomorphism f ?*

We put the word “generic” in brackets because as the reader will see the answer depends on notion of genericity.

For technical reasons one sometimes counts only *isolated* points of period n ; let

$$P_n^i(f) = \#\{x \in M : x = f^n(x) \text{ and } y \neq f^n(y) \text{ for } y \neq x \text{ in some neighborhood of } x\}. \quad (1.2)$$

We call a diffeomorphism $f \in \text{Diff}^r(M)$ an *Artin-Mazur diffeomorphism* (or simply *A-M diffeomorphism*) if the number of isolated periodic orbits of f grows at most exponentially fast, *i.e.* for some number $C > 0$ we have

$$P_n^i(f) \leq \exp(Cn) \quad \text{for all } n \in \mathbb{Z}_+. \quad (1.3)$$

Artin & Mazur [AM] proved the following result.

Theorem 1.1.2. *For $0 \leq r \leq \infty$, A-M diffeomorphisms are dense in $\text{Diff}^r(M)$ with the uniform C^r -topology.*

We say that a point $x \in M$ of period n for f is hyperbolic if $df^n(x)$, the linearization of f^n at x , has no eigenvalues with modulus 1. (Notice that a hyperbolic solution to $f^n(x) = x$ must also be isolated.) We call $f \in \text{Diff}^r(M)$ a strongly Artin-Mazur diffeomorphism if for some number $C > 0$,

$$P_n(f) \leq \exp(Cn) \quad \text{for all } n \in \mathbb{Z}_+, \quad (1.4)$$

and all periodic points of f are hyperbolic (whence $P_n(f) = P_n^i(f)$). In [K1] an elementary proof of the following extension of the Artin-Mazur result is given.

Theorem 1.1.3. *For $0 \leq r < \infty$, strongly A-M diffeomorphisms are dense in $\text{Diff}^r(M)$ with the uniform C^r -topology.*

According to the standard terminology a set in $\text{Diff}^r(M)$ is called residual if it contains a countable intersection of open dense sets and a property is called (*Baire*) *generic* if diffeomorphisms with that property form a residual set. It turns out the A-M property is not generic, as is shown in [K2]. Moreover:

Theorem 1.1.4. *[K2] For any $2 \leq r < \infty$ there is an open set $\mathcal{N} \subset \text{Diff}^r(M)$ such that for any given sequence $a = \{a_n\}_{n \in \mathbb{Z}_+}$ there is a Baire generic set \mathcal{R}_a in \mathcal{N} depending on the sequence a_n with the property if $f \in \mathcal{R}_a$, then for infinitely many $n_k \in \mathbb{Z}_+$ we have $P_{n_k}^i(f) > a_{n_k}$.*

Of course since $P_n(f) \geq P_n^i(f)$, the same statement can be made about $P_n(f)$. But in fact it is shown in [K2] that $P_n(f)$ is infinite for n sufficiently large, due to a continuum of periodic points, for at least a dense set of $f \in \mathcal{N}$.

The proof of this Theorem is based on a result of Gonchenko-Shilnikov-Turaev [GST1]. Two slightly different detailed proofs of their result are given in [K2] and [GST2]. The proof in [K2] relies on a strategy outlined in [GST1].

However, it seems unnatural that if you pick a diffeomorphism at random then it may have an arbitrarily fast growth of number of periodic points. Moreover, Baire generic sets in Euclidean spaces can have zero Lebesgue measure. Phenomena that are Baire generic, but have a small probability are well-known in dynamical systems, KAM theory, number theory, etc. (see [O], [HSY], [K3] for various examples). This partially motivates the problem posed by Arnold [A]:

Problem 1.1.5. *Prove that “with probability one” $f \in \text{Diff}^r(M)$ is an A-M diffeomorphism.*

Arnold suggested the following interpretation of “with probability one”: *for a (Baire) generic finite parameter family of diffeomorphisms $\{f_\varepsilon\}$, for Lebesgue almost every ε we have that f_ε is A-M* (compare with [K3]). As Theorem 1.3 shows, a result on the genericity of the set of A-M diffeomorphisms based on (Baire) topology is likely to be extremely subtle, if possible at all ¹. We use instead a notion of “probability one” based on prevalence [HSY, K3], which is independent of Baire genericity. We also are able to state the result in the form Arnold suggested for generic families using this measure-theoretic notion of genericity.

For a rough understanding of prevalence, consider a Borel measure μ on a Banach space V . We say that a property holds “ μ -almost surely for perturbations” if it holds on a Borel set $P \subset V$ such that *for all $v \in V$ we have $v + w \in P$ for almost every w with respect to μ* . Notice that if $V = \mathbb{R}^k$ and μ is Lebesgue measure, then “almost surely with respect to perturbations by μ ” is equivalent to “Lebesgue almost everywhere”. Moreover, the Fubini/Tonelli Theorem implies that if μ is any Borel probability measure on \mathbb{R}^k , then a property that holds almost surely with respect to perturbations by μ must also hold Lebesgue almost everywhere. Based on this observation, we call a property on a Banach space “prevalent” if it holds almost surely with respect to perturbations by μ for some Borel probability measure μ on V , which for technical reasons we require to have compact support. In order to apply this notion to the Banach manifold $\text{Diff}^r(M)$, we must describe how we make perturbations in this space, which we will do in the next Section.

Our first main result is a partial solution to Arnold’s problem. It says that *for a prevalent diffeomorphism $f \in \text{Diff}^r(M)$, with $1 < r \leq \infty$, and all $\delta > 0$ there exists*

¹For example, using technique from [GST2] and [K2] one can prove that for a (Baire) generic finite-parameter family $\{f_\varepsilon\}$ and a (Baire) generic parameter value ε the corresponding diffeomorphism f_ε is not A-M. Unfortunately, how to estimate the measure of non-A-M diffeomorphisms from below is a so far unreachable question

$C = C(\delta) > 0$ such that for all $n \in \mathbb{Z}_+$,

$$P_n(f) \leq \exp(Cn^{1+\delta}). \quad (1.5)$$

The results of this paper have been announced in [KH].

The Kupka-Smale Theorem (see e.g.[PM]) states that for a generic diffeomorphism all periodic points are hyperbolic and all associated stable and unstable manifolds intersect one another transversally. [K3] shows that the Kupka-Smale Theorem also holds on a prevalent set. So, the Kupka-Smale Theorem, in particular, says that a Baire generic (resp. prevalent) diffeomorphism has only hyperbolic periodic points, but *how hyperbolic are the periodic points, as function of their period, for a Baire generic (resp. prevalent) diffeomorphism f ?* This is the second main problem we deal with in this paper.

Recall that a linear operator $L : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is *hyperbolic* if it has no eigenvalues on the unit circle $\{|z| = 1\} \subset \mathbb{C}$. Denote by $|\cdot|$ the Euclidean norm in \mathbb{C}^N . Then we define the *hyperbolicity* of a linear operator L by

$$\gamma(L) = \inf_{\phi \in [0,1)} \inf_{|v|=1} |Lv - \exp(2\pi i\phi)v|. \quad (1.6)$$

We also say that L is γ -hyperbolic if $\gamma(L) \geq \gamma$. In particular, if L is γ -hyperbolic, then its eigenvalues $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ are at least γ -distant from the unit circle, i.e. $\min_j \|\lambda_j - 1\| \geq \gamma$. The *hyperbolicity* of a periodic point $x = f^n(x)$ of period n , denoted by $\gamma_n(x, f)$, equals the hyperbolicity of the linearization $df^n(x)$ of f^n at points x , i.e. $\gamma_n(x, f) = \gamma(df^n(x))$. Similarly the number of periodic points $P_n(f)$ of period n is defined, one can define

$$\gamma_n(f) = \min_{\{x: x=f^n(x)\}} \gamma_n(x, f). \quad (1.7)$$

The idea of Gromov [G] and Yomdin [Y] of measuring hyperbolicity is that a γ -hyperbolic point of period n of a C^2 diffeomorphism f has an $M_2^{-2n}\gamma$ -neighborhood (where $M_2 = \|f\|_{C^2}$) free from periodic points of the same period². In Appendix A we prove the following result.

Proposition 1.1.6. *Let M be a compact manifold of dimension N , let $f : M \rightarrow M$ be a $C^{1+\rho}$ diffeomorphism, where $0 < \rho \leq 1$, that has only hyperbolic periodic points,*

²In [Y] hyperbolicity is introduced as the minimal distance of eigenvalues to the unit circle. This way of defining hyperbolicity does not guarantee the existence of a $M_2^{-2n}\gamma$ -neighborhood free from periodic points of the same period; see Appendix A

and let $M_{1+\rho} = \max\{\|f\|_{C^{1+\rho}}, 2^{1/\rho}\}$. Then there is a constant $C = C(M) > 0$ such that for each $n \in \mathbb{Z}_+$ we have

$$P_n(f) \leq C (M_{1+\rho})^{nN(1+\rho)/\rho} \gamma_n(f)^{-N/\rho}. \quad (1.8)$$

Proposition 1.1.6 implies that a lower estimate on a decay of hyperbolicity $\gamma_n(f)$ gives an upper estimate on growth of number of periodic points $P_n(f)$. Therefore, a natural question is:

Question 1.1.7. *How quickly can $\gamma_n(f)$ decay with n for a “generic” C^r diffeomorphism f ?*

For a Baire generic $f \in \text{Diff}^r(M)$, the existence of lower bound on a rate of decay of $\gamma_n(f)$ would imply the existence of an upper bound on a rate of growth of the number of periodic points $P_n(f)$, whereas no such bound exists by Theorem 1.1.4. Thus again we consider genericity in the measure-theoretic sense of prevalence. Our second main result, which in view of Proposition 1.1.6 implies the first main result, is that *for a prevalent diffeomorphism $f \in \text{Diff}^r(M)$, with $1 < r \leq \infty$, and for any $\delta > 0$ there exists $C = C(\delta) > 0$ such that*

$$\gamma_n(f) \geq \exp(-Cn^{1+\delta}). \quad (1.9)$$

Now we shall discuss in more detail our definition of prevalence (“probability one”) in the space of diffeomorphisms $\text{Diff}^r(M)$.

1.2 Prevalence in the space of diffeomorphisms $\text{Diff}^r(M)$

The space of C^r diffeomorphisms $\text{Diff}^r(M)$ of a compact manifold M is a Banach manifold. Locally we can identify it with a Banach space, which gives it a local linear structure in the sense that we can perturb a diffeomorphism by “adding” small elements of the Banach space. As we described in the previous Section, the notion of prevalence requires us to make additive perturbations with respect to a probability measure that is independent of the place where we make the perturbation. Thus although there is not a unique way to put a linear structure on $\text{Diff}^r(M)$, it is important to make a choice that is consistent throughout the Banach manifold.

The way we make perturbations on $\text{Diff}^r(M)$ by small elements of a Banach space is as follows. First we embed M into the interior of the closed unit ball $B^N \subset \mathbb{R}^N$,

which we can do for N sufficiently large by the Whitney Embedding Theorem [W]. We emphasize that our results hold for *every* possible choice of an embedding of M into \mathbb{R}^N . We then consider a closed neighborhood $U \subset B^N$ of M and Banach space $C^r(U, \mathbb{R}^N)$ of C^r functions from U to \mathbb{R}^N . Next we *extend* every element $f \in \text{Diff}^r(M)$ to an element $F \in C^r(U, \mathbb{R}^N)$ that is strongly contracting in *all* the directions transverse to M ³. Again the particular choice of how we make this extension is not important to our results; in Appendix C we describe how to extend a diffeomorphism and what conditions we need to ensure that the results of Sacker [Sac] and Fenichel [F] apply as follows. Since F has M as an invariant manifold, if we add to F a small perturbation in $g \in C^r(U, \mathbb{R}^N)$, the perturbed map $F + g$ has an invariant manifold in U that is C^r -close to M . Then $F + g$ restricted to its invariant manifold corresponds in a natural way to an element of $\text{Diff}^r(M)$, which we consider to be the perturbation of $f \in \text{Diff}^r(M)$ by $g \in C^r(U, \mathbb{R}^N)$. The details of this construction are described in Appendix C.

In this way we reduce the problem to the study of maps in $\text{Diff}^r(U)$, the open subset of $C^r(U, \mathbb{R}^N)$ consisting of those elements that are diffeomorphisms from U to some subset of its interior. The construction we described in the previous paragraph ensures that the number of periodic points $P_n(f)$ and their hyperbolicity $\gamma_n(f)$ for elements of $\text{Diff}^r(M)$ are the same for the corresponding elements of $\text{Diff}^r(U)$, so the bounds that we prove on these quantities for almost every perturbation of any element of $\text{Diff}^r(U)$ hold as well for almost every perturbation of any element of $\text{Diff}^r(M)$. Another justification for considering diffeomorphisms in Euclidean space is that the problem of exponential/superexponential growth of the number of periodic points $P_n(f)$ for a prevalent $f \in \text{Diff}^r(M)$ is a *local problem* on M and is not affected by a global shape of M .

The results stated in the next Section apply to any compact domain $U \subset \mathbb{R}^N$, but for simplicity we state them for the closed unit ball B^N . In the previous Section, we said that a property is *prevalent* on a Banach space such as $C^r(B^N)$ if it holds on a Borel subset S for which there exists a Borel probability measure μ on $C^r(B^N)$ with compact support such that for all $f \in C^r(B^N)$ we have $f + g \in S$ for almost every g with respect to μ . The complement of a prevalent set is said to be *shy*. We then say that a property is prevalent on an open subset of $C^r(B^N)$ such as $\text{Diff}^r(B^N)$ if the exceptions to the property in $\text{Diff}^r(B^N)$ form a shy subset of $C^r(B^N)$.

In this paper the perturbation measure μ that we use is supported within the analytic functions in $C^r(B^N)$. In this sense we foliate $\text{Diff}^r(B^N)$ by analytic leaves

³The existence of such extension is not obvious, as pointed out by C. Carminati.

that are compact and overlapping. The main result then says that *for every analytic leaf* $L \subset \text{Diff}(B^N)$ and every $\delta > 0$, for almost every diffeomorphism $f \in L$ in the leaf L both (1.5) and (1.9) are satisfied. Now we define an analytic leaf as a ‘‘Hilbert Brick’’ in the space of analytic functions and a natural Lebesgue product probability measure μ on it.

1.3 Formulation of the main result in the multidimensional case

Fix a coordinate system $x = (x_1, \dots, x_N) \in \mathbb{R}^N \supset B^N$ and the scalar product $\langle x, y \rangle = \sum_i x_i y_i$. Let $\alpha = (\alpha_1, \dots, \alpha_N)$ be a multiindex from \mathbb{Z}_+^N , and let $|\alpha| = \sum_i \alpha_i$. For a point $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ we write $x^\alpha = \prod_{i=1}^N x_i^{\alpha_i}$. Associate to a real analytic function $\phi : B^N \rightarrow \mathbb{R}^N$ the set of coefficients of its expansion:

$$\phi_{\vec{\varepsilon}}(x) = \sum_{\alpha \in \mathbb{Z}_+^N} \vec{\varepsilon}_\alpha x^\alpha. \quad (1.10)$$

Denote by $W_{k,N}$ the space of N -component homogeneous vector-polynomials of degree k in N variables and by $\nu(k, N) = \dim W_{k,N}$ the dimension of $W_{k,N}$. According to the notation of the expansion (1.10), denote coordinates in $W_{k,N}$ by

$$\vec{\varepsilon}_k = (\{\vec{\varepsilon}_\alpha\}_{|\alpha|=k}) \in W_{k,N}. \quad (1.11)$$

In $W_{k,N}$ we use a scalar product that is invariant with respect to orthogonal transformation of $\mathbb{R}^N \supset B^N$ (see Appendix B), defined as follows:

$$\langle \vec{\varepsilon}_k, \vec{\nu}_k \rangle_k = \sum_{|\alpha|=k} \binom{k}{\alpha}^{-1} \langle \vec{\varepsilon}_\alpha, \vec{\nu}_\alpha \rangle, \quad \|\vec{\varepsilon}_k\|_k = (\langle \vec{\varepsilon}_k, \vec{\varepsilon}_k \rangle_k)^{1/2}. \quad (1.12)$$

Denote by

$$B_k^N(r) = \{\vec{\varepsilon}_k \in W_{k,N} : \|\vec{\varepsilon}_k\|_k \leq r\} \quad (1.13)$$

the closed r -ball in $W_{k,N}$ centered at the origin. Let $Leb_{k,N}$ be Lebesgue measure on $W_{k,N}$ induced by the scalar product (1.12) and normalized by a constant so that the volume of the unit ball is one: $Leb_{k,N}(B_k^N(1)) = 1$.

Fix a nonincreasing sequence of positive numbers $\vec{r} = (\{r_k\}_{k=0}^\infty)$ such that $r_k \rightarrow 0$ as $k \rightarrow \infty$ and define a Hilbert Brick of size \vec{r}

$$\begin{aligned} HB^N(\vec{r}) &= \{\vec{\varepsilon} = \{\vec{\varepsilon}_\alpha\}_{\alpha \in \mathbb{Z}_+^N} : \text{for all } k \in \mathbb{Z}_+, \|\vec{\varepsilon}_k\|_k \leq r_k\} \\ &= B_0^N(r_0) \times B_1^N(r_1) \times \cdots \times B_k^N(r_k) \times \cdots \\ &\subset W_{0,N} \times W_{1,N} \times \cdots \times W_{k,N} \times \cdots \end{aligned} \quad (1.14)$$

Define a Lebesgue product probability measure $\mu_{\vec{r}}^N$ associated to the Hilbert Brick $HB^N(\vec{r})$ of size \vec{r} by normalizing for each $k \in \mathbb{Z}_+$ the corresponding Lebesgue measure $Leb_{k,N}$ on $W_{k,N}$ to the Lebesgue probability measure on the r_k -ball $B_k^N(r_k)$:

$$\mu_{k,r}^N = r^{-\nu(k,N)} Leb_{k,N} \quad \text{and} \quad \mu_{\vec{r}}^N = \times_{k=0}^\infty \mu_{k,r_k}^N. \quad (1.15)$$

Definition 1.3.1. Let $f \in \text{Diff}^r(B^N)$ be a C^r diffeomorphism of B^N into its interior. We call $HB^N(\vec{r})$ a Hilbert Brick of an admissible size $\vec{r} = (\{r_k\}_{k=0}^\infty)$ with respect to f if

A) for each $\vec{\varepsilon} \in HB^N(\vec{r})$, the corresponding function $\phi_{\vec{\varepsilon}}(x) = \sum_{\alpha \in \mathbb{Z}_+^N} \vec{\varepsilon}_\alpha x^\alpha$ is analytic on B^N ;

B) for each $\vec{\varepsilon} \in HB^N(\vec{r})$, the corresponding map $f_{\vec{\varepsilon}}(x) = f(x) + \phi_{\vec{\varepsilon}}(x)$ is a diffeomorphism from B^N into its interior, i.e. $\{f_{\vec{\varepsilon}}\}_{\vec{\varepsilon} \in HB^N(\vec{r})} \subset \text{Diff}^r(B^N)$;

C) for all $\delta > 0$ and all $C > 0$, the sequence $r_k \exp(Ck^{1+\delta}) \rightarrow \infty$ as $k \rightarrow \infty$.

Remark 1.3.2. The first and second conditions ensure that the family $\{f_{\vec{\varepsilon}}\}_{\vec{\varepsilon} \in HB^N(\vec{r})}$ lie inside an analytic leaf within the class of diffeomorphisms $\text{Diff}^r(B^N)$. The third condition provides us enough freedom to perturb. It is important for our method to have infinitely many parameters to perturb. If r_k 's were decaying too fast to zero it would make our family of perturbations essentially finite-dimensional.

An example of an admissible sequence $\vec{r} = (\{r_k\}_{k=0}^\infty)$ is $r_k = \tau/k!$, where τ depends on f and is chosen sufficiently small to ensure that condition (B) holds. Notice that the diameter of $HB^N(\vec{r})$ is then propotional to τ , so that τ can be chosen as some multiple of the distance from f to the boundary of $\text{Diff}^r(B^N)$.

Main Theorem. For any $0 < \rho \leq \infty$ (or even $1 + \rho = \omega$) and any $C^{1+\rho}$ diffeomorphism $f \in \text{Diff}^{1+\rho}(B^N)$, consider a Hilbert Brick $HB^N(\vec{r})$ of an admissible size \vec{r} with respect to f and the family of analytic perturbations of f

$$\{f_{\vec{\varepsilon}}(x) = f(x) + \phi_{\vec{\varepsilon}}(x)\}_{\vec{\varepsilon} \in HB^N(\vec{r})} \quad (1.16)$$

with the Lebesgue product probability measure $\mu_{\vec{r}}^N$ associated to $HB^N(\vec{r})$. Then for every $\delta > 0$ and $\mu_{\vec{r}}^N$ -a.e. $\vec{\varepsilon}$ there is $C = C(\vec{\varepsilon}, \delta) > 0$ such that for all $n \in \mathbb{Z}_+$

$$\gamma_n(f_{\vec{\varepsilon}}) > \exp(-Cn^{1+\delta}), \quad P_n(f_{\vec{\varepsilon}}) < \exp(Cn^{1+\delta}). \quad (1.17)$$

Remark 1.3.3. *The fact that the measure $\mu_{\vec{r}}^N$ depends on f does not conform to our definition of prevalence. However, we can decompose $\text{Diff}^r(B^N)$ into a nested countable union of sets \mathcal{S}_j that are each a positive distance from the boundary of $\text{Diff}^r(B^N)$ and for each $j \in \mathbb{Z}^+$ choose an admissible sequence \vec{r}_j that is valid for all $f \in \mathcal{S}_j$. Since a countable intersection of prevalent subsets of a Banach space is prevalent [HSY], the Main Theorem implies the results stated in terms of prevalence in the introduction.*

Remark 1.3.4. *Recently the first author along with A. Gorodetski [GK] applied the technique developed here and obtained partial solution of Palis' conjecture about finiteness of the number of coexisting sinks for surface diffeomorphisms.*

In Appendix C we deduce from the Main Theorem the following result.

Theorem 1.3.5. *Let $\{f_\varepsilon\}_{\varepsilon \in B^m} \subset \text{Diff}^{1+\rho}(M)$ be a generic m -parameter family of $C^{1+\rho}$ diffeomorphisms of a compact manifold M for some $\rho > 0$. Then for every $\delta > 0$ and a.e. $\varepsilon \in B^m$ there is a constant $C = C(\varepsilon, \delta)$ such that (1.17) is satisfied for every $n \in \mathbb{Z}_+$.*

In Appendix C we also give a precise meaning to the term *generic*. Now we formulate the most general result we shall prove.

Definition 1.3.6. *Let $\gamma \geq 0$ and $f \in \text{Diff}^{1+\rho}(B^N)$ be a $C^{1+\rho}$ diffeomorphism for some $\rho > 0$. A point $x \in B^N$ is called (n, γ) -periodic if $|f^n(x) - x| \leq \gamma$ and (n, γ) -hyperbolic if $\gamma_n(x, f) = \gamma(df^n(x)) \geq \gamma$.*

(Notice that a point can be (n, γ) -hyperbolic regardless of its periodicity, but this property is of interest primarily for (n, γ) -periodic points.) For positive C and δ let $\gamma_n(C, \delta) = \exp(-Cn^{1+\delta})$.

Theorem 1.3.7. *Given the hypotheses of the Main Theorem, for every $\delta > 0$ and for $\mu_{\vec{r}}^N$ -a.e. \vec{r} there is $C = C(\vec{r}, \delta) > 0$ such that for all $n \in \mathbb{Z}_+$, every $(n, \gamma_n^{1/\rho}(C, \delta))$ -periodic point $x \in B^N$ is $(n, \gamma_n(C, \delta))$ -hyperbolic. (Here we assume $0 < \rho \leq 1$; in a space $\text{Diff}^{1+\rho}(B^N)$ with $\rho > 1$, the statement holds with ρ replaced by 1.)*

This result together with Proposition 1.1.6 implies the Main Theorem, because any periodic point of period n is (n, γ) -periodic for any $\gamma > 0$.

Remark 1.3.8. *In the statement of the Main Theorem and Theorem 1.3.7 the unit ball B^N can be replaced by an bounded open set $U \subset \mathbb{R}^N$. After scaling U can be considered as a subset of the unit ball B^N .*

One can define a distance on a compact manifold M and almost periodic points of diffeomorphisms of M . Then one can cover $M = \cup_i U_i$ by coordinate charts and define hyperbolicity for almost periodic points using these charts $\{U_i\}_i$ (see [Y] for details). This gives a precise meaning to the following result.

Theorem 1.3.9. *Let $\{f_\varepsilon\}_{\varepsilon \in B^m} \subset \text{Diff}^{1+\rho}(M)$ be a generic m -parameter family of diffeomorphisms of a compact manifold M for some $\rho > 0$. Then for every $\delta > 0$ and almost every $\varepsilon \in B^m$ there is a constant $C = C(\varepsilon, \delta)$ such that every $(n, \gamma_n^{1/\rho}(C, \delta))$ -periodic point x in B^N is $(n, \gamma_n(C, \delta))$ -hyperbolic. (Here again we assume $0 < \rho \leq 1$, replacing ρ with 1 in the conclusion if $\rho > 1$.)*

The meaning of the term generic is the same as in Theorem 1.3.5 and is discussed in Appendix C.

1.4 Formulation of the main result in the 1-dimensional case

The proof of the main multidimensional result (Theorem 1.3.7) is quite long and complicated. In order to describe the general approach we develop in this paper we apply our method to the 1-dimensional maps which represent a nontrivial simplified model for the multidimensional problem. The statement of the main result for the 1-dimensional maps has another important feature: it clarifies the statement of the main multidimensional result.

Fix the interval $I = [-1, 1]$. Associate to a real analytic function $\phi : I \rightarrow \mathbb{R}$ the set of coefficients of its expansion

$$\phi_\varepsilon(x) = \sum_{k=0}^{\infty} \varepsilon_k x^k. \tag{1.18}$$

For a nonincreasing sequence of positive numbers $\vec{r} = (\{r_k\}_{k=0}^{\infty})$ such that $r_k \rightarrow 0$ as $k \rightarrow \infty$ following the multidimensional notations we define a Hilbert Brick of size \vec{r}

$$HB^1(\vec{r}) = \{\varepsilon = \{\varepsilon_k\}_{k=0}^{\infty} : \text{for all } k \in \mathbb{Z}_+, |\varepsilon_k| \leq r_k\} \tag{1.19}$$

and the product probability measure $\mu_{\vec{r}}^1$ associated to the Hilbert Brick $HB^1(\vec{r})$ of size \vec{r} which considers each ε_k as a random variable independent from the other ε_k 's and uniformly distributed on $[-r_k, r_k]$.

Main 1–dimensional Theorem. For any $0 < \rho \leq \infty$ (or even $1 + \rho = \omega$) and any $C^{1+\rho}$ map $f : I \rightarrow I$ of the interval $I = [-1, 1]$ consider a Hilbert Brick $HB^1(\vec{\mathbf{r}})$ of an admissible size $\vec{\mathbf{r}}$ with respect to f and the family of analytic perturbations of f

$$\{f_\varepsilon(x) = f(x) + \phi_\varepsilon(x)\}_{\varepsilon \in HB^1(\vec{\mathbf{r}})} \quad (1.20)$$

with the Lebesgue product probability measure $\mu_{\vec{\mathbf{r}}}^1$ associated to $HB^1(\vec{\mathbf{r}})$. Then for every $\delta > 0$ and $\mu_{\vec{\mathbf{r}}}^1$ -a.e. ε there is $C = C(\varepsilon, \delta) > 0$ such that for all $n \in \mathbb{Z}_+$

$$\gamma_n(f_\varepsilon) > \exp(-Cn^{1+\delta}), \quad P_n(f_\varepsilon) < \exp(Cn^{1+\delta}). \quad (1.21)$$

Moreover, for $\mu_{\vec{\mathbf{r}}}^1$ -a.e. ε , every $(n, \exp(-Cn^{1+\delta}))$ -periodic point is $(n, \exp(-Cn^{1+\delta}))$ -hyperbolic.

In [MMS] Martens-de Melo-Van Strien prove a stronger statement for C^2 maps. They show that for any C^2 map f of an interval without flat critical points there are $\gamma > 0$ and $n_0 \in \mathbb{Z}_+$ such that for any $n > n_0$ we have $|\gamma_n(f)| > 1 + \gamma$. This also implies that the number of periodic points is bounded by an exponential function of the period. The notion of a flat critical point used in [MMS] is a nonstandard one from a point of view of singularity theory, in particular, if 0 is a critical point, then distance of $f(x)$ to $f(0)$ does not have to decay to 0 as $x \rightarrow 0$ faster than any degree of x .

In [KK] an example of a C^r -unimodal map with a critical point having tangency of order $2r + 2$ and an arbitrary fast rate of growth of the number of periodic points is presented.

Let us point out again that the main purpose of discussing the 1-dimensional case in details is to highlight ideas and explain the general method without overloading presentation by technical details. The general N -dimensional case is highly involved and excessive amount of technical details make understanding of general ideas of the method not easily accessible.

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⁴This paper is based on the first author's Ph.D. thesis.

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Chapter 2

Strategy of the Proof

Here we describe the strategy of the proof of the Main Result (Theorem 1.3.7). The general idea is to fix $C > 0$ and prove an upper bound on the measure of the set of “bad” parameter values $\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}})$ for which the conclusion of the Theorem does not hold. The upper bound we obtain will approach zero as $C \rightarrow \infty$, from which it follows immediately that the set of $\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}})$ that are “bad” for all $C > 0$ has measure zero. For a given $C > 0$, we bound the measure of “bad” parameter values inductively as follows.

Stage 1. We delete all parameter values $\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}})$ for which the corresponding diffeomorphism $f_{\vec{\varepsilon}}$ has an almost fixed point which is not sufficiently hyperbolic and bound the measure of the deleted set.

Stage 2. We consider only parameter values for which each almost fixed point is sufficiently hyperbolic. Then we delete all parameter values $\vec{\varepsilon}$ for which $f_{\vec{\varepsilon}}$ has an almost periodic point of period 2 which is not sufficiently hyperbolic and bound the measure of that set.

Stage n . We consider only parameter values for which each almost periodic point of period at most $n - 1$ is sufficiently hyperbolic (we shall call this *the Inductive Hypothesis*). Then we delete all parameter values $\vec{\varepsilon}$ for which $f_{\vec{\varepsilon}}$ has an almost periodic point of period n which is not sufficiently hyperbolic and bound the measure of that set.

The main difficulty in the proof is then to prove a bound on the measure of “bad” parameter values at stage n such that the bounds are summable over n and that the sum approaches zero as $C \rightarrow \infty$. Let us formalize the problem. Fix positive ρ , δ , and C , and recall that $\gamma_n(C, \delta) = \exp(-Cn^{1+\delta})$ for $n \in \mathbb{Z}_+$. Assume $\rho \leq 1$; if not, change its value to 1.

Definition 2.0.1. A diffeomorphism $f \in \text{Diff}^{1+\rho}(B^N)$ satisfies the *Inductive Hypothesis of order n* with constants (C, δ, ρ) , denoted $f \in IH(n, C, \delta, \rho)$, if for all $k \leq n$, every $(k, \gamma_k^{1/\rho}(C, \delta))$ -periodic point is $(k, \gamma_k(C, \delta))$ -hyperbolic.

For $f \in \text{Diff}^{1+\rho}(M)$, consider the sequence of sets in the parameter space $HB^N(\vec{\mathbf{r}})$

$$\begin{aligned} B_n(C, \delta, \rho, \vec{\mathbf{r}}, f) &= \{\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}}) : \\ &f_{\vec{\varepsilon}} \in IH(n-1, C, \delta, \rho) \text{ but } f_{\vec{\varepsilon}} \notin IH(n, C, \delta, \rho)\} \end{aligned} \quad (2.1)$$

In other words, $B_n(C, \delta, \rho, \vec{\mathbf{r}}, f)$ is the set of “bad” parameter values $\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}})$ for which all almost periodic points of $f_{\vec{\varepsilon}}$ with period strictly less than n are sufficiently hyperbolic, but there is an almost periodic point of period n that is not sufficiently hyperbolic. Let

$$\begin{aligned} M_1 &= \sup_{\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}})} \max\{\|f_{\vec{\varepsilon}}\|_{C^1}, \|f_{\vec{\varepsilon}}^{-1}\|_{C^1}\}; \\ M_{1+\rho} &= \sup_{\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}})} \max\{\|f_{\vec{\varepsilon}}\|_{C^{1+\rho}}, M_1, 2^{1/\rho}\}. \end{aligned} \quad (2.2)$$

Our goal is to find an upper bound

$$\mu_{\vec{\mathbf{r}}}^N \{B_n(C, \delta, \rho, \vec{\mathbf{r}}, f)\} \leq \mu_n(C, \delta, \rho, \vec{\mathbf{r}}, M_{1+\rho}). \quad (2.3)$$

for the measure of the set of “bad” parameter values. Then the sum over n of (2.3) gives an upper bound

$$\mu_{\vec{\mathbf{r}}}^N \{\cup_{n=1}^{\infty} B_n(C, \delta, \rho, \vec{\mathbf{r}}, f)\} \leq \sum_{n=1}^{\infty} \mu_n(C, \delta, \rho, \vec{\mathbf{r}}, M_{1+\rho}) \quad (2.4)$$

on the measure of the set of all parameters $\vec{\varepsilon}$ for which $f_{\vec{\varepsilon}}$ has for at least one n an $(n, \gamma_n^{1/\rho}(C, \delta))$ -periodic point that is not $(n, \gamma_n(C, \delta))$ -hyperbolic. If this sum converges and

$$\sum_{n=1}^{\infty} \mu_n(C, \delta, \rho, \vec{\mathbf{r}}, M_{1+\rho}) = \mu(C, \delta, \rho, \vec{\mathbf{r}}, M_{1+\rho}) \rightarrow 0 \text{ as } C \rightarrow \infty \quad (2.5)$$

for every positive ρ, δ , and $M_{1+\rho}$, then Theorem 1.3.7 follows. In the remainder of this Chapter we describe the key construction we use to obtain a bound $\mu_n(C, \delta, \rho, \vec{\mathbf{r}}, M_{1+\rho})$ that meets condition (2.5).

2.1 Various perturbations of recurrent trajectories by Newton interpolation polynomials

The approach we take to estimate the measure of “bad” parameter values in the space of perturbations $HB^N(\vec{r})$ is to choose a coordinate system for this space and for a finite subset of the coordinates to estimate the amount that we must change a particular coordinate to make a “bad” parameter value “good”. Actually we will choose a coordinate system that depends on a particular point $x_0 \in B^N$, the idea being to use this coordinate system to estimate the measure of “bad” parameter values corresponding to initial conditions in some neighborhood of x_0 , then cover B^N with a finite number of such neighborhoods and sum the corresponding estimates. For a particular set of initial conditions, a diffeomorphism will be “good” if every point in the set is either sufficiently nonperiodic or sufficiently hyperbolic.

In order to keep the notations and formulas simple as we formalize this approach, we consider the case of 1-dimensional maps, but the reader should always have in mind that our approach is designed for multidimensional diffeomorphisms. Let $f : I \rightarrow I$ be a C^1 map on the interval $I = [-1, 1]$. Recall that a trajectory $\{x_k\}_{k \in \mathbb{Z}}$ of f is called *recurrent* if it returns arbitrarily close to its initial position – that is, for all $\gamma > 0$ we have $|x_0 - x_n| < \gamma$ for some $n > 0$. A very basic question is how much one should perturb f to make x_0 periodic. Here is an elementary Closing Lemma that gives a simple partial answer to this question.

Closing Lemma. *Let $\{x_k = f^k(x_0)\}_{k=0}^n$ be a trajectory of length $n + 1$ of a map $f : I \rightarrow I$. Let $u = (x_0 - x_n) / \prod_{k=0}^{n-2} (x_{n-1} - x_k)$. Then x_0 is a periodic point of period n of the map*

$$f_u(x) = f(x) + u \prod_{k=0}^{n-2} (x - x_k) \tag{2.6}$$

Of course f_u is close to f if and only if u is sufficiently small, meaning that $|x_0 - x_n|$ should be small compared to $\prod_{k=0}^{n-2} |x_{n-1} - x_k|$. However, this product is likely to contain small factors for recurrent trajectories. In general, it is difficult to control the effect of perturbations for recurrent trajectories. The simple reason why it is so is because *one can not perturb f at two nearby points independently*.

The Closing Lemma above also gives an idea of how much we must change the parameter u to make a point x_0 that is (n, γ) -periodic not be (n, γ) -periodic for a given $\gamma > 0$, which as we described above is one way to make a map that is “bad” for the initial condition x_0 become “good”. To make use of the other part of our

alternative we must determine how much we need to perturb a map f to make a given x_0 be (n, γ) -hyperbolic for some $\gamma > 0$.

Perturbation of hyperbolicity. Let $\{x_k = f^k(x_0)\}_{k=0}^{n-1}$ be a trajectory of length n of a C^1 map $f : I \rightarrow I$. Then for the map

$$f_v(x) = f(x) + v(x - x_{n-1}) \prod_{k=0}^{n-2} (x - x_k)^2 \quad (2.7)$$

such that $v \in \mathbb{R}$ and

$$\left| \left| \prod_{k=0}^{n-1} f'(x_k) + v \prod_{k=0}^{n-2} (x_{n-1} - x_k)^2 \prod_{k=0}^{n-2} f'(x_k) \right| - 1 \right| > \gamma \quad (2.8)$$

we have that x_0 is an (n, γ) -hyperbolic point of f_v .

One more time we can see the product of distances $\prod_{k=0}^{n-2} |x_{n-1} - x_k|$ along the trajectory is an important quantitative characteristic of how much freedom we have to perturb.

The perturbations (2.6) and (2.7) are reminiscent of Newton interpolation polynomials. Let us put these formulas into a general setting using singularity theory.

Given $n > 0$ and a C^1 function $f : I \rightarrow \mathbb{R}$ we define an associated function $j^{1,n}f : I^n \rightarrow I^n \times \mathbb{R}^{2n}$ by

$$j^{1,n}f(x_0, \dots, x_{n-1}) = \left(x_0, \dots, x_{n-1}, f(x_0), \dots, f(x_{n-1}), \right. \\ \left. f'(x_0), \dots, f'(x_{n-1}) \right). \quad (2.9)$$

In singularity theory this function is called the n -tuple 1-jet of f . The ordinary 1-jet of f , usually denoted by $j^1f(x) = (x, f(x), f'(x))$, maps I to the 1-jet space $\mathcal{J}^1(I, \mathbb{R}) \simeq I \times \mathbb{R}^2$. The product of n copies of $\mathcal{J}^1(I, \mathbb{R})$, called the *multijet space*, is denoted by

$$\mathcal{J}^{1,n}(I, \mathbb{R}) = \underbrace{\mathcal{J}^1(I, \mathbb{R}) \times \dots \times \mathcal{J}^1(I, \mathbb{R})}_{n \text{ times}} \quad (2.10)$$

and is equivalent to $I^n \times \mathbb{R}^{2n}$ after rearranging coordinates. The n -tuple 1-jet of f associates with each n -tuple of points in I^n all the information necessary to determine how close the n -tuple is to being a periodic orbit, and if so, how close it is to being nonhyperbolic.

The set

$$\Delta_n(I) = \left\{ \{x_0, \dots, x_{n-1}\} \times I^n \times \mathbb{R}^n \subset \mathcal{J}^{1,n}(I, \mathbb{R}) : \right. \\ \left. \exists i \neq j \text{ such that } x_i = x_j \right\} \quad (2.11)$$

is called the *diagonal* (or sometimes the *generalized diagonal*) in the space of multijets. In singularity theory the space of multijets is defined outside of the diagonal $\Delta_n(I)$ and is usually denoted by $\mathcal{J}_n^1(I, \mathbb{R}) = \mathcal{J}^{1,n}(I, \mathbb{R}) \setminus \Delta_n(I)$ (see [GG]). It is easy to see that a recurrent trajectory $\{x_k\}_{k \in \mathbb{Z}_+}$ is located in a neighborhood of the diagonal $\Delta_n(I) \subset \mathcal{J}^{1,n}(I, \mathbb{R})$ in the space of multijets for a sufficiently large n . If $\{x_k\}_{k=0}^{n-1}$ is a part of a recurrent trajectory of length n , then the product of distances along the trajectory

$$\prod_{k=0}^{n-2} |x_{n-1} - x_k| \quad (2.12)$$

measures how close $\{x_k\}_{k=0}^{n-1}$ to the diagonal $\Delta_n(I)$, or how independently one can perturb points of a trajectory. One can also say that (2.12) is a quantitative characteristic of how recurrent a trajectory of length n is. Introduction of this *product of distances along a trajectory into analysis of recurrent trajectories* is a new point of our paper.

2.2 Newton interpolation and blow-up along the diagonal in multijet space

Now we present a construction due to Grigoriev and Yakovenko [GY] which puts the “Closing Lemma” and “Perturbation of Hyperbolicity” statements above into a general framework. It is an interpretation of Newton interpolation polynomials as an algebraic blow-up along the diagonal in the multijet space. In order to keep the notations and formulas simple we continue in this section to consider only the 1-dimensional case.

Consider the $2n$ -parameter family of perturbations of a C^1 map $f : I \rightarrow I$ by polynomials of degree $2n - 1$

$$f_\varepsilon(x) = f(x) + \phi_\varepsilon(x), \quad \phi_\varepsilon(x) = \sum_{k=0}^{2n-1} \varepsilon_k x^k, \quad (2.13)$$

where $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{2n-1}) \in \mathbb{R}^{2n}$. The perturbation vector ε consists of coordinates from the Hilbert Brick $HB^1(\bar{\mathbf{r}})$ of analytic perturbations defined in Section 1.3. Our goal now is to describe how such perturbations affect the n -tuple 1-jet of f , and since the operator $j^{1,n}$ is linear in f , for the time being we consider only the perturbations ϕ_ε and their n -tuple 1-jets. For each n -tuple $\{x_k\}_{k=0}^{n-1}$ there is a natural transformation $\mathcal{J}^{1,n} : I^n \times \mathbb{R}^{2n} \rightarrow \mathcal{J}^{1,n}(I, \mathbb{R})$ from ε -coordinates to jet-coordinates, given by

$$\mathcal{J}^{1,n}(x_0, \dots, x_{n-1}, \varepsilon) = j^{1,n}\phi_\varepsilon(x_0, \dots, x_{n-1}). \quad (2.14)$$

Instead of working directly with the transformation $\mathcal{J}^{1,n}$, we introduce intermediate u -coordinates based on Newton interpolation polynomials. The relation between ε -coordinates and u -coordinates is given implicitly by

$$\phi_\varepsilon(x) = \sum_{k=0}^{2n-1} \varepsilon_k x^k = \sum_{k=0}^{2n-1} u_k \prod_{j=0}^{k-1} (x - x_{j \pmod n}). \quad (2.15)$$

Based on this identity, we will define functions $\mathcal{D}^{1,n} : I^n \times \mathbb{R}^{2n} \rightarrow \mathcal{D}^{1,n}(I, \mathbb{R})$ and $\pi^{1,n} : \mathcal{D}^{1,n}(I, \mathbb{R}) \rightarrow \mathcal{J}^{1,n}(I, \mathbb{R})$ so that $\mathcal{J}^{1,n} = \pi^{1,n} \circ \mathcal{D}^{1,n}$, or in other words the diagram in Figure 1 commutes. We will show later that $\mathcal{D}^{1,n}$ is invertible, while $\pi^{1,n}$ is invertible away from the diagonal $\Delta_n(I)$ and defines a blow-up along it in the space of multijets $\mathcal{J}^{1,n}(I, \mathbb{R})$.

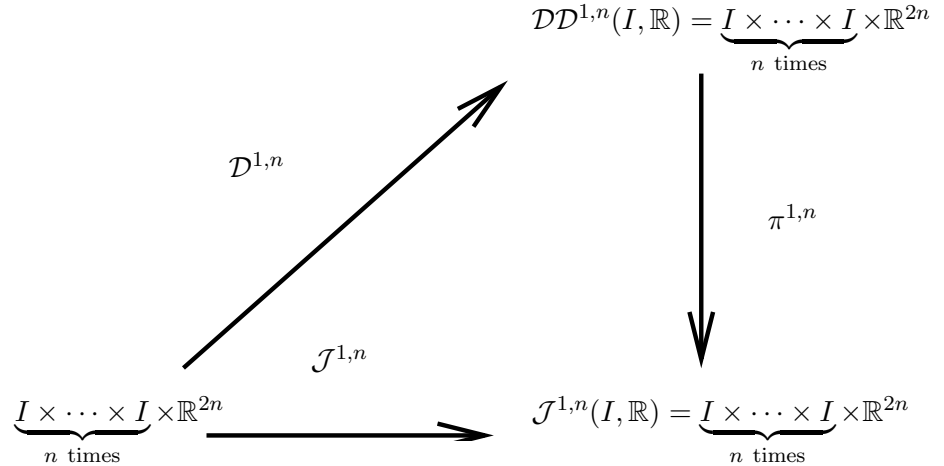


Figure 2.1: Algebraic Blow-up along the Diagonal $\Delta_n(I)$

The intermediate space, which we denote by $\mathcal{D}\mathcal{D}^{1,n}(I, \mathbb{R})$, is called *the space of divided differences* and consists of n -tuples of points $\{x_k\}_{k=0}^{n-1}$ and $2n$ real coefficients $\{u_k\}_{k=0}^{2n-1}$. Here are explicit coordinate-by-coordinate formulas defining $\pi^{1,n} :$

$\mathcal{DD}^{1,n}(I, \mathbb{R}) \rightarrow \mathcal{J}^{1,n}(I, \mathbb{R})$. This mapping is given by

$$\begin{aligned} \pi^{1,n}(x_0, \dots, x_{n-1}, u_0, \dots, u_{2n-1}) = \\ \left(x_0, \dots, x_{n-1}, \phi_\varepsilon(x_0), \dots, \phi_\varepsilon(x_{n-1}), \phi'_\varepsilon(x_0), \dots, \phi'_\varepsilon(x_{n-1}) \right), \end{aligned} \quad (2.16)$$

where

$$\begin{aligned} \phi_\varepsilon(x_0) &= u_0, \\ \phi_\varepsilon(x_1) &= u_0 + u_1(x_1 - x_0), \\ \phi_\varepsilon(x_2) &= u_0 + u_1(x_2 - x_0) + u_2(x_2 - x_0)(x_2 - x_1), \\ &\vdots \\ \phi_\varepsilon(x_{n-1}) &= u_0 + u_1(x_{n-1} - x_0) + \dots \\ &\quad + u_{n-1}(x_{n-1} - x_0) \dots (x_{n-1} - x_{n-2}), \\ \phi'_\varepsilon(x_0) &= \frac{\partial}{\partial x} \left(\sum_{k=0}^{2n-1} u_k \prod_{j=0}^k (x - x_{j(\text{mod } n)}) \right) \Big|_{x=x_0}, \\ &\vdots \\ \phi'_\varepsilon(x_{n-1}) &= \frac{\partial}{\partial x} \left(\sum_{k=0}^{2n-1} u_k \prod_{j=0}^k (x - x_{j(\text{mod } n)}) \right) \Big|_{x=x_{n-1}}, \end{aligned} \quad (2.17)$$

These formulas are very useful for dynamics. For a given base map f and initial point x_0 , the image $f_\varepsilon(x_0) = f(x_0) + \phi_\varepsilon(x_0)$ of x_0 depends only on u_0 . Furthermore the image can be set to any desired point by choosing u_0 appropriately – we say then that it depends only and nontrivially on u_0 . If x_0 , x_1 , and u_0 are fixed, the image $f_\varepsilon(x_1)$ of x_1 depends only on u_1 , and as long as $x_0 \neq x_1$ it depends nontrivially on u_1 . More generally for $0 \leq k \leq n-1$, if distinct points $\{x_j\}_{j=0}^k$ and coefficients $\{u_j\}_{j=0}^{k-1}$ are fixed, then the image $f_\varepsilon(x_k)$ of x_k depends only and nontrivially on u_k .

Suppose now that an n -tuple of points $\{x_j\}_{j=0}^n$ not on the diagonal $\Delta_n(I)$ and Newton coefficients $\{u_j\}_{j=0}^{n-1}$ are fixed. Then derivative $f'_\varepsilon(x_0)$ at x_0 depends only and nontrivially on u_n . Likewise for $0 \leq k \leq n-1$, if distinct points $\{x_j\}_{j=0}^{n-1}$ and Newton coefficients $\{u_j\}_{j=0}^{n+k-1}$ are fixed, then the derivative $f'_\varepsilon(x_k)$ at x_k depends only and nontrivially on u_{n+k} .

As Figure 2 illustrates, these considerations show that for any map f and any desired trajectory of distinct points with any given derivatives along it, one can choose Newton coefficients $\{u_k\}_{k=0}^{2n-1}$ and explicitly construct a map $f_\varepsilon = f + \phi_\varepsilon$ with such a

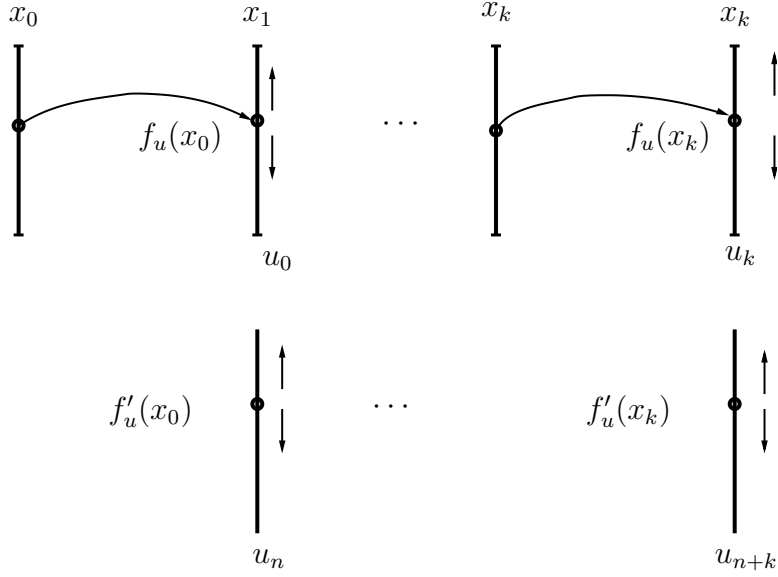


Figure 2.2: Newton Coefficients and their action

trajectory. Thus we have shown that $\pi^{1,n}$ is invertible away from the diagonal $\Delta_n(I)$ and defines a blow-up along it in the space of multijets $\mathcal{J}^{1,n}(I, \mathbb{R})$.

Next we define the function $\mathcal{D}^{1,n} : I^n \times \mathbb{R}^{2n} \rightarrow \mathcal{DD}^{1,n}(I, \mathbb{R})$ explicitly using so-called divided differences. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a C^r function of one real variable.

Definition 2.2.1. *The first order divided difference of g is defined as*

$$\Delta g(x_0, x_1) = \frac{g(x_1) - g(x_0)}{x_1 - x_0} \quad (2.18)$$

for $x_1 \neq x_0$ and extended by its limit value as $g'(x_0)$ for $x_1 = x_0$. Iterating this construction we define divided differences of the m -th order for $2 \leq m \leq r$,

$$\Delta^m g(x_0, \dots, x_m) = \frac{\Delta^{m-1} g(x_0, \dots, x_{m-2}, x_m) - \Delta^{m-1} g(x_0, \dots, x_{m-2}, x_{m-1})}{x_m - x_{m-1}} \quad (2.19)$$

for $x_{m-1} \neq x_m$ and extended by its limit value for $x_{m-1} = x_m$.

A function loses at most one derivative of smoothness with each application of Δ , so $\Delta^m g$ is at least C^{r-m} if g is C^r . Notice that Δ^m is linear as a function of g ,

and one can show that it is a symmetric function of x_0, \dots, x_m ; in fact, by induction it follows that

$$\Delta^m g(x_0, \dots, x_m) = \sum_{i=0}^m \frac{g(x_i)}{\prod_{j \neq i} (x_i - x_j)} \quad (2.20)$$

Another identity that is proved by induction will be more important for us, namely

$$\Delta^m x^k(x_0, \dots, x_m) = p_{k,m}(x_0, \dots, x_m), \quad (2.21)$$

where $p_{k,m}(x_0, \dots, x_m)$ is 0 for $m > k$ and for $m \leq k$ is the sum of all degree $k - m$ monomials in x_0, \dots, x_m with unit coefficients,

$$p_{k,m}(x_0, \dots, x_m) = \sum_{r_0 + \dots + r_m = k - m} \prod_{j=0}^m x_j^{r_j}. \quad (2.22)$$

The divided differences are the right coefficients for the Newton interpolation formula. For all C^∞ functions $g : \mathbb{R} \rightarrow \mathbb{R}$ we have

$$\begin{aligned} g(x) = & \Delta^0 g(x_0) + \Delta^1 g(x_0, x_1)(x - x_0) + \dots \\ & + \Delta^{n-1} g(x_0, \dots, x_{n-1})(x - x_0) \dots (x - x_{n-2}) \\ & + \Delta^n g(x_0, \dots, x_{n-1}, x)(x - x_0) \dots (x - x_{n-1}) \end{aligned} \quad (2.23)$$

identically for all values of x, x_0, \dots, x_{n-1} . All terms of this representation are polynomial in x except for the last one which we view as a remainder term. The sum of the polynomial terms is the degree $(n - 1)$ *Newton interpolation polynomial* for g at $\{x_k\}_{k=0}^{n-1}$. To obtain a degree $2n - 1$ interpolation polynomial for g and its derivative at $\{x_k\}_{k=0}^{n-1}$, we simply use (2.23) with n replaced by $2n$ and the $2n$ -tuple of points $\{x_{k \pmod n}\}_{k=0}^{2n-1}$.

Recall that $\mathcal{D}^{1,n}$ was defined implicitly by (2.15). We have described how to use divided differences to construct a degree $2n - 1$ interpolating polynomial of the form on the right-hand side of (2.15) for an arbitrary C^∞ function g . Our interest then is in the case $g = \phi_\varepsilon$, which as a degree $2n - 1$ polynomial itself will have no remainder term and coincide exactly with the interpolating polynomial. Thus $\mathcal{D}^{1,n}$ is given coordinate-by-coordinate by

$$\begin{aligned} u_m = & \Delta^m \left(\sum_{k=0}^{2n-1} \varepsilon_k x^k \right) (x_0, \dots, x_{m \pmod n}) \\ = & \varepsilon_m + \sum_{k=m+1}^{2n-1} \varepsilon_k p_{k,m}(x_0, \dots, x_{m \pmod n}) \end{aligned} \quad (2.24)$$

for $m = 0, \dots, 2n - 1$.

Equation (2.24) defines a transformation $(u_0, \dots, u_{2n-1}) = \mathcal{L}_{\mathbf{X}_n}^1(\varepsilon)$ on \mathbb{R}^{2n} , where $\mathbf{X}_n = (x_0, \dots, x_{n-1}) \in I^n$. We call $\mathcal{L}_{\mathbf{X}_n}^1$ the *Newton map*. This map is simply a restriction of $\mathcal{D}^{1,n}$ to its final $2n$ coordinates:

$$\mathcal{D}^{1,n}(\mathbf{X}_n, \varepsilon) = (\mathbf{X}_n, \mathcal{L}_{\mathbf{X}_n}^1(\varepsilon)). \quad (2.25)$$

Notice that for fixed \mathbf{X}_n , the Newton map is linear and given by an upper triangular matrix with units on the diagonal. Hence it is Lebesgue measure-preserving and invertible, whether or not \mathbf{X}_n lies on the diagonal $\Delta_n(I)$.

Furthermore, the Newton map $\mathcal{L}_{\mathbf{X}_n}^1$ preserves the class of *scaled Lebesgue product measures* introduced in (1.15). In general, a measure μ on \mathbb{R}^{2n} is a scaled Lebesgue product measure if it is the product $\mu = \mu_0 \times \dots \times \mu_{2n-1}$, where each μ_j is Lebesgue measure on \mathbb{R} scaled by a constant factor (which may depend on the coordinate j). Since the $\mathcal{L}_{\mathbf{X}_n}^1$ only shears in coordinate directions, we have the following lemma.

Lemma 2.2.2. *The Newton map $\mathcal{L}_{\mathbf{X}_n}^1$ given by (2.24) preserves all scaled Lebesgue product measures.*

This lemma will be used in Chapter 3. In the next section, we will introduce the particular scaled Lebesgue product measure that the lemma will be applied to.

We call the basis of monomials

$$\prod_{j=0}^k (x - x_{j(\text{mod } n)}) \quad \text{for } k = 0, \dots, 2n - 1 \quad (2.26)$$

in the space of polynomials of degree $2n - 1$ the *Newton basis* defined by the n -tuple $\{x_k\}_{k=0}^{n-1}$. The Newton map and the Newton basis, and their analogues in dimension N , are useful tools for perturbing trajectories and estimating the measure $\mu_n(C, \delta, \rho, \vec{\mathbf{r}}, M_{1+\rho})$ of “bad” parameter values $\vec{\varepsilon} \in HB^N(\mathbf{r})$.

2.3 Estimates of the measure of “bad” parameters and Fubini reduction to finite-dimensional families

We return now to the the general case of $C^{1+\rho}$ diffeomorphisms on \mathbb{R}^N . In order to bound $\mu_{\vec{\mathbf{r}}}^N\{B_n(C, \delta, \rho, \vec{\mathbf{r}}, f)\}$ we decompose the infinite-dimensional Hilbert Brick

$HB^N(\vec{\mathbf{r}})$ into the direct sum of a finite-dimensional brick of polynomials of degree $2n - 1$ in N variables and its orthogonal complement.

Recall that $\vec{\mathbf{r}} = (\{r_m\}_{m=0}^\infty)$ denotes the nonincreasing sequence $\{r_m\}_{m \in \mathbb{Z}_+}$ of sizes of the Hilbert Brick. With the notation (1.11) and (1.12), define

$$\begin{aligned} HB_{<k}^N(\vec{\mathbf{r}}) &= \{ \{ \vec{\varepsilon}_m \}_{m < k} : \text{for every } 0 \leq m < k, \|\vec{\varepsilon}_m\|_m \leq r_m \} \\ &= B_0^N(r_0) \times \cdots \times B_{k-1}^N(r_{k-1}) \subset W_{0,N} \times W_{1,N} \times \cdots \times W_{k-1,N}; \\ HB_{\geq k}^N(\vec{\mathbf{r}}) &= \{ \{ \vec{\varepsilon}_m \}_{m \geq k} : \text{for every } m \geq k, \|\vec{\varepsilon}_m\|_m \leq r_m \} \\ &= B_k^N(r_k) \times B_{k+1}^N(r_{k+1}) \times \cdots \subset W_{k,N} \times W_{k+1,N} \times \cdots; \\ HB^N(\vec{\mathbf{r}}) &= HB_{<k}^N(\vec{\mathbf{r}}) \oplus HB_{\geq k}^N(\vec{\mathbf{r}}). \end{aligned} \quad (2.27)$$

Each parameter $\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}})$ has a unique decomposition into

$$\begin{aligned} \vec{\varepsilon} &= (\vec{\varepsilon}_{<k}, \vec{\varepsilon}_{\geq k}) \in HB_{<k}^N(\vec{\mathbf{r}}) \oplus HB_{\geq k}^N(\vec{\mathbf{r}}), \\ \phi_{\vec{\varepsilon}}(x) &= \phi_{\vec{\varepsilon}_{<k}}(x) + \phi_{\vec{\varepsilon}_{\geq k}}(x) = \sum_{|\alpha| < k} \vec{\varepsilon}_\alpha x^\alpha + \sum_{|\alpha| \geq k} \vec{\varepsilon}_\alpha x^\alpha, \end{aligned} \quad (2.28)$$

where $\phi_{\vec{\varepsilon}_{<k}}(x)$ is a vector-polynomial of degree $k - 1$ and $\phi_{\vec{\varepsilon}_{\geq k}}(x)$ is an analytic function with all Taylor coefficients of order less than k being equal to zero. Recall the notation (1.15), and decompose the measure $\mu_{\vec{\mathbf{r}}}^N$ on the brick $HB^N(\vec{\mathbf{r}})$ into the product

$$\mu_{<k, \vec{\mathbf{r}}}^N = \times_{m=0}^{k-1} \mu_{m, r_m}^N, \quad \mu_{\geq k, \vec{\mathbf{r}}}^N = \times_{m=k}^\infty \mu_{m, r_m}^N, \quad \mu_{\vec{\mathbf{r}}}^N = \mu_{<k, \vec{\mathbf{r}}}^N \times \mu_{\geq k, \vec{\mathbf{r}}}^N. \quad (2.29)$$

Thus, each component of the decomposition of the brick $HB_{<k}^N(\vec{\mathbf{r}})$ (resp. $HB_{\geq k}^N(\vec{\mathbf{r}})$) is supplied with the Lebesgue product probability measure $\mu_{<k, \vec{\mathbf{r}}}^N$ (resp. $\mu_{\geq k, \vec{\mathbf{r}}}^N$). Denote by

$$W_{<k, N} = \times_{m=0}^{k-1} W_{m, N}, \quad W_{\geq k, N} = \times_{m=k}^\infty W_{m, N} \quad (2.30)$$

the spaces to which the brick $HB_{<k}^N(\vec{\mathbf{r}})$ and the Hilbert Brick $HB_{\geq k}^N(\vec{\mathbf{r}})$ belong.

Consider the decomposition with $k = 2n - 1$. Suppose we can get an estimate

$$\mu_{<2n, \vec{\mathbf{r}}}^N \{ B_n(C, \delta, \rho, \vec{\mathbf{r}}, f, \vec{\varepsilon}_{\geq 2n}) \} \leq \mu_n(C, \delta, \rho, M_{1+\rho}) \quad (2.31)$$

of the measure of the “bad” set

$$\begin{aligned} B_n(C, \delta, \rho, \vec{\mathbf{r}}, f, \vec{\varepsilon}_{\geq 2n}) &= \{ \vec{\varepsilon}_{<2n} \in HB_{<2n}^N(\vec{\mathbf{r}}) : \\ f_{\vec{\varepsilon}} &\in IH(n - 1, C, \delta, \rho) \text{ but } f_{\vec{\varepsilon}} \notin IH(n, C, \delta, \rho) \}. \end{aligned} \quad (2.32)$$

in each slice $HB_{<2n}^N(\vec{\mathbf{r}}) \times \{ \vec{\varepsilon}_{\geq 2n} \} \subset HB^N(\vec{\mathbf{r}})$, uniformly over $\vec{\varepsilon}_{\geq 2n} \in HB_{\geq 2n}^N(\vec{\mathbf{r}})$. Then by the Fubini/Tonelli Theorem and by the choice of the probability measure (2.29),

estimate (2.31) implies (2.3). Thus we reduce the problem of estimating measure of the “bad” set (2.1) in the infinite-dimensional Hilbert Brick $HB^N(\vec{\mathbf{r}})$ to estimating measure of the “bad” set (2.32) in the finite-dimensional brick $HB_{<2n}^N(\vec{\mathbf{r}})$ of vector-polynomials of degree $2n - 1$. Now our main goal is to get an estimate for the right-hand side of (2.31).

Fix a parameter value $\vec{\varepsilon}_{\geq 2n} \in HB_{\geq 2n}^N(\vec{\mathbf{r}})$ and the corresponding parameter slice $HB_{<2n}^N(\vec{\mathbf{r}}) \times \{\vec{\varepsilon}_{\geq 2n}\}$ in the Hilbert Brick $HB^N(\vec{\mathbf{r}})$. Let $\tilde{f} = f_{(0, \vec{\varepsilon}_{\geq 2n})}$ be the center of this slice. In this slice we have the finite-parameter family

$$\{\tilde{f}_{\vec{\varepsilon}_{<2n}}\}_{\vec{\varepsilon}_{<2n} \in HB_{<2n}^N(\vec{\mathbf{r}})} = \{f_{(\vec{\varepsilon}_{<2n}, \vec{\varepsilon}_{\geq 2n})}\}_{\vec{\varepsilon}_{<2n} \in HB_{<2n}^N(\vec{\mathbf{r}})} \quad (2.33)$$

of perturbations by polynomials of degree $2n - 1$. This is the family we consider at the n -th stage of the induction. We redenote the “bad” set of parameter values $B_n(C, \delta, \rho, \vec{\mathbf{r}}, f, \vec{\varepsilon}_{\geq 2n})$ by $B_n(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f})$.

2.4 Simple trajectories and the Inductive Hypothesis

Based on the discussion in Section 2.1, we make the following definition.

Definition 2.4.1. *A trajectory x_0, \dots, x_{n-1} of length n of a diffeomorphism $f \in \text{Diff}^r(B^N)$, where $x_k = f^k(x_0)$, is called (n, γ) -simple if*

$$\prod_{k=0}^{n-2} |x_{n-1} - x_k| \geq \gamma^{1/4N}. \quad (2.34)$$

A point x_0 is called (n, γ) -simple if its trajectory $\{x_k = f^k(x_0)\}_{k=0}^{n-1}$ of length n is (n, γ) -simple. Otherwise a point (resp. a trajectory) is called non- (n, γ) -simple.

If a trajectory is simple, then perturbation of this trajectory by Newton Interpolation Polynomials is effective as the Closing Lemma and Perturbation of hyperbolicity examples of Section 2.1 show. To evaluate the product of distances it is important to choose a “good” starting point x_0 of an almost periodic trajectory $\{x_k\}_k$ in order to have the largest possible value of the product in (2.34); for some starting points the product of distances may be artificially small.

Consider the following example of a homoclinic intersection: Let $f : B^2 \hookrightarrow B^2$ be a diffeomorphism with a hyperbolic saddle point at the origin $f(0) = 0$. Suppose that the stable manifold $W^s(0)$ and the unstable manifold $W^u(0)$ intersect at some

point $q \in W^s(0) \cap W^u(0)$. Then for a sufficiently large n there is a periodic point x of period n in a neighborhood of q going once nearby 0. It is clear that the trajectory $\{f^k(x)\}_{k=1}^n$ spends a lot of time in a neighborhood of the origin. Choose two starting points $x'_0 = f^{k'}(x)$ and $x''_0 = f^{k''}(x)$ for the product (2.34). If x'_0 is not in an $\exp(-\varepsilon n)$ -neighborhood of the origin for some $\varepsilon > 0$, but x''_0 is, then it might happen that $\prod_{k=0}^{n-2} |f^{n-1}(x'_0) - f^k(x'_0)| \sim \exp(-\delta n)$ and $\prod_{k=0}^{n-2} |f^{n-1}(x''_0) - f^k(x''_0)| \sim \exp(-\delta' n^2)$ for some $\delta, \delta' > 0$. Indeed, if we pick out of n points of $\{f^k(x)\}_{k=1}^n$ only $n/2$ closest to the origin, then simple calculation shows that all of them are in $\exp(-\varepsilon n)$ -neighborhood of the origin. ε is some positive number depending on the eigenvalues of $df(0)$. So the first product might be significantly larger than the second one. This is because the trajectory $\{f^k(x''_0)\}_{k=0}^{n-1}$ has many points in a neighborhood of the origin and all of the corresponding terms in the product are small. This shows that sometimes the product of distances along a trajectory (2.34) might be small not because the trajectory is too recurrent, but because we choose a “bad” starting point. This motivates the following definition.

Definition 2.4.2. *A point x is called essentially (n, γ) -simple if for some nonnegative $j < n$, the point $f^j(x)$ is (n, γ) -simple. Otherwise a point is called essentially non- (n, γ) -simple.*

Let us return to the strategy of the proof of Theorem 1.3.7. At the n -th stage of the induction over the period we consider the family of polynomial perturbations $\{\tilde{f}_{\vec{\varepsilon} < 2n}\}_{\vec{\varepsilon} < 2n \in HB_{< 2n}^N(\vec{\mathbf{r}})}$ of the form (2.33) of the diffeomorphism $\tilde{f} \in \text{Diff}^{1+\rho}(B^N)$ by polynomials of degree $2n - 1$. Consider among them only diffeomorphisms $\tilde{f}_{\vec{\varepsilon} < 2n}$ that satisfies Inductive Hypothesis of order $n - 1$ with constants (C, δ, ρ) , i.e. $\tilde{f}_{\vec{\varepsilon} < 2n} \in IH(n - 1, C, \delta, \rho)$ as we proposed earlier. To simplify notations we redenote the set $B_n(C, \delta, \rho, \vec{\mathbf{r}}, f, \vec{\varepsilon}_{\geq 2n})$ by $B_n(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f})$ with $\tilde{f} = f_{\vec{\varepsilon}_{\geq 2n}}$. Our main goal is to estimate measure of “bad” parameter values $B_n(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f})$, given by (2.32), for which the corresponding diffeomorphism has an $(n, \gamma_n^{1/\rho}(C, \delta))$ -periodic, but not $(n, \gamma_n(C, \delta))$ -hyperbolic, point $x \in B^N$.

We split the set of all possible almost periodic points of period n into two classes: *essentially $(n, \gamma_n^{1/N}(C, \delta))$ -simple* and *essentially non- $(n, \gamma_n^{1/N}(C, \delta))$ -simple*. Decompose the set of “bad” parameters $B_n(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f})$ into two sets of “bad” parameters with simple and nonsimple almost periodic points that are not sufficiently hyperbolic:

$$\begin{aligned} B_n^{\text{sim}}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}) &= \{\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}}) : \tilde{f}_{\vec{\varepsilon} < 2n} \in IH(n - 1, C, \delta, \rho), \\ \tilde{f}_{\vec{\varepsilon} < 2n} &\text{ has an } (n, \gamma_n^{1/\rho}(C, \delta))\text{-periodic, essentially} \\ (n, \gamma_n^{1/N}(C, \delta))\text{-simple, but not } (n, \gamma_n(C, \delta))\text{-hyperbolic point } x\} \end{aligned} \quad (2.35)$$

and

$$\begin{aligned}
B_n^{\text{non}}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}) &= \{\tilde{\varepsilon} \in HB^N(\vec{\mathbf{r}}) : \tilde{f}_{\tilde{\varepsilon} < 2n} \in IH(n-1, C, \delta, \rho), \\
&\tilde{f}_{\tilde{\varepsilon} < 2n} \text{ has an } (n, \gamma_n^{1/\rho}(C, \delta))\text{-periodic, essentially} \\
&\text{non-}(n, \gamma_n^{1/N}(C, \delta))\text{-simple, but not } (n, \gamma_n(C, \delta))\text{-hyperbolic point } x\}
\end{aligned} \tag{2.36}$$

It is clear that we have

$$B_n(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}) = B_n^{\text{sim}}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}) \cup B_n^{\text{non}}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}). \tag{2.37}$$

We shall estimate the measures of the sets of simple orbits $B_n^{\text{sim}}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f})$ and nonsimple orbits $B_n^{\text{non}}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f})$ separately, but using very similar methods.

Let $\tilde{f}_{\tilde{\varepsilon} < 2n} \in IH(n-1, C, \delta, \rho)$ be a diffeomorphism that satisfies the Inductive Hypothesis of order $n-1$ with constants (C, δ, ρ) . It turns out that if $\tilde{f}_{\tilde{\varepsilon} < 2n}$ has an $(n, \gamma_n^{1/\rho}(C, \delta))$ -periodic and essentially non- $(n, \gamma_n^{1/N}(C, \delta))$ -simple point x_0 , then the trajectory of x_0 has a close return $\tilde{f}_{\tilde{\varepsilon} < 2n}^k(x_0) = x_k$ for $k < n$ such that distance $|x_0 - x_k|$ is much smaller of all the previous $|x_0 - x_j|$, $1 \leq j < k$. Let us formulate more precisely what we mean here by ‘‘much smaller’’.

Definition 2.4.3. *Let $g \in \text{Diff}^{1+\rho}(B^N)$ be a diffeomorphism and let $D > 1$ be some number. A point $x_0 \in B^N$ (resp. a trajectory $x_0, \dots, x_{n-1} = g^{n-1}(x_0) \subset B^N$ of length n) has a weak (D, n) -gap at a point $x_k = g^k(x_0)$ if*

$$|x_k - x_0| \leq D^{-n} \min_{0 < j \leq k-1} |x_0 - x_j|. \tag{2.38}$$

and there is no $m < k$ such that x_0 has a weak (D, n) -gap at $x_m = g^m(x_0)$.

Remark 2.4.4. *The term ‘‘gap’’ arises by considering of the sequence $-\log|x_0 - x_1|$, $-\log|x_0 - x_2|$, \dots , $-\log|x_0 - x_k|$. Definition 2.4.3 implies that the last term is significantly larger than all the previous terms.*

Let’s show that n should be divisible by k for an almost periodic point of period n with a weak gap at x_k . This feature of a weak gap allows us to treat almost periodic trajectories of length n with a weak gap at x_k as n/k almost identical parts of length k each.

Lemma 2.4.5. *Let $g \in \text{Diff}^{1+\rho}(B^N)$ be a diffeomorphism, M_1 be an upper bound on C^1 -norm of g and g^{-1} , $D > M_1^2$, and let x_0 have a weak (D, n) -gap at x_k and $|x_0 - x_n| \leq |x_0 - x_k|$. Then n is divisible by k .*

Sketch of Proof: Denote by $\gcd(k, n)$ the greatest common divisor of k and n . Then using the bound on C^1 -norm of g and g^{-1} for any $x, y \in B^N$ we have

$$M_1^{-1} |g^{-1}(x) - g^{-1}(y)| \leq |x - y| \leq M_1 |g(x) - g(y)| \quad (2.39)$$

Using the Euclidean division algorithm developed in Part II of this paper, one can show that

$$|x_0 - x_{\gcd(k, n)}| \leq M_1^{2n} D^{-n} \min_{0 < j \leq k-1} |x_0 - x_j|. \quad (2.40)$$

Since $D > M_1^2$, we cannot have $\gcd(k, n) < k$, so n must be divisible by k . Q.E.D.

In Part II of this paper we prove the following result.

Theorem 2.4.6. *Let $g \in \text{Diff}^{1+\rho}(B^N)$ be a diffeomorphism for some $\rho > 0$ and satisfy the Inductive Hypothesis of order $n-1$ with constants (C, δ, ρ) , i.e. $g \in IH(n-1, C, \delta, \rho)$ and let $M_{1+\rho} = \max\{\|g^{-1}\|_{C^1}, \|g\|_{C^{1+\rho}}, 2^{1/\rho}\}$, $C > 100\rho^{-1}\delta^{-1} \log M_{1+\rho}$, and $D = \max\{M_{1+\rho}^{30/\rho}, \exp(C/100)\}$. Suppose the diffeomorphism g has an $(n, \gamma_n^{1/\rho}(C, \delta))$ -periodic and essentially non- $(n, \gamma_n^{1/N}(C, \delta))$ -simple point $x_0 \in B^N$. Then either x_0 is $(n, \gamma_n(C, \delta))$ -hyperbolic or x_0 has a weak (D, n) -gap at $x_k = g^k(x_0)$ for some k dividing n and x_j is $(k, \gamma_n^{1/N}(C, \delta))$ -simple for some $j < n$.*

Remark 2.4.7. *As a matter of fact we need a sharper result, but Theorem 2.4.6 is a nice starting point.*

Theorem 2.4.6 implies that the set of “bad” parameters with an essentially non-simple trajectory can be decomposed into the following finite union: Define the set of parameters with an almost periodic point of period n with a weak gap at the k -th point of its trajectory.

$$\begin{aligned} B_n^{\text{wgap}(k)}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}; D) &= \{\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}}) : \tilde{f}_{\vec{\varepsilon}} \in IH(n-1, C, \delta, \rho), \\ \tilde{f}_{\vec{\varepsilon}} &\text{ has an } (n, \gamma_n^{1/\rho}(C, \delta))\text{-periodic, but not} \\ (n, \gamma_n(C, \delta))\text{-hyperbolic point } x_0 &\text{ with a weak } (D, n)\text{-gap at } x_k = \tilde{f}_{\vec{\varepsilon}}^k(x_0)\} \end{aligned} \quad (2.41)$$

Then for $D = \max\{M_{1+\rho}^{30/\rho}, \exp(C/100)\}$, Theorem 2.4.6 gives

$$B_n^{\text{non}}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}) \subseteq \left(\bigcup_{k|n} B_n^{\text{wgap}(k)}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}; D) \right). \quad (2.42)$$

Combining inclusions (2.35) and (2.42), we have

$$\begin{aligned} B_n(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}) &\subseteq \\ B_n^{\text{sim}}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}) &\cup \left(\bigcup_{k|n} B_n^{\text{wgap}(k)}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}; D) \right). \end{aligned} \quad (2.43)$$

Thus we need to get estimates on the measures of bad parameters associated with essentially simple trajectories $B_n^{\text{sim}}(C, \delta, \rho, \vec{r}, \tilde{f})$ and trajectories with a weak gap $B_n^{\text{wgap}^{(k)}}(C, \delta, \rho, \vec{r}, \tilde{f}; D)$, where k divides n . In Chapter 3, we describe the Discretization method for the 1-dimensional model problem. This method will allow us to estimate the measure of parameters $B_n^{\text{sim}}(C, \delta, \rho, \vec{r}, \tilde{f})$ associated with simple almost periodic points. At the end of Chapter 3, we show how using the Discretization method one can estimate the measure of parameters $B_n^{\text{wgap}^{(k)}}(C, \delta, \rho, \vec{r}, \tilde{f}; D)$ associated with almost periodic trajectories with a weak gap. Loosely speaking, it is because those trajectories have simple parts of length k (see the end of Theorem 2.4.6), and hyperbolicity of simple part of length k enforces hyperbolicity of the trajectories of length n .

Chapter 3

A Model Problem: C^2 -smooth Maps of the Interval $I = [-1, 1]$

In Section 2.4 we concluded that the key to the proof of Theorem 1.3.7 (which implies the Main Theorem) is to get an estimate of the measure of “bad” parameters. Recall that the set of “bad” parameters (2.32) consists of those parameters $\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}})$ for which the corresponding diffeomorphism $f_{\vec{\varepsilon}}: B^N \hookrightarrow B^N$ has an almost periodic point x of period n that is not sufficiently hyperbolic. In this chapter we present a detailed discussion of C^2 -smooth 1-dimensional noninvertible maps ($N = 1$ and $\rho = 1$) with a Hilbert Brick of a “nice” size. This 1-dimensional model gives a useful insight into the general approach of estimating the measure of “bad” parameters for the N -dimensional $C^{1+\rho}$ -smooth diffeomorphisms and allows us to avoid several technical complications that will arise in Part II of this paper [K5]. These complications are outlined in the next chapter.

3.1 Setting up of the model

Let $C^2(I, I)$ be the space of C^2 -smooth maps of the interval $I = [-1, 1]$ into its interior. Consider a C^2 -smooth map of the interval $f \in C^2(I, I)$ and the family of perturbations of f by analytic functions represented as their power series

$$f_{\varepsilon}(x) = f(x) + \sum_{k=0}^{\infty} \varepsilon_k x^k. \quad (3.1)$$

Fix a positive $\tau > 0$. Define a range of parameters of this family in the form of

a Hilbert Brick

$$HB^{\text{st}}(\tau) = \left\{ \{\varepsilon_m\}_{m=0}^{\infty} : \forall m \geq 0, |\varepsilon_m| < \frac{\tau}{m!} \right\}. \quad (3.2)$$

We call $HB^{\text{st}}(\tau)$ a *Hilbert Brick of standard thickness* with width τ . If we choose τ small enough, then the whole family $\{f_\varepsilon\}_{\varepsilon \in HB^{\text{st}}(\tau)} \subset C^2(I, I)$ consists of C^2 -smooth maps of the interval I .

Define the Lebesgue product probability measure, denoted by μ_τ^{st} , on the Hilbert Brick of parameters $HB^{\text{st}}(\tau)$ by normalizing the 1-dimensional Lebesgue measure along each component to the 1-dimensional Lebesgue probability measure

$$\mu_{m,\tau}^{\text{st}} = \left(\frac{m!}{2\tau} \right) Leb_1, \quad \mu_{<k,\tau}^{\text{st}} = \times_{m=0}^{k-1} \mu_{m,\tau}^{\text{st}} \quad \mu_\tau^{\text{st}} = \times_{m=0}^{\infty} \mu_{m,\tau}^{\text{st}}. \quad (3.3)$$

The main result of this Chapter is the following 1-dimensional analogue of Theorem 1.3.7.

Theorem 3.1.1. *Let $f \in C^2(I, I)$ be a C^2 -smooth map of the interval I into its interior and let $\tau > 0$ be so small that the family of analytic perturbations $\{f_\varepsilon\}_{\varepsilon \in HB^{\text{st}}(\tau)} \subset C^2(I, I)$ consists of C^2 -smooth maps of the interval I . Then for any $\delta > 0$ and μ_τ^{st} -a.e. $\varepsilon \in HB^{\text{st}}(\tau)$ there exists $C = C(\varepsilon, \delta) > 0$ such that the number of periodic points $P_n(f_\varepsilon)$ of f_ε of period n and their minimal hyperbolicity $\gamma_n(f_\varepsilon)$, defined in (1.7), for all $n \in \mathbb{Z}_+$ satisfy*

$$\gamma_n(f_\varepsilon) > \exp(-Cn^{1+\delta}), \quad P_n(f_\varepsilon) < \exp(Cn^{1+\delta}). \quad (3.4)$$

The strategy for the proof of this Theorem is the same as the strategy of the proof of Theorem 1.3.7 described in Chapter 2. Denote the supremum C^2 and C^1 -norms of the family (3.1)

$$M_1 = \sup_{\varepsilon \in HB^{\text{st}}(\tau)} \{\|f_\varepsilon\|_{C^1}\}, \quad M_2 = \sup_{\varepsilon \in HB^{\text{st}}(\tau)} \{\|f_\varepsilon\|_{C^2}, M_1, 2\} \quad (3.5)$$

By analogy with the direct decomposition of the Hilbert Brick in the N -dimensional case (2.27), for each positive integer $k \in \mathbb{Z}_+$ define the direct decomposition of the Hilbert Brick of standard thickness $HB^{\text{st}}(\tau)$

$$HB_{<k}^{\text{st}}(\tau) = \left\{ \{\varepsilon_m\}_{m=0}^{k-1} : \forall 0 \leq m < k, |\varepsilon_m| < \frac{\tau}{m!} \right\}, \quad (3.6)$$

$$HB_{\geq k}^{\text{st}}(\tau) = \left\{ \{\varepsilon_m\}_{m=k}^{\infty} : \forall m \geq k, |\varepsilon_m| < \frac{\tau}{m!} \right\}.$$

We call $HB_{<k}^{\text{st}}(\tau)$ a (k -dimensional) *Brick of standard thickness* with width τ . The product measure μ_τ^{st} on the whole Hilbert Brick $HB^{\text{st}}(\tau)$ induces the measure of product of Lebesgue probability $\mu_{<k,\tau}^{\text{st}}$ on the k -dimensional Brick $HB_{<k}^{\text{st}}(\tau)$.

Fix $n \in \mathbb{Z}_+$ and consider the n -th stage of the induction over the period (see the beginning of Chapter 2). Let

$$\tilde{f}(x) = f(x) + \sum_{k=2n}^{\infty} \varepsilon_k x^k \quad (3.7)$$

for some $\{\varepsilon_k\}_{k=2n}^{\infty} \in HB_{\geq 2n}^{\text{st}}(\tau)$, and consider the $2n$ -parameter family of perturbations by polynomials of degree $2n - 1$ with coefficients in the brick of standard thickness $HB_{<2n}^{\text{st}}(\tau)$,

$$\tilde{f}_\varepsilon(x) = \tilde{f}(x) + \sum_{k=0}^{2n-1} \varepsilon_k x^k, \quad \varepsilon = (\varepsilon_0, \dots, \varepsilon_{2n-1}) \in HB_{<2n}^{\text{st}}(\tau). \quad (3.8)$$

The bounds M_1 and M_2 from (3.5) apply to this subfamily of (3.1).

Using the Fubini reduction to finite-dimensional families from Section 2.3 right after (2.31), for the proof of Theorem 3.1.1 it is sufficient to estimate the measure of “bad” parameters in each such a family.

To fit the notations of our model we choose a sufficiently small positive γ_n and we introduce sets of all “bad” parameters (compare with (2.32))

$$\begin{aligned} B_{n,\tau}(C, \delta, \tilde{f}, \gamma_n) &= \{\varepsilon \in HB_{<2n}^{\text{st}}(\tau) : \tilde{f}_\varepsilon \in IH(n-1, C, \delta, 1), \\ &\tilde{f}_\varepsilon \text{ has an } (n, \gamma_n)\text{-periodic, but not } (n, \gamma_n)\text{-hyperbolic point } x_0\}, \end{aligned} \quad (3.9)$$

and define the sets $B_{n,\tau}^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n)$ and $B_{n,\tau}^{\text{non}}(C, \delta, \tilde{f}, \gamma_n)$ of “bad” parameters with essentially simple (respectively nonsimple) trajectories as in (2.35) and (2.36):

$$\begin{aligned} B_n^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n) &= \{\vec{\varepsilon} \in HB_{<2n}^{\text{st}}(\tau) : \tilde{f}_{\vec{\varepsilon}} \in IH(n-1, C, \delta, 1), \\ &\tilde{f}_{\vec{\varepsilon}} \text{ has an } (n, \gamma_n)\text{-periodic, essentially} \\ &(n, \gamma_n)\text{-simple, but not } (n, \gamma_n)\text{-hyperbolic point } x_0\}, \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} B_n^{\text{non}}(C, \delta, \tilde{f}, \gamma_n) &= \{\vec{\varepsilon} \in HB_{<2n}^{\text{st}}(\tau) : \tilde{f}_{\vec{\varepsilon}} \in IH(n-1, C, \delta, 1), \\ &\tilde{f}_{\vec{\varepsilon}} \text{ has an } (n, \gamma_n)\text{-periodic, essentially} \\ &\text{non-}(n, \gamma_n)\text{-simple, but not } (n, \gamma_n)\text{-hyperbolic point } x_0\}. \end{aligned} \quad (3.11)$$

For sufficiently small γ_n , e.g., $\gamma_n \leq \gamma_n(C, \delta)$, similarly to (2.37) we have the following decomposition

$$B_{n,\tau}(C, \delta, \tilde{f}, \gamma_n) = B_{n,\tau}^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n) \cup B_{n,\tau}^{\text{non}}(C, \delta, \tilde{f}, \gamma_n). \quad (3.12)$$

The main result of the next three sections is the following estimate.

Proposition 3.1.2. *Let $\{f_\varepsilon\}_{\varepsilon \in HB_{<2n}^{\text{st}}(\tau)}$ be the family of polynomial perturbations (3.8) with bound M_2 on the C^2 -norm. Then with the notation above, for any $C > 2$, $\delta > 0$, and $\tau > 0$ and a sufficiently small positive γ_n , e.g., $\gamma_n \leq \gamma_n(C, \delta)$, the following estimate on the measure of parameters associated with maps \tilde{f}_ε with an (n, γ_n) -periodic, essentially (n, γ_n) -simple, but not (n, γ_n) -hyperbolic, point holds*

$$\mu_{<2n,\tau}^{\text{st}}\{B_{n,\tau}^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n)\} \leq 6^{2n} M_2^{6n+1} \frac{(n-1)!}{\tau} \frac{(2n-1)!}{\tau} \gamma_n^{1/4}. \quad (3.13)$$

It is clear that for any $C > 0$ and $\delta > 0$, if $\gamma_n = \exp(-Cn^{1+\delta})$, then the right-hand side of (3.13) tends to 0 as $n \rightarrow \infty$ superexponentially fast in n . An estimate on the measure of essentially nonsimple trajectories $\mu_{<2n,\tau}^{\text{st}}\{B_n^{\text{non}}(C, \delta, \tilde{f}, \gamma_n)\}$ is obtained in Section 3.5, Proposition 3.5.1. Application of these two Propositions and arguments (2.3–2.4) will prove Theorem 3.1.1. The method of obtaining an estimate for $\mu_{<2n,\tau}^{\text{st}}\{B_{n,\tau}^{\text{non}}(C, \delta, \tilde{f}, \gamma_n)\}$ is similar to the one we shall develop now to prove (3.13).

3.2 Decomposition into pseudotrajectories

In this section, we decompose the set of “bad” parameters $B_{n,\tau}^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n)$ for which there exists a simple, almost periodic, but not sufficiently hyperbolic trajectory into a finite union of sets of “bad” parameters. Each set will be associated with a particular simple, almost periodic, but not sufficiently hyperbolic pseudotrajectory. In the next section we will estimate the measure of “bad” parameters associated with a *particular* trajectory, and in the subsequent section we will extend this estimate to the set of “bad” parameters associated with *all possible* simple trajectories, obtaining estimate (3.13).

Fix a sufficiently small $\gamma_n > 0$ and $\tilde{\gamma}_n = \gamma_n M_2^{-2n}$. Consider the $2\tilde{\gamma}_n$ -grid in the interval I

$$I_{\tilde{\gamma}_n} = \{x \in I : \exists k \in \mathbb{Z} \text{ such that } x = (2k+1)\tilde{\gamma}_n\} \subset I \quad (3.14)$$

Definition 3.2.1. We call a k -tuple $\{x_j\}_{j=0}^{k-1} \in I_{\tilde{\gamma}_n}^k$ a $\tilde{\gamma}_n$ -pseudotrajectory associated to ε (or to the map \tilde{f}_ε) if for each $j = 1, \dots, k-1$ we have $|\tilde{f}_\varepsilon(x_{j-1}) - x_j| \leq \tilde{\gamma}_n$, and we call it a $\tilde{\gamma}_n$ -pseudotrajectory associated to $HB_{<2n}^{\text{st}}(\tau)$ (or to the family $\{\tilde{f}_\varepsilon\}_{\varepsilon \in HB_{<2n}^{\text{st}}(\tau)}$) if it is associated to some $\varepsilon \in HB_{<2n}^{\text{st}}(\tau)$.

A $\tilde{\gamma}_n$ -pseudotrajectory x_0, \dots, x_{n-1} of length n associated to some parameter $\varepsilon \in HB_{<2n}^{\text{st}}(\tau)$ for some $\gamma > 0$ is

- (n, γ) -periodic if $|\tilde{f}_\varepsilon(x_{n-1}) - x_0| \leq \gamma$,
- (n, γ) -simple if $\prod_{j=0}^{n-2} |x_{n-1} - x_j| \geq \gamma^{1/4}$,
- (n, γ) -hyperbolic if $\left| \prod_{j=0}^{n-1} |(\tilde{f}_\varepsilon)'(x_j)| - 1 \right| \geq \gamma$.

Remark 3.2.2. For fixed $\varepsilon \in HB_{<2n}^{\text{st}}(\tau)$, each initial point $x_0 \in I_{\tilde{\gamma}_n}$ generates a $\tilde{\gamma}_n$ -pseudotrajectory $\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{n-1}$ of length n as follows. For each successive $k = 1, \dots, n-1$, choose $\tilde{x}_k \in I_{\tilde{\gamma}_n}$ such that $|\tilde{x}_k - \tilde{f}_\varepsilon(\tilde{x}_{k-1})| \leq \tilde{\gamma}_n$. Notice that this choice is unique unless $\tilde{f}_\varepsilon(\tilde{x}_{k-1})$ happens to lie halfway between two points of $I_{\tilde{\gamma}_n}$. It may be helpful in understanding the upcoming arguments to think of each initial point $x_0 \in I_{\tilde{\gamma}_n}$ as generating a unique $\tilde{\gamma}_n$ -pseudotrajectory for a given f_ε , though for a measure zero set of ε there are exceptions to this rule. In fact, for our estimates it is important only that the number of $\tilde{\gamma}_n$ -pseudotrajectories per initial point be bounded by an exponential function of n , which is true in this case even if there is a choice of two grid points at each iteration.

We would like to contain the set of “bad” parameters $B_{n,\tau}^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n)$ in a finite collection of subsets each of “bad” parameters corresponding to a single $\tilde{\gamma}_n$ -pseudotrajectory

$$B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; x_0, \dots, x_{n-1}) = \{\varepsilon \in HB_{<2n}^{\text{st}}(\tau) : \{x_k\}_{k=0}^{n-1} \text{ is a } \tilde{\gamma}_n\text{-pseudotrajectory associated to } \varepsilon \text{ and is } \left(n, \frac{\gamma_n}{2}\right)\text{-simple and } (n, M_2^n \gamma_n)\text{-periodic, but not } (n, M_2^{3n} \gamma_n)\text{-hyperbolic}\} \quad (3.15)$$

Introduce the union of all “bad” sets associated with $\tilde{\gamma}_n$ -pseudotrajectories

$$B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2) = \cup_{\{x_0, \dots, x_{n-1}\} \subset I_{\tilde{\gamma}_n}^n} B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; x_0, \dots, x_{n-1}). \quad (3.16)$$

Most of the sets in the right hand side are empty, and one of our goals is to estimate the number of nonempty ones.

In comparison to the definition (3.10) of $B_{n,\tau}^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n)$, we increase periodicity and hyperbolicity for pseudotrajectories and decrease simplicity. This will allow us to prove that

$$B_{n,\tau}^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n) \subset B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2). \quad (3.17)$$

Intuitively this is true because each trajectory of length n can be approximated by a pseudotrajectory of length n which has almost the same periodicity, simplicity, and hyperbolicity as the original one. We will make this argument precise at the end of Section 3.4.

Remark 3.2.3. *Unlike $B_{n,\tau}^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n)$, we do not assume in the definition (3.15) of $B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2)$ that $\tilde{f}_\varepsilon \in IH(n-1, C, \delta, 1)$. This is because we only need the Inductive Hypothesis to estimate the measure of “bad” parameters in the case of nonsimple trajectories.*

Our goal is then to estimate the measure $\mu_{<2n,\tau}^{\text{st}}\{B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2)\}$ in order to prove Proposition 3.1.2. Loosely speaking, this measure will be estimated in two steps:

Step 1. Estimate the number of different $\tilde{\gamma}_n$ -pseudotrajectories $\#_n(\tilde{\gamma}_n, \tau)$ associated to $HB_{<2n}^{\text{st}}(\tau)$;

Step 2. Estimate the measure

$$\mu_{<2n,\tau}^{\text{st}}\{B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; x_0, \dots, x_{n-1})\} \leq \mu_n(M_2, \gamma_n, \tilde{\gamma}_n, \tau) \quad (3.18)$$

uniformly for an (n, γ_n) -simple $\tilde{\gamma}_n$ -pseudotrajectory $\{x_0, \dots, x_{n-1}\} \in I_{\tilde{\gamma}_n}^n$.

Then the product of two numbers $\#_n(\tilde{\gamma}_n, \tau)$ and $\mu_n(M_2, \gamma_n, \tilde{\gamma}_n, \tau)$ that are obtained in Steps 1 and 2 gives the required estimate (3.13).

Actually the procedure of estimating $\mu_{<2n,\tau}^{\text{st}}\{B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2)\}$ is a little more complicated. Based on the definition (3.15) of the set $B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; x_0, \dots, x_{n-1})$ of parameters ε for which the diffeomorphism f_ε has a prescribed $\tilde{\gamma}_n$ -pseudotrajectory $\{x_0, \dots, x_{n-1}\} \in I_{\tilde{\gamma}_n}^n$ that is almost periodic and not sufficiently hyperbolic, define a set of parameters ε for which only a part of the $\tilde{\gamma}_n$ -pseudotrajectory $\{x_0, \dots, x_{m-1}\} \in I_{\tilde{\gamma}_n}^m$ is prescribed for f_ε :

$$\begin{aligned} B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; x_0, \dots, x_{m-1}) &= \{\varepsilon \in HB_{<2n}^{\text{st}}(\tau) : \text{there exist} \\ x_m, \dots, x_{n-1} \in I_{\tilde{\gamma}_n} &\text{ such that } \{x_j\}_{j=0}^{n-1} \text{ is a } \tilde{\gamma}_n\text{-pseudotrajectory} \\ &\text{ associated to } \varepsilon, \text{ and } \varepsilon \in B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; x_0, \dots, x_{n-1})\}. \end{aligned} \quad (3.19)$$

For each $m = 1, 2, \dots, n-1$ it is clear that

$$B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; x_0, \dots, x_{m-1}) = \cup_{x_m \in I_{\tilde{\gamma}_n}} B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; x_0, \dots, x_m). \quad (3.20)$$

Inductive application of this formula to the definition (3.16) yields

$$B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2) = \cup_{\tilde{x}_0 \in I_{\tilde{\gamma}_n}} B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; \tilde{x}_0). \quad (3.21)$$

The estimate of Step 1 then breaks down as follows:

$$\#_n(\tilde{\gamma}_n, \tau) \approx \boxed{\begin{array}{c} \# \text{ of initial} \\ \text{points of } I_{\tilde{\gamma}_n} \end{array}} \times \boxed{\begin{array}{c} \# \text{ of } \tilde{\gamma}_n\text{-pseudotrajectories} \\ \text{per initial point} \end{array}} \quad (3.22)$$

And up to an exponential function of n , the estimate of Step 2 breaks down like:

$$\mu_n(M_2, \gamma_n, \tilde{\gamma}_n, \tau) \approx \frac{\boxed{\begin{array}{c} \text{Measure of} \\ \text{periodicity} \end{array}} \times \boxed{\begin{array}{c} \text{Measure of} \\ \text{hyperbolicity} \end{array}}}{\boxed{\begin{array}{c} \# \text{ of } \tilde{\gamma}_n\text{-pseudotrajectories} \\ \text{per initial point} \end{array}}} \quad (3.23)$$

(Roughly speaking, the terms in the numerator represent respectively the measure of parameters for which a given initial point will be (n, γ_n) -periodic and the measure of parameters for which a given n -tuple is (n, γ_n) -hyperbolic; they correspond to estimates (3.30) and (3.33) in the next section.) Thus after cancellation, the estimate of the measure of “bad” set $B_{n,\tau}^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n)$ associated to simple, almost periodic, not sufficiently hyperbolic trajectories becomes:

$$\boxed{\begin{array}{c} \# \text{ of initial} \\ \text{points of } I_{\tilde{\gamma}_n} \end{array}} \times \boxed{\begin{array}{c} \text{Measure of} \\ \text{periodicity} \end{array}} \times \boxed{\begin{array}{c} \text{Measure of} \\ \text{hyperbolicity} \end{array}} \leq \boxed{\begin{array}{c} \text{Measure of bad} \\ \text{parameters} \end{array}} \quad (3.24)$$

The first term on the right hand side of (3.24) is of order γ_n^{-1} (up to an exponential function in n). In Section 3.3, we will show that the second term is at most of order $n!\gamma_n^{3/4}$, and the third term is at most of order $(2n)!\gamma_n^{1/2}$, so that the product on the right-hand side of (3.24) is of order at most $n!(2n)!\gamma_n^{1/4}$ (up to an exponential function in n) and is superexponentially small in n . These bounds use the change of parameter coordinates by Newton interpolation polynomials that was introduced in Section 2.2, and they *do not depend on whether the parameters are associated with the brick* $HB_{<2n}^{\text{st}}(\tau)$, except in that we use the bound M_2 on the C^2 norm of the maps involved.

In Section 3.4, we complete the proof of Proposition 3.1.2 by bounding the total measure of “bad” parameters for all pseudotrajectories associated to $HB_{<2n}^{\text{st}}(\tau)$. Since we use the Fubini theorem in the Newton coordinates u_0, \dots, u_{2n-1} , we need to know the maximum range of each of these parameters in the image of $HB_{<2n}^{\text{st}}(\tau)$ under this coordinate change. In the “Distortion Lemma”, we show that the image of $HB_{<2n}^{\text{st}}(\tau)$ is contained in a brick 3 times as large in each direction. Then, in the “Collection Lemma”, we show in effect that the cancellation in going from (3.22) and (3.23) to (3.24) is valid. In fact, the number of $\tilde{\gamma}_n$ -pseudotrajectories for a given initial point may depend significantly on the initial point, and we do not bound it explicitly. Rather, we show that in the decomposition (3.21), the measure of each term $B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; \tilde{x}_0)$ is bounded (up to a factor exponential in n) by the product of the “measure of periodicity” and “measure of hyperbolicity” derived in Section 3.3, thus yielding a final estimate of the form (3.24).

3.3 Application of Newton interpolation polynomials to estimate the measure of “bad” parameters for a single trajectory

In this section we fix an n -tuple of points $\{x_j\}_{j=0}^{n-1} \in I^n$, denoted by \mathbf{X}_n , and estimate the measure of “bad” parameters $B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; x_0, \dots, x_{n-1})$ associated with *this particular* trajectory. Recall that $\tilde{\gamma}_n = M_2^{-2n} \gamma_n$.

Problem 3.3.1. *Estimate the measure of $\varepsilon \in HB_{<2n}^{\text{st}}(\tau)$ for which the n -tuple $\{x_j\}_{j=0}^{n-1}$ is*

- A) a $\tilde{\gamma}_n$ -pseudotrajectory, i.e., $|\tilde{f}_\varepsilon(x_j) - x_{j+1}| \leq \tilde{\gamma}_n$ for $j = 0, \dots, n-2$;
 - B) (n, γ_n) -periodic, i.e., $|\tilde{f}_\varepsilon(x_{n-1}) - x_0| \leq \gamma_n$; and
 - C) not (n, γ_n) -hyperbolic, i.e., $\left| \prod_{j=0}^{n-1} |(f_\varepsilon)'(x_j)| - 1 \right| \leq \gamma_n$.
- (3.25)

Recall the definitions and notation of Sections 2.2 and 2.3. In particular, $W_{<2n,1}$ is the space of polynomials of degree $2n-1$ with the standard basis $\{x^m\}_{m=0}^{2n-1}$. The measure $\mu_{<2n,\tau}^{\text{st}}$ defined on the brick $HB_{<2n}^{\text{st}}(\tau) \in W_{<2n,1}$ by (3.3) extends naturally to $W_{<2n,1}$ using the same formulas. Denote by $W_{<2n,1}^{u,\mathbf{X}_n}$ the same space of polynomials of degree $2n-1$, but with the Newton basis (2.26). Lemma 2.2.2 implies that the Newton Map $\mathcal{L}_{\mathbf{X}_n}^1 : W_{<2n,1} \rightarrow W_{<2n,1}^{u,\mathbf{X}_n}$ defined by (2.24) preserves the measure $\mu_{<2n,\tau}^{\text{st}}$.

In other words, the definition (3.3) produces the same measure whether the standard basis or Newton basis is used.

Now we will estimate the measure of “bad” parameters for a particular trajectory using the Newton basis, without regard (except in the final hyperbolicity estimate) to whether the parameters $u = (u_0, u_1, \dots, u_{2n-1})$ lie in the image $\mathcal{L}_{\mathbf{X}_n}^1(HB_{<2n}^{\text{st}}(\tau))$ of the brick we are concerned with. For a fixed n -tuple of points $\mathbf{X}_n = \{x_j\}_{j=0}^{n-1}$, consider the Newton family of polynomial perturbations

$$\tilde{f}_{u, \mathbf{X}_n}(x) = \tilde{f}(x) + \sum_{m=0}^{2n-1} u_m \prod_{j=0}^{m-1} (x - x_{j \pmod{n}}). \quad (3.26)$$

Notice that in (2.17) and Figure 2.2, the image $\tilde{f}_{u, \mathbf{X}_n}(x_0)$ of x_0 is independent of u_k for all $k > 0$. Therefore, the position of $\tilde{f}_{u, \mathbf{X}_n}(x_0)$ depends only on u_0 . Recall that $\mu_{m, \tau}^{\text{st}}$ is 1-dimensional Lebesgue measure scaled by $m!/(2\tau)$. This gives

$$\mu_{0, \tau}^{\text{st}} \left\{ u_0 : \left| \tilde{f}_{u, \mathbf{X}_n}(x_0) - x_1 \right| = \left| \tilde{f}(x_0) + u_0 - x_1 \right| \leq \tilde{\gamma}_n \right\} \leq \frac{0!}{2\tau} 2\tilde{\gamma}_n = \frac{0!}{\tau} \tilde{\gamma}_n \quad (3.27)$$

Fix u_0 . Similarly, the position of $\tilde{f}_u(x_1)$ depends only on u_1 (see (2.17) and Fig. 2.2). Thus, we have

$$\begin{aligned} \mu_{1, \tau}^{\text{st}} \left\{ u_1 : \left| \tilde{f}_{u, \mathbf{X}_n}(x_1) - x_2 \right| = \right. \\ \left. \left| \tilde{f}(x_1) + u_0 + u_1(x_1 - x_0) - x_2 \right| \leq \tilde{\gamma}_n \right\} \leq \frac{1!}{\tau} \frac{\tilde{\gamma}_n}{|x_1 - x_0|} \end{aligned} \quad (3.28)$$

Inductively for $k = 2, \dots, n-1$, fix u_0, \dots, u_{k-1} . Then the position of $\tilde{f}_{u, \mathbf{X}_n}(x_k)$ depends only on u_k . Moreover, for $k = 2, \dots, n-2$ we have

$$\begin{aligned} \mu_{k, \tau}^{\text{st}} \left\{ u_k : \left| \tilde{f}_{u, \mathbf{X}_n}(x_k) - x_{k+1} \right| = \right. \\ \left. \left| \tilde{f}(x_k) + \sum_{m=0}^k u_m \prod_{j=0}^{m-1} (x_k - x_j) - x_{k+1} \right| \leq \tilde{\gamma}_n \right\} \leq \frac{k!}{\tau} \frac{\tilde{\gamma}_n}{\prod_{j=0}^{k-1} |x_k - x_j|}, \end{aligned} \quad (3.29)$$

and for $k = n-1$ we have

$$\mu_{n-1, \tau}^{\text{st}} \left\{ u_{n-1} : \left| \tilde{f}_{u, \mathbf{X}_n}(x_{n-1}) - x_0 \right| \leq \gamma_n \right\} \leq \frac{(n-1)!}{\tau} \frac{\gamma_n}{\prod_{j=0}^{n-2} |x_{n-1} - x_j|}. \quad (3.30)$$

In particular, the parameter u_{n-1} is responsible for (n, γ_n) -periodicity of the n -tuple \mathbf{X}_n . Formula (3.30) estimates the “measure of periodicity” discussed in the previous section.

Choose u_0, \dots, u_{n-1} so that the n -tuple \mathbf{X}_n is a $\tilde{\gamma}_n$ -pseudotrajectory and is (n, γ_n) -periodic. Notice that parameters $u_n, u_{n+1}, \dots, u_{2n-1}$ do not change the $\tilde{\gamma}_n$ -pseudotrajectory $\{x_k\}_{k=0}^{n-1}$. Fix now parameters u_0, \dots, u_{2n-2} and vary only u_{2n-1} . Then for any C^1 -smooth map $g : I \rightarrow I$, consider the 1-parameter family

$$g_{u_{2n-1}}(x) = g(x) + (x - x_{n-1}) \prod_{j=0}^{n-2} (x - x_j)^2 \quad (3.31)$$

Since the corresponding monomial $(x - x_{n-1}) \prod_{j=0}^{n-2} (x - x_j)^2$ has zeroes of the second order at all points x_k , except the last one x_{n-1} , we have

$$\prod_{j=0}^{n-1} (g_{u_{2n-1}})'(x_j) = \left(g'(x_{n-1}) + u_{2n-1} \prod_{j=0}^{n-2} |x_{n-1} - x_j|^2 \right) \prod_{j=0}^{n-2} g'(x_j). \quad (3.32)$$

To get the final estimate, we use the fact that we are interested only in maps from the family $\{\tilde{f}_\varepsilon\}_{\varepsilon \in HB_{<2n}^{\text{st}}(\tau)}$. Therefore, we have $|\tilde{f}'_{u, \mathbf{X}_n}(x_{n-1})| \leq M_1 \leq M_2$. For condition (C) of (3.25) to hold, $\left| \prod_{j=0}^{n-1} \tilde{f}'_{u, \mathbf{X}_n}(x_j) \right|$ must lie in $[1 - \gamma_n, 1 + \gamma_n]$. If this occurs for any $u_{2n-1} = \varepsilon_{2n-1} \in HB_{2n}^{\text{st}}(\tau)$, then $\left| \prod_{j=0}^{n-2} \tilde{f}'_{u, \mathbf{X}_n}(x_j) \right| \geq (1 - \gamma_n)/M_2$ for all u_{2n-1} , because this product does not depend on u_{2n-1} . Using (3.32) and the fact that $1 - \gamma_n \geq 1/2$, we get

$$\begin{aligned} \mu_{2n-1, \tau}^{\text{st}} \left\{ u_{2n-1} : \left| \prod_{j=0}^{n-1} |(\tilde{f}_{u, \mathbf{X}_n})'(x_j)| - 1 \right| \leq \gamma_n \right\} \leq \\ 2M_2 \frac{(2n-1)!}{2\tau} \frac{4\gamma_n}{\prod_{j=0}^{n-2} |x_{n-1} - x_j|^2} = 4M_2 \frac{(2n-1)!}{\tau} \frac{\gamma_n}{\prod_{j=0}^{n-2} |x_{n-1} - x_j|^2}. \end{aligned} \quad (3.33)$$

This formula estimates the ‘‘measure of hyperbolicity’’ discussed in the previous section.

By Lemma 2.2.2, we can combine all these estimates and get

$$\begin{aligned} \mu_{<n, \tau}^{\text{st}} \times \mu_{2n-1, \tau}^{\text{st}} \{ (u_0, \dots, u_{n-1}, u_{2n-1}) \in W_{<n, 1}^{u, \mathbf{X}_n} \times W_{2n-1, 1}^{u, \mathbf{X}_n} : \\ \tilde{f}_{u, \mathbf{X}_n} \text{ satisfies conditions (3.25) and } \|\tilde{f}_{u, \mathbf{X}_n}\|_{C^2} \leq M_2 \} \leq \\ 4M_2 \frac{(n-1)! \gamma_n}{\tau \prod_{j=0}^{n-2} |x_{n-1} - x_j|} \frac{(2n-1)! \gamma_n}{\tau \prod_{j=0}^{n-2} |x_{n-1} - x_j|^2} \prod_{m=0}^{n-2} \frac{m! \tilde{\gamma}_n}{\tau \prod_{j=0}^{m-1} |x_m - x_j|}, \end{aligned} \quad (3.34)$$

where the spaces $W_{<n, 1}^{u, \mathbf{X}_n}$ and $W_{2n-1, 1}^{u, \mathbf{X}_n}$ are discussed in the beginning of this section. This estimate corresponds loosely to (3.23) in the previous section. The final term

is an upper bound on measure of parameters for which \mathbf{X}_n is a $\tilde{\gamma}_n$ -pseudotrajectory for f_{u, \mathbf{X}_n} . Roughly speaking, since almost every initial point x_0 has exactly one $\tilde{\gamma}_n$ -pseudotrajectory $\mathbf{X}_n \in I_{\tilde{\gamma}_n}^n$ for each set of parameters, and the total measure of parameters in $HB_{<2n}^{\text{st}}(\tau)$ is 1, the sum over all $\tilde{\gamma}_n$ -pseudotrajectories \mathbf{X}_n associated to x_0 and $HB_{<2n}^{\text{st}}(\tau)$ of the parameter measure associated with \mathbf{X}_n should be 1. Thus the final term on the right-hand side of (3.34) also represents an upper bound on the inverse of the number of $\tilde{\gamma}_n$ -pseudotrajectories per initial point, which appears in (3.23). However, we need the upper bound to be sharp in order to cancel this term with that in (3.22), and the heuristic explanation of this paragraph is complicated by the fact that the parameterization we are using *depends on the $\tilde{\gamma}_n$ -pseudotrajectory \mathbf{X}_n* . These challenges will be resolved in the Collection Lemma of the next section.

3.4 The Distortion and Collection Lemmas

In this section we formulate the Distortion Lemma for the Newton map $\mathcal{L}_{\mathbf{X}_n}^1$, and complete the estimate of the measure of all “bad” parameters with a simple, almost periodic, but not sufficiently hyperbolic trajectory (3.13) by collecting all possible “bad” pseudotrajectories (see the Collection Lemma below).

Consider an ordered n -tuple of points $\mathbf{X}_n = \{x_j\}_{j=0}^{n-1} \in I^n$ and the Newton map $\mathcal{L}_{\mathbf{X}_n}^1 : W_{<2n,1} \rightarrow W_{<2n,1}^{u, \mathbf{X}_n}$, defined by (2.24). We now estimate the distortion of the Newton map $\mathcal{L}_{\mathbf{X}_n}^1$ as the map from the standard basis $\{\varepsilon_k\}_{k=0}^{2n-1}$ in $W_{<2n,1}$ to the Newton basis $\{u_k\}_{k=0}^{2n-1}$ in $W_{<2n,1}^{u, \mathbf{X}_n}$. It helps to have in mind the following picture characterizing the distortion of the Newton map.

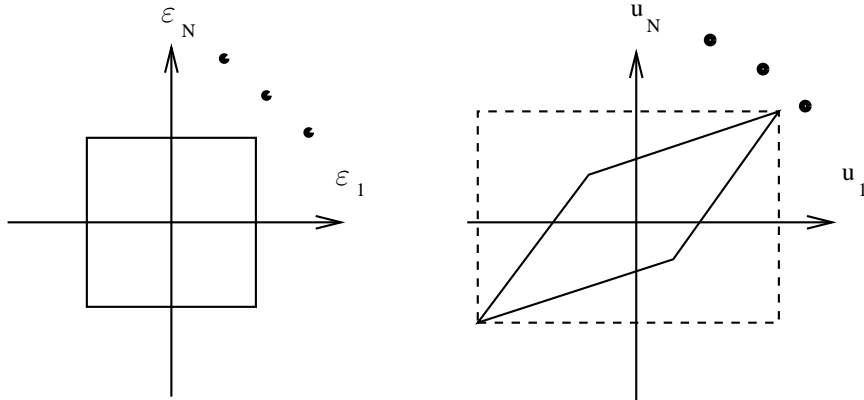


Figure 3.1: Distortion by the Newton map

The Distortion Lemma. Let $\mathbf{X}_n = \{x_j\}_{j=0}^{n-1} \in I^n$ be an ordered n -tuple of points in the interval $I = [-1, 1]$ and $\mathcal{L}_{\mathbf{X}_n}^1 : W_{<2n,1} \rightarrow W_{<2n,1}^{u, \mathbf{X}_n}$ be the Newton map, defined by (2.24). Then the image of the Brick of standard thickness $HB_{<2n}^{\text{st}}(\tau)$ with width $\tau > 0$ is contained in the Brick of standard thickness $HB_{<2n}^{\text{st}}(3\tau)$ with width 3τ :

$$\mathcal{L}_{\mathbf{X}_n}^1(HB_{<2n}^{\text{st}}(\tau)) \subset HB_{<2n}^{\text{st}}(3\tau) \subset W_{<2n,1}^{u, \mathbf{X}_n}. \quad (3.35)$$

In other words, independently of the choice of an n -tuple $\{x_j\}_{j=0}^{n-1} \in I^n$ for any $0 \leq m < 2n$, the coefficient u_m has at most the range of values $|u_m| \leq \frac{3\tau}{m!}$ in the image $\mathcal{L}_{\mathbf{X}_n}^1(HB_{<2n}^{\text{st}}(\tau))$.

Remark 3.4.1. For this lemma, the sides of the Brick $HB_{<2n}^{\text{st}}(\tau)$ have to decay at least as fast as a factorial in the order of the side, i.e., $r_n \leq \frac{\tau}{n!}$ for some $\tau > 0$. If the sides of a Brick under investigation decay, say, as an exponential function, i.e., $r_n = \exp(-Kn)$ for some $K > 0$, then the Distortion Lemma fails and there is no uniform estimate on distortion. In terms of formula (2.24), if the range of values of ε_k does not decay fast enough with k , then u_m depends significantly on ε_k with k much larger than m .

Proof: Recall that for $\{\varepsilon_m\}_{m=0}^{2n-1} \in HB_{<2n}^{\text{st}}(\tau)$ we have for each m that $|\varepsilon_m| \leq \tau/m!$. By definition (2.24) of the Newton map $\mathcal{L}_{\mathbf{X}_n}^1$ we have

$$u_m = \varepsilon_m + \sum_{k=m+1}^{2n-1} \varepsilon_k p_{k,m}(x_0, \dots, x_m \pmod{n}), \quad (3.36)$$

where $p_{k,m}$ is the homogeneous polynomial of degree $k - m$ defined by (2.22). Notice that every monomial of $p_{k,m}$ is uniformly bounded by 1, provided all points $\{x_j\}_{j=0}^{n-1}$ are bounded in absolute value by 1. Therefore, $|p_{k,m}|$ is uniformly bounded by the number of its monomials $\binom{k}{m}$. This implies that

$$\begin{aligned} |u_m| &\leq |\varepsilon_m| + \sum_{k=m+1}^{2n-1} |\varepsilon_k| \binom{k}{m} \leq \\ &\frac{\tau}{m!} \left(1 + \sum_{k=m+1}^{2n-1} \frac{1}{(k-m)!} \right) \leq \frac{3\tau}{m!}. \end{aligned} \quad (3.37)$$

This completes the proof of the Lemma. Q.E.D.

For a given n -tuple $\mathbf{X}_n = \{x_j\}_{j=0}^{n-1} \in I^n$, the parallelepiped

$$\mathcal{P}_{<2n, \mathbf{X}_n}^{\text{st}}(\tau) = \mathcal{L}_{\mathbf{X}_n}^1(HB_{<2n}^{\text{st}}(\tau)) \subset W_{<2n,1}^{u, \mathbf{X}_n} \quad (3.38)$$

is the set of parameters (u_0, \dots, u_{2n-1}) that correspond to parameters $(\varepsilon_0, \dots, \varepsilon_{2n-1}) \in HB_{<2n}^{\text{st}}(\tau)$. In other words, these are the Newton parameters *allowed by the family* (3.8) for the n -tuple \mathbf{X}_n . We already knew by Lemma 2.2.2 that $\mathcal{P}_{<2n, \mathbf{X}_n}^{\text{st}}(\tau)$ has the same volume as $HB_{<2n}^{\text{st}}(\tau)$, but the Distortion Lemma tells us in addition that the projection of $\mathcal{P}_{<2n, \mathbf{X}_n}^{\text{st}}(\tau)$ onto any coordinate axis is at most a factor of 3 longer than the projection of $HB_{<2n}^{\text{st}}(\tau)$.

Let $\mathbf{X}_m = \{x_j\}_{j=0}^{m-1}$ be the m -tuple of first m points of the n -tuple \mathbf{X}_n . We now consider which Newton parameters are allowed by the family (3.8) when \mathbf{X}_m is fixed but x_m, \dots, x_{n-1} are arbitrary. Since we will only be using the definitions below for discretized n -tuples $\mathbf{X}_n \in I_{\tilde{\gamma}_n}^n$, we consider only the (finite number of) possibilities $x_m, \dots, x_{n-1} \in I_{\tilde{\gamma}_n}$. Let

$$\pi_{<2n, \leq m}^{u, \mathbf{X}_n} : W_{<2n, 1}^{u, \mathbf{X}_n} \rightarrow W_{\leq m, 1}^{u, \mathbf{X}_m} \quad \text{and} \quad \pi_{<2n, m}^{u, \mathbf{X}_n} : W_{<2n, 1}^{u, \mathbf{X}_n} \rightarrow W_{m, 1}^{u, \mathbf{X}_m}$$

be the natural projections onto the space $W_{\leq m, 1}^{u, \mathbf{X}_m} \simeq \mathbb{R}^m$ of polynomials of degree m and the space $W_{m, 1}^{u, \mathbf{X}_m} \simeq \mathbb{R}$ of homogeneous polynomials of degree m respectively. Denote the unions over all $x_m, \dots, x_{n-1} \in I_{\tilde{\gamma}_n}$ of the images of $\mathcal{P}_{<2n, \mathbf{X}_n}^{\text{st}}(\tau)$ under the respective projections $\pi_{<2n, \leq m}^{u, \mathbf{X}_n}$ and $\pi_{<2n, m}^{u, \mathbf{X}_n}$ by

$$\begin{aligned} \mathcal{P}_{<2n, \leq m, \mathbf{X}_m}^{\text{st}}(\tau) &= \bigcup_{x_m, \dots, x_{n-1} \in I_{\tilde{\gamma}_n}} \pi_{<2n, \leq m}^{u, \mathbf{X}_n}(\mathcal{P}_{<2n, \mathbf{X}_n}^{\text{st}}(\tau)) \subset W_{\leq m, 1}^{u, \mathbf{X}_m} \\ \mathcal{P}_{<2n, m, \mathbf{X}_m}^{\text{st}}(\tau) &= \bigcup_{x_m, \dots, x_{n-1} \in I_{\tilde{\gamma}_n}} \pi_{<2n, m}^{u, \mathbf{X}_n}(\mathcal{P}_{<2n, \mathbf{X}_n}^{\text{st}}(\tau)) \subset W_{m, 1}^{u, \mathbf{X}_m}. \end{aligned} \quad (3.39)$$

For each $m < n$, the set $\mathcal{P}_{<2n, \leq m, \mathbf{X}_m}^{\text{st}}(\tau)$ is a polyhedron and $\mathcal{P}_{<2n, m, \mathbf{X}_m}^{\text{st}}(\tau)$ is a segment of length at most $6\tau/m!$ by the Distortion Lemma. Both depend only on the m -tuple \mathbf{X}_m and width τ . The set $\mathcal{P}_{<2n, \leq m, \mathbf{X}_m}^{\text{st}}(\tau)$ consists of all Newton parameters $\{u_j\}_{j=0}^m \in W_{m, 1}^{u, \mathbf{X}_m}$ that are allowed by the family (3.8) for the m -tuple \mathbf{X}_m .

For each $m < n$, we introduce the family of diffeomorphisms

$$\tilde{f}_{u(m), \mathbf{X}_m}(x) = \tilde{f}(x) + \sum_{s=0}^m u_s \prod_{j=0}^{s-1} (x - x_j), \quad (3.40)$$

where $u(m) = (u_0, \dots, u_m) \in \mathcal{P}_{<2n, \leq m, \mathbf{X}_m}^{\text{st}}(\tau)$. For each possible continuation \mathbf{X}_n of \mathbf{X}_m , the family $\tilde{f}_{u(m), \mathbf{X}_m}$ includes the subfamily of $\tilde{f}_{u, \mathbf{X}_n}$ (with $u \in \mathcal{P}_{<2n, \mathbf{X}_n}^{\text{st}}(\tau)$) corresponding to $u_{m+1} = u_{m+2} = \dots = u_{2n-1} = 0$. However, the action of $\tilde{f}_{u, \mathbf{X}_n}$ on x_0, \dots, x_m doesn't depend on u_{m+1}, \dots, u_{2n-1} , so for these points the family $\tilde{f}_{u(m), \mathbf{X}_m}$ is representative of the entire family $\tilde{f}_{u, \mathbf{X}_n}$. This motivates the definition

$$\begin{aligned} T_{<2n, \leq m, \tau}^{1, \tilde{\gamma}_n}(\tilde{f}; x_0, \dots, x_{m-1}, x_m, x_{m+1}) &= \\ \left\{ u(m) \in \mathcal{P}_{<2n, \leq m, \mathbf{X}_m}^{\text{st}}(\tau) \subset W_{\leq m, 1}^{u, \mathbf{X}_m} : \right. & \\ \left. |\tilde{f}_{u(m), \mathbf{X}_m}(x_{j-1}) - x_j| \leq \tilde{\gamma}_n \text{ for } j = 1, \dots, m+1 \right\}. & \end{aligned} \quad (3.41)$$

The set $T_{<2n, \leq m, \tau}^{1, \tilde{\gamma}_n}(\tilde{f}; x_0, \dots, x_{m-1}, x_m, x_{m+1})$ represents the set of Newton parameters $u(m) = (u_0, \dots, u_m)$ allowed by the family (3.8) for which x_0, \dots, x_{m+1} is a $\tilde{\gamma}_n$ -pseudotrajectory of $\tilde{f}_{u(m), \mathbf{X}_m}$ (and hence of $\tilde{f}_{u, \mathbf{X}_n}$ for all valid extensions u and \mathbf{X}_n of $u(m)$ and \mathbf{X}_m).

In the following lemma, we collect all possible $\tilde{\gamma}_n$ -pseudotrajectories and estimates of “bad” measure corresponding to those $\tilde{\gamma}_n$ -pseudotrajectories.

The Collection Lemma. *With the notations above, for all $x_0 \in I_{\tilde{\gamma}_n}$ the measure of the “bad” parameters satisfies*

$$\begin{aligned} & \mu_{<2n, \tau}^{\text{st}} \{B_{n, \tau}^{\text{sim}, \tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; x_0)\} \leq \\ & 6^{2n} M_2^{4n+1} \frac{(n-1)!}{\tau} \frac{(2n-1)!}{\tau} \gamma_n^{5/4} . \end{aligned} \quad (3.42)$$

Proof of the Collection Lemma: We prove by backward induction on m that for $x_0, \dots, x_m \in I_{\tilde{\gamma}_n}$,

$$\begin{aligned} & \mu_{<2n, \tau}^{\text{st}} \{B_{n, \tau}^{\text{sim}, \tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; x_0, \dots, x_m)\} \\ & \leq 6^{2n-m} M_2^{4n+1} \frac{(n-1)!}{\tau} \frac{(2n-1)!}{\tau} \\ & \mu_{<m, \tau}^{\text{st}} \{T_{<2n, \leq m-1, \tau}^{1, \tilde{\gamma}_n}(\tilde{f}; x_0, \dots, x_m)\} \gamma_n^{5/4}, \end{aligned} \quad (3.43)$$

resulting when $m = 0$ in (3.42).

Consider the case $m = n - 1$. Fix an $(n, \gamma_n/2)$ -simple n -tuple $\mathbf{X}_n = \{x_j\}_{j=0}^{n-1} \in I_{\tilde{\gamma}_n}^n$. Using formulas (3.30) and (3.33), we get

$$\begin{aligned} & \mu_{n-1, \tau}^{\text{st}} \{u_{n-1} : \left| \tilde{f}_{u, \mathbf{X}_n}(x_{n-1}) - x_0 \right| \leq M_2^n \gamma_n\} \leq \\ & \frac{(n-1)!}{\tau} \frac{M_2^n \gamma_n}{\prod_{m=0}^{n-2} |x_{n-1} - x_m|} \leq \frac{2^{1/4} M_2^n (n-1)!}{\tau} \gamma_n^{3/4} \end{aligned} \quad (3.44)$$

and

$$\begin{aligned} & \mu_{2n-1, \tau}^{\text{st}} \{u_{2n-1} : \left| \prod_{j=0}^{n-1} |(\tilde{f}_{u, \mathbf{X}_n})'(x_j)| - 1 \right| \leq M_2^{3n} \gamma_n\} \leq \\ & 4M_2 \frac{(2n-1)!}{\tau} \frac{M_2^{3n} \gamma_n}{\prod_{m=0}^{n-2} |x_{n-1} - x_m|^2} \leq \frac{2^{5/2} M_2^{3n+1} (2n-1)!}{\tau} \gamma_n^{1/2}. \end{aligned} \quad (3.45)$$

The Fubini Theorem, Lemma 2.2.2, and the definition of the product measure $\mu_{<2n, \tau}^{\text{st}}$

imply that

$$\begin{aligned}
& \mu_{<2n,\tau}^{\text{st}} \left\{ B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2; x_0, \dots, x_{n-1}) \right\} \leq \\
& \quad \mu_{<n-1,\tau}^{\text{st}} \left\{ T_{<2n,\leq n-2,\tau}^{1,\tilde{\gamma}_n}(\tilde{f}; x_0, \dots, x_{n-1}) \right\} \times \\
& \quad \mu_{n-1,\tau}^{\text{st}} \left\{ u_{n-1} : \left| \tilde{f}_{u,\mathbf{X}_n}(x_{n-1}) - x_0 \right| \leq M_2^n \gamma_n \right\} \times \prod_{s=n}^{2n-2} \mu_{s,\tau}^{\text{st}} \{ \mathcal{P}_{<2n,s,\mathbf{X}_n}^{\text{st}}(\tau) \} \times \\
& \quad \mu_{2n-1,\tau}^{\text{st}} \left\{ u_{2n-1} : \left| \prod_{j=0}^{n-1} |(f_{u,\mathbf{X}_n})'(x_j)| - 1 \right| \leq M_2^{3n} \gamma_n \right\} \\
& \leq 2^{11/4} 3^{n-1} M_2^{4n+1} \frac{(n-1)!}{\tau} \frac{(2n-1)!}{\tau} \mu_{<n-1,\tau}^{\text{st}} \{ T_{<2n,\leq n-2,\tau}^{1,\tilde{\gamma}_n}(\tilde{f}; x_0, \dots, x_{n-1}) \} \gamma_n^{5/4}
\end{aligned} \tag{3.46}$$

The last inequality follows from the Distortion Lemma, which says that for each $s = n, n+1, \dots, 2n-2$

$$\mu_{s,\tau}^{\text{st}} \{ \mathcal{P}_{<2n,s,\mathbf{X}_n}^{\text{st}}(\tau) \} \leq 3. \tag{3.47}$$

Since $2^{11/4} 3^{n-1} < 6^{n+1}$, this yields the required estimate (3.43) for $m = n-1$.

Suppose now that for $m+1$, (3.43) is true and we would like to prove it for m . Denote by $G_{<2n,m,\tau}^{1,\tilde{\gamma}_n}(\tilde{f}, u(m-1); x_0, \dots, x_m) \subset I_{\tilde{\gamma}_n}$ the set of points x_{m+1} of the $2\tilde{\gamma}_n$ -grid $I_{\tilde{\gamma}_n}$ such that the $(m+2)$ -tuple x_0, \dots, x_{m+1} is a $\tilde{\gamma}_n$ -pseudotrajectory associated to some extension $u(m) \in \mathcal{P}_{<2n,\leq m,\mathbf{X}_m}^{\text{st}}(\tau)$ of $u(m-1)$. In other words, $G_{<2n,m,\tau}^{1,\tilde{\gamma}_n}(\tilde{f}, u(m-1); x_0, \dots, x_m)$ is the set of all possible continuations of the $\tilde{\gamma}_n$ -pseudotrajectory x_0, \dots, x_m using all possible Newton parameters u_m allowed by the family (3.8).

Now if x_0, \dots, x_m is a $\tilde{\gamma}_n$ -pseudotrajectory associated to $u(m) = (u_0, \dots, u_m)$, then at most 2 values of $x_{m+1} \in I_{\tilde{\gamma}_n}$ are within $\tilde{\gamma}_n$ of $\tilde{f}_{u(m),\mathbf{X}_m}(x_m)$. Thus for fixed $u(m-1) = (u_0, \dots, u_{m-1}) \in \mathcal{P}_{<2n,\leq m-1,\mathbf{X}_n}^{\text{st}}(\tau)$, each value of $u_m \in \mathcal{P}_{<2n,m,\mathbf{X}_n}^{\text{st}}(\tau)$ corresponds to at most 2 points in $G_{<2n,m,\tau}^{1,\tilde{\gamma}_n}(\tilde{f}, u(m-1); x_0, \dots, x_m)$. It follows that

$$\begin{aligned}
& \sum_{x_{m+1} \in G_{<2n,m,\tau}^{1,\tilde{\gamma}_n}(\tilde{f}, u(m-1); x_0, \dots, x_m)} \mu_{\leq m,\tau}^{\text{st}} \{ T_{<2n,\leq m,\tau}^{1,\tilde{\gamma}_n}(\tilde{f}; x_0, \dots, x_{m+1}) \} \leq \\
& 2 \mu_{m,\tau}^{\text{st}} \{ \mathcal{P}_{<2n,m,\mathbf{X}_n}^{\text{st}}(\tau) \} \mu_{\leq m-1,\tau}^{\text{st}} \{ T_{<2n,\leq m-1,\tau}^{1,\tilde{\gamma}_n}(\tilde{f}; x_0, \dots, x_m) \}.
\end{aligned} \tag{3.48}$$

The Distortion Lemma then implies that

$$\begin{aligned}
& \sum_{x_{m+1} \in G_{<2n,m,\tau}^{1,\tilde{\gamma}_n}(\tilde{f}, u(m-1); x_0, \dots, x_m)} \mu_{\leq m,\tau}^{\text{st}} \{ T_{<2n,\leq m,\tau}^{1,\tilde{\gamma}_n}(\tilde{f}; x_0, \dots, x_{m+1}) \} \leq \\
& 6 \mu_{\leq m-1,\tau}^{\text{st}} \{ T_{<2n,\leq m-1,\tau}^{1,\tilde{\gamma}_n}(\tilde{f}; x_0, \dots, x_m) \}.
\end{aligned} \tag{3.49}$$

Inductive application of this formula completes the proof of the Collection Lemma. Q.E.D.

Proof of Proposition 3.1.2: The number of starting points $\tilde{x}_0 \subset I_{\tilde{\gamma}_n}$ for a $\tilde{\gamma}_n$ -pseudotrajectory equals $1/\tilde{\gamma}_n$. Therefore, multiplying the estimate (3.42) by $1/\tilde{\gamma}_n = M_2^{2n}/\gamma_n$ and using (3.21) we get

$$\mu_{<2n,\tau}^{\text{st}}\{B_{n,\tau}^{\text{sim},\tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2)\} \leq 6^{2n} M_2^{6n+1} \frac{(n-1)!}{\tau} \frac{(2n-1)!}{\tau} \gamma_n^{1/4}. \quad (3.50)$$

To prove the required inequality (3.13), it remains only to prove (3.17).

If a parameter ε belongs to $B_{n,\tau}^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n)$, then ε is associated with an essentially (n, γ_n) -simple, (n, γ_n) -periodic, and not (n, γ_n) -hyperbolic trajectory $\{x_k = f_\varepsilon^k(x_0)\}_{k=0}^{n-1}$. Our goal is to show that $\varepsilon \in B_{n,\tau}^{\text{sim}}(\tilde{f}, \gamma_n, M_2)$; recall the definitions (3.15) and (3.16).

The bound M_2 on the C^2 -norm of \tilde{f}_ε implies that for all $x, y \in I$ we have $|\tilde{f}_\varepsilon(x) - \tilde{f}_\varepsilon(y)| < M_2|x - y|$. Essential simplicity of the trajectory $\{x_m\}_{m=0}^{n-1}$ implies that for some $j < n$, the shifted trajectory $\{x_{j+m} = \tilde{f}_\varepsilon^{j+m}(x_0)\}_{m=0}^{n-1}$ is (n, γ_n) -simple and $(n, M_2^j \gamma_n)$ -periodic. Let's approximate the shifted trajectory $\{x_{j+m}\}_{m=0}^{n-1}$ by a $\tilde{\gamma}_n$ -pseudotrajectory $\{\tilde{x}_{j+m}\}_{m=0}^{n-1} \in I_{\tilde{\gamma}_n}^n$ associated to the (fixed above) parameter ε . Consider the $\tilde{\gamma}_n$ -pseudotrajectory $\{\tilde{x}_{j+m}\}_{m=0}^{n-1} \in I_{\tilde{\gamma}_n}^n$ starting at \tilde{x}_j . Choose \tilde{x}_j such that $|x_j - \tilde{x}_j| \leq \tilde{\gamma}_n$ and choose \tilde{x}_{j+1} such that $|\tilde{f}_\varepsilon(\tilde{x}_j) - \tilde{x}_{j+1}| \leq \tilde{\gamma}_n$. Then

$$\begin{aligned} |x_{j+1} - \tilde{x}_{j+1}| &\leq |\tilde{f}_\varepsilon(x_j) - \tilde{f}_\varepsilon(\tilde{x}_j)| + |\tilde{f}_\varepsilon(\tilde{x}_j) - \tilde{x}_{j+1}| \\ &\leq (M_2 + 1)\tilde{\gamma}_n = \frac{M_2^2 - 1}{M_2 - 1}\tilde{\gamma}_n \end{aligned} \quad (3.51)$$

By induction on m , choosing \tilde{x}_{j+m} such that $|\tilde{f}_\varepsilon(\tilde{x}_{j+m-1}) - \tilde{x}_{j+m}| \leq \tilde{\gamma}_n$, we have

$$\begin{aligned} |x_{j+m} - \tilde{x}_{j+m}| &\leq |\tilde{f}_\varepsilon(x_{j+m-1}) - \tilde{f}_\varepsilon(\tilde{x}_{j+m-1})| + |\tilde{f}_\varepsilon(\tilde{x}_{j+m-1}) - \tilde{x}_{j+m}| \\ &\leq \frac{M_2^{m+1} - 1}{M_2 - 1}\tilde{\gamma}_n \end{aligned} \quad (3.52)$$

Using this estimate with $m = n - 1$, we have

$$\begin{aligned} |\tilde{f}_\varepsilon(\tilde{x}_{j+n-1}) - \tilde{x}_j| &\leq |\tilde{f}_\varepsilon(\tilde{x}_{j+n-1}) - \tilde{f}_\varepsilon(x_{j+n-1})| + \\ |\tilde{f}_\varepsilon(x_{j+n-1}) - x_j| + |x_j - \tilde{x}_j| &\leq M_2^{n+1}\tilde{\gamma}_n + M_2^{n-1}\gamma_n \leq M_2^n\gamma_n. \end{aligned} \quad (3.53)$$

So, the $\tilde{\gamma}_n$ -pseudotrajectory $\{\tilde{x}_{j+m}\}_{m=0}^{n-1}$ is $(n, M_2^n \gamma_n)$ -periodic.

Next, since $\{x_{j+m}\}_{m=0}^{n-1}$ is (n, γ_n) -simple, this means $\prod_{m=0}^{n-2} |x_{j+n-1} - x_{j+m}| \geq \gamma_n^{1/4}$. Each term in the product must then be at least $2^{-(n-2)}\gamma_n^{1/4}$. For $m \leq n-1$, we have already shown that $|x_{j+m} - \tilde{x}_{j+m}| \leq M_2^n \tilde{\gamma}_n$. Then recalling that $\tilde{\gamma}_n = M_2^{-2n} \gamma_n$,

$$\begin{aligned} |\tilde{x}_{j+n-1} - \tilde{x}_{j+m}| &\geq |x_{j+n-1} - x_{j+m}| - 2M_2^n \tilde{\gamma}_n \\ &\geq \frac{2^{-(n-2)}\gamma_n^{1/4} - 2M_2^n \tilde{\gamma}_n}{2^{-(n-2)}\gamma_n^{1/4}} |x_{j+n-1} - x_{j+m}| \\ &= (1 - 2^{n-1}M_2^{-n}\gamma_n^{3/4}) |x_{j+n-1} - x_{j+m}| \\ &\geq (1 - \gamma_n^{3/4}) |x_{j+n-1} - x_{j+m}|. \end{aligned} \tag{3.54}$$

Since $\gamma_n \leq \gamma_n(C, \delta) = \exp(-Cn^{1+\delta})$ with $C > 2$, a simple calculation shows that $(1 - \gamma_n^{3/4})^{n-1} \geq 2^{-1/4}$. Then taking the product of (3.54) over $m = 0, 1, \dots, n-2$ proves that the approximating pseudotrajectory $\{\tilde{x}_{j+m}\}_{m=0}^{n-1}$ is $(n, \gamma_n/2)$ -simple.

Now consider the difference of derivatives

$$\left| \prod_{m=0}^{n-j-1} \tilde{f}'_\varepsilon(\tilde{x}_{j+m}) \prod_{m=0}^{j-1} \tilde{f}'_\varepsilon(\tilde{x}_{n+m}) - \prod_{m=0}^{n-1} \tilde{f}'_\varepsilon(x_m) \right|. \tag{3.55}$$

Since $\|\tilde{f}\|_{C^2} \leq M_2$, for $0 \leq m \leq n-j-1$ we have

$$|\tilde{f}'_\varepsilon(\tilde{x}_{j+m}) - \tilde{f}'_\varepsilon(x_{j+m})| \leq M_2 |\tilde{x}_{j+m} - x_{j+m}| \leq M_2^{n+1} \tilde{\gamma}_n \leq \gamma_n \tag{3.56}$$

and for $0 \leq m \leq j-1$ we have

$$\begin{aligned} |\tilde{f}'_\varepsilon(\tilde{x}_{n+m}) - \tilde{f}'_\varepsilon(x_m)| &\leq M_2 |\tilde{x}_{n+m} - x_m| \leq \\ M_2 (|\tilde{x}_{n+m} - x_{n+m}| + |x_{n+m} - x_m|) &\leq M_2^{2n} \tilde{\gamma}_n + M_2^n \gamma_n \leq 2M_2^n \gamma_n. \end{aligned} \tag{3.57}$$

Then since $|\tilde{f}'_\varepsilon(x)| \leq M_2$ for all $x \in I$, we get that the difference of derivatives (3.55) is bounded by $2nM_2^{2n-1}\gamma_n \leq M_2^{3n-1}\gamma_n$. Therefore, if the initial exact trajectory $\{x_k\}_{k=0}^{n-1}$ is not (n, γ_n) -hyperbolic, then the pseudotrajectory is not $(n, M_2^{3n}\gamma_n)$ -hyperbolic, and the parameter ε belongs to the set $B_{n,\tau}^{\text{sim}, \tilde{\gamma}_n}(\tilde{f}, \gamma_n, M_2)$. Q.E.D.

3.5 Discretization Method for trajectories with a gap

Recall that we consider the C^2 -smooth 1-dimensional model, defined in Section 3.1. Our goal is to estimate the measure of the set (3.9) of all “bad” parameters $B_{n,\tau}(C, \delta, \tilde{f}, \gamma_n)$. This set belongs to the union (3.12) of “bad” parameters $B_{n,\tau}^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n)$ and $B_{n,\tau}^{\text{non}}(C, \delta, \tilde{f}, \gamma_n)$

associated with essentially simple and essentially nonsimple almost periodic points. In the last three sections, we developed the Discretization Method and estimated the measure of the set $B_{n,\tau}^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n)$ associated with essentially simple almost periodic trajectories (3.13). In this and the next section, we consider “bad” parameters $B_{n,\tau}^{\text{non}}(C, \delta, \tilde{f}, \gamma_n)$ associated with essentially nonsimple almost periodic trajectories (3.11) and get an estimate on the measure of this set.

Proposition 3.5.1. *Let $\{\tilde{f}_\varepsilon\}_{\varepsilon \in HB_{<2n}^{\text{st}}(\tau)}$ be the family of polynomial perturbations (3.8) and M_2 be an upper bound on the C^2 -norm of the family. Then with the notation above, for any $\delta > 0$, $C > 100\delta^{-1} \log M_2$, $\tau > 0$ and a sufficiently small positive γ_n , e.g., $\gamma_n \leq \gamma_n(C, \delta)$, we have the following estimate on the measure of parameters associated with maps \tilde{f}_ε with an (n, γ_n) -periodic, essentially non- (n, γ_n) -simple, but not (n, γ_n) -hyperbolic point:*

$$\mu_{<2n,\tau}^{\text{st}}\{B_{n,\tau}^{\text{non}}(C, \delta, \tilde{f}, \gamma_n)\} \leq 16n^{1+\delta} 6^{2n} M_2^{10n+1} \exp(-Cn/200). \quad (3.58)$$

Remark 3.5.2. *Though we have stated Proposition 3.5.1 only for the C^2 case in \mathbb{R} , we will make our upcoming definitions for the general $C^{1+\rho}$ case in \mathbb{R}^N so that they may be used later. For the time being, the reader may keep in mind the case $\rho = N = 1$. The scheme of the proof of this Proposition is in Section 3.5.2.*

According to the decomposition (2.42), the set $B_{n,\tau}^{\text{non}}(C, \delta, \tilde{f}, \gamma_n)$ of parameters associated with essentially nonsimple trajectories can be decomposed into a finite union of sets of parameters with a trajectory that has a weak gap. This decomposition follows from Theorem 2.4.6. We need to improve this theorem. Let us sharpen Definition 2.4.3 of a weak gap for almost periodic trajectories.

Definition 3.5.3. *Let $g \in \text{Diff}^{1+\rho}(B^N)$ be a $C^{1+\rho}$ -smooth diffeomorphism of the ball B^N (respectively $g \in C^{1+\rho}(I, I)$ be a $C^{1+\rho}$ -smooth map of the interval I) and let $\bar{\mathbf{r}} = \{r_k\}_{k=0}^\infty$ be a nonincreasing sequence of sizes of the Brick $HB^N(\bar{\mathbf{r}})$ that tend to zero, and $D > 2$ be some number. A point $x_0 \in B^N$ (respectively $x_0 \in I$) or a trajectory $x_0, \dots, x_{n-1} = g^{n-1}(x_0)$ in B^N (respectively I) of length n has a (D, n, r_{2k}) -gap at a point $x_k = g^k(x_0)$ if*

$$|x_k - x_0| \leq \min \left\{ D^{-n \log_2 n} \min_{0 < j \leq k-1} |x_0 - x_j|, r_{2k}^{4(N+N^2)}, \left(\prod_{j=0}^{k-2} |x_{k-1} - x_j| \right)^{4(N+2)} \right\}. \quad (3.59)$$

In the case of the model when $r_k = \frac{\tau}{k!}$, we denote a (D, n, r_{2k}) -gap by a (D, n, τ) -gap.

This definition is designed to fit the induction over the period n outlined in Chapter 2. Recall that the first term in the minimum (3.59) corresponds to Definition 2.4.3 of a weak (D, n) -gap. For $D = M_{1+\rho}^{30/\rho}$, if a trajectory $\{x_j\}_{j=0}^{n-1}$ is $(n, |x_k - x_0|)$ -periodic and has a weak (D, n) -gap at x_k , then by Lemma 2.4.5 the fraction $p = n/k$ is an integer, and one can split the trajectory of length n into p almost identical parts of length k each. Then by perturbing the linearization $dg_u^k(x_0)$ at x_{k-1} using the family $g_u(x) = g(x) + u(x - x_{k-1}) \prod_{j=0}^{k-2} (x - x_j)^2$, one can reach sufficient hyperbolicity for $dg_u^k(x_0)$. Moreover, the first and the third terms in the minimum (3.59) allow us to extract sufficient hyperbolicity of $dg_u^n(x_0)$ from sufficient hyperbolicity of $dg_u^k(x_0)$. Roughly, it is because we have $dg_u^n(x_0) \approx (dg_u^k(x_0))^p$.

The second and the third terms guarantee that with respect to the normalized Lebesgue measure $\mu_{\mathbb{F}}^N$, “bad” parameters occupy a small measure set; loosely speaking, they counteract terms like $(2n - 1)!/\tau$ and $\prod_{j=0}^{n-2} |x_{n-1} - x_j|^{-1}$ in (3.30) and (3.33).

Recall that in our notation, $\gamma_n(C, \delta) = \exp(-Cn^{1+\delta})$ for each $n \in \mathbb{Z}_+$. It is important to have in mind that the scale is $\gamma_k(C, \delta)$ not $\gamma_n(C, \delta)$ in the definition below.

Definition 3.5.4. *Let $g \in \text{Diff}^{1+\rho}(B^N)$ be a $C^{1+\rho}$ -smooth diffeomorphism of the ball B^N (respectively $g \in C^{1+\rho}(I, I)$ be a $C^{1+\rho}$ -smooth map of the interval I) for some $\rho > 0$. Let also $C > 0$, $\delta > 0$ and $k < n$ be positive integers. We say that a point x_0 has a (k, n, C, δ, ρ) -leading saddle if $|x_0 - x_k| \leq n^{-1/\rho} \gamma_k^{4N/\rho}(C, \delta)$. Also if x_0 is $(n, \gamma_n^{1/\rho}(C, \delta))$ -periodic, we say that x_0 has no (n, C, δ, ρ) -leading saddles if for all $k < n$ we have that x_0 has no (k, n, C, δ, ρ) -leading saddles.*

Lemma 3.5.5. *Let a $C^{1+\rho}$ -smooth diffeomorphism (respectively $C^{1+\rho}$ -smooth map) g satisfy the Inductive Hypothesis of some order $n - 1$ with some constants $\rho > 0$, $C > 30\rho^{-1} \log M_{1+\rho}$, and $\delta > 0$, i.e., $g \in IH(n - 1, C, \delta, \rho)$. Assume that g has a point $x_0 \in B^N$ (respectively $x \in I$) that has a (k, n, C, δ, ρ) -leading saddle. Then there is a periodic point $x^* = g^k(x^*)$ of period k such that $|x^* - x_0| \leq 2n^{-1/\rho} \gamma_k^{3N/\rho}(C, \delta)$. Moreover, by the Inductive Hypothesis, x^* is $(k, \gamma_k(C, \delta))$ -hyperbolic.*

This lemma follows from a lemma in Part II of this paper [K5] that is used to prove Theorem 2.4.6 and the Shift Theorem below. It turns out that Theorem 2.4.6 can be improved to

The Shift Theorem. *Let $g \in \text{Diff}^{1+\rho}(B^N)$ be a $C^{1+\rho}$ -smooth diffeomorphism (respectively $g \in C^{1+\rho}(I, I)$ be a $C^{1+\rho}$ -smooth map of the interval I) with some $\rho > 0$, $M_{1+\rho} = \max\{\|g^{-1}\|_{C^1}, \|g\|_{C^{1+\rho}}, 2^{1/\rho}\}$, and let $\mathbf{r} = \{r_k\}_{k=0}^\infty$ be an admissible sequence*

(see Definition 1.3.1). Assume that g satisfies the Inductive Hypothesis of some order $n - 1$ with some constants $\delta > 0$, $\rho > 0$, and $C > 100\rho^{-1}\delta^{-1}\log M_{1+\rho}$, i.e., $g \in IH(n - 1, C, \delta, \rho)$. If a point $x_0 \in B^N$ (respectively $x_0 \in I$) is $(n, \gamma_n^{1/\rho}(C, \delta))$ -periodic, then either x_0 is $(n, \gamma_n(C, \delta))$ -hyperbolic, or for some $j < n \log_2 n$ the point $x_j = g^j(x_0)$ has no (n, C, δ, ρ) -leading saddles.

Moreover, if x_0 has no (n, C, δ, ρ) -leading saddles, then either it is $(n, \gamma_n(C, \delta))$ -simple or it is $(k, \gamma_k(C, \delta))$ -simple and has a (D, n, r_{2k}) -gap at x_k for some k dividing n and $D = \max\{M_{1+\rho}^{30/\rho}, \exp(C/100)\}$.

Remark 3.5.6. *The Shift Theorem is the key for splitting all almost periodic trajectories of period n into groups and is the main reason why the estimate on the number of periodic points is $\exp(Cn^{1+\delta})$, not say $\exp(Cn \log n)$. In view of the importance of the theorem, we present an outline of its proof.*

Outline of the proof of the Shift Theorem: Let x_0 be $(n, \gamma_n^{1/\rho}(C, \delta))$ -periodic.

- If x_0 has a (k, n, C, δ, ρ) -leading saddle, then by Lemma 3.5.5 there is a periodic point $x^* = g^k(x^*)$ such that $|x_0 - x^*| \leq 2n^{-1/\rho}\gamma_n^{3N/\rho}(C, \delta)$, and by the Inductive Hypothesis x^* is $(k, \gamma_k(C, \delta))$ -hyperbolic. If x_{pk} remains in the $2n^{-1/\rho}\gamma_n^{3N/\rho}(C, \delta)$ -neighborhood of x^* for $1 \leq p < n/k$, then the approximation $df^n(x_0) \approx df^n(x^*) = (df^k(x^*))^{n/k}$ is good enough to extract from $(k, \gamma_k(C, \delta))$ -hyperbolicity of x^* sufficient hyperbolicity of $df^n(x_0)$. (If n/k is not an integer, we use a similar argument with k replaced by $\gcd(n, k)$; the Euclidean Algorithm in Part II of this paper [K5] shows that x_0 is almost $\gcd(n, k)$ -periodic.)
- If x_0 has a (k, n, C, δ, ρ) -leading saddle, but x_{pk} leaves the $2n^{-1/\rho}\gamma_n^{3N/\rho}(C, \delta)$ -neighborhood of x^* for some $p < n/k$, then we show (assuming k to be the smallest almost-period of x_0 with this property) that x_{pk} has no (k', n, C, δ, ρ) -leading saddles for $k' < 2k$. We then proceed inductively, either proving $(k', \gamma_{k'}(C, \delta))$ -hyperbolicity of x_{pk} using a k' -leading saddle at x_{pk} for some $k' \geq 2k$, or concluding that with a further shift we can eliminate all leading saddles of period up to $2k'$. Sufficient $(k', \gamma_{k'}(C, \delta))$ -hyperbolicity of x_{pk} implies the necessary $(n, \gamma_n(C, \delta))$ -hyperbolicity of x_0 , because x_{pk} is still quite close to x_0 . Either we prove the necessary hyperbolicity at some step of the induction, or after at most $\log_2 n$ shifts we eliminate all (n, C, δ, ρ) -leading saddles.
- In the case we can not prove $(n, \gamma_n(C, \delta))$ -hyperbolicity of x_0 using the arguments above, for some $j < p \log_2 n < n \log_2 n$ we have that $x_j = \tilde{x}_0$ has no

(n, C, δ, ρ) -leading saddles. Notice that \tilde{x}_0 is $(n, M_{1+\rho}^{n \log_2 n} \gamma_n^{1/\rho}(C, \delta))$ -periodic. Put $D = \max\{M_{1+\rho}^{30/\rho}, \exp(C/100)\}$. It turns out that \tilde{x}_0 can have at most N weak (D, n) -gaps at $\tilde{x}_{k_1}, \dots, \tilde{x}_{k_s}$ for $k_1 < \dots < k_s < n$, $s < N$. In the 1-dimensional case it trivializes to have at most one weak (D, n) -gap. The reason is that each weak (D, n) -gap x_{k_j} after the first one at k_1 implies that the linearization $df^{k_1}(\tilde{x}_0)$ has to have an almost eigenvalue that is a k_j/k_1 -root of unity with almost eigenvector $(\tilde{x}_{k_j} - \tilde{x}_0)$. Heuristically, this follows from replacing f^{k_1} in a neighborhood of \tilde{x}_0 by its linearization. Once we know that there are no weak (D, n) -gaps, the Euclidian Algorithm gives that

$$\min_{0 < j \leq n-1} |\tilde{x}_0 - \tilde{x}_j| \geq D^{-n \log_2 n}. \quad (3.60)$$

If there exist weak (D, n) -gap, then there are at most N of them. If we can prove that for any $k < n$ absence of $(D, n, r_{2k'})$ -gaps for $k' < k$ implies x_0 is $(k, \gamma_n(C, \delta))$ -simple, then occurrence of (D, n, r_{2k}) -gap at x_k completes the proof of this part of the Theorem. Assuming now that each of weak (D, n) -gaps is not a (D, n, r_{2k}) -gap for the corresponding $k < n$ we get a lower bound on $\min_{0 < j \leq n-1} |\tilde{x}_0 - \tilde{x}_j|$.

At this point the proof follows the following scheme. First, we discuss the case of no weak (D, n) -gaps and show that x_0 is $(n, \gamma_n(C, \delta))$ -simple. It turns out that the proof of $(n, \gamma_n(C, \delta))$ -simplicity is stable enough to handle the case when x_0 has no (C, n, δ, ρ) -leading saddles and $s(\leq N)$ weak (D, n) -gaps which are not (D, n, r_{2k}) -gap for the corresponding $k < n$. Thus, we extend calculations of the former one to the latter case.

- Consider concentric balls B_k centered at \tilde{x}_0 of radii $r_k = D^{-k \log_2 k}$. Pigeon-hole arguments show that the number of visits m_k of $\{\tilde{x}_j\}_{j=1}^{n-1}$ to B_k is not too large. Otherwise, there are consecutive visits \tilde{x}_j and \tilde{x}_{j+m} to B_k with m being small. Thus, \tilde{x}_j has an (m, n, C, δ, ρ) -leading saddle and, therefore, a periodic point $\tilde{x}_j^* = g^k(\tilde{x}_j^*)$ close to \tilde{x}_j and also to \tilde{x}_0 . This implies that \tilde{x}_0 has an (m, n, C, δ, ρ) -leading saddle too. This is a contradiction.

To make estimates more precise let

$$A_k = \{1 \leq j < n : |\tilde{x}_0 - \tilde{x}_j|\}$$

be the collection of indices of points of the trajectory $\{\tilde{x}_j\}_{j=0}^n$ which visit the

r_k -ball B_k around \tilde{x}_0 . In the part II of this paper [K5] we show that

$$m_k = \#\{A_k\} \leq \frac{16NC}{\rho \log D} \frac{n}{(k \log_2 k)^{\frac{1}{1+\delta}}}. \quad (3.61)$$

- Knowing that there are no visits to B_n by (3.60) and a bound on the number of visits m_k to B_k for $1 \leq k < n$, we can get a lower estimate on $\prod_{j=1}^{n-1} |\tilde{x}_0 - \tilde{x}_j|$ according the following scheme. By (3.60) we have $A_k = \emptyset$ and $m_k = 0$. Rewrite the product of distance as follows

$$\prod_{j=1}^{n-1} |\tilde{x}_j - \tilde{x}_0| = \prod_{j \notin A_1} |\tilde{x}_j - \tilde{x}_0| \prod_{j \in A_1 \setminus A_2} |\tilde{x}_j - \tilde{x}_0| \cdots \prod_{j \in A_{n-1}} |\tilde{x}_j - \tilde{x}_0|. \quad (3.62)$$

By definition of A_k for each $j \in A_{k-1} \setminus A_k$ we have $|\tilde{x}_0 - \tilde{x}_j| \geq r_k$. Put $a_k = k \log_2 k$. Then the product (3.62) admits the following lower bound

$$\begin{aligned} & r_1^{k-1-m_1} r_2^{m_1-m_2} \cdots r_k^{m_{k-1}-m_k} = \\ & \exp\left(-\log D [a_0(k-1-m_1) + \right. \\ & \left. a_1(m_1-m_2) + \cdots + a_k(m_{k-1}-m_k)]\right) \end{aligned} \quad (3.63)$$

Using Abel's resummation we can rewrite the last expression in the form

$$\exp(-\log D [m_1(a_1 - a_0) + m_2(a_2 - a_1) + \cdots + m_k(a_k - a_{k-1})]). \quad (3.64)$$

By definition of a_k we have $a_{k+1} - a_k \leq 2 \log_2 k$. Using inequality (3.61) above we get the following lower bound for the product (3.64)

$$\begin{aligned} \exp\left(-\frac{32NC}{\rho} n \log_2 n \sum_{k=1}^n k^{-\frac{1}{1+\delta}}\right) & \geq \\ & \geq \frac{1}{n} \exp\left(-\frac{64NC}{\rho} n^{1+\frac{\delta}{1+\delta}} \log_2 n\right) \geq \\ & \geq \exp(-Cn^{1+\delta}), \end{aligned} \quad (3.65)$$

where $16N n^{-\frac{\delta^2}{1+\delta}} \log_2 n < \rho$. This show that \tilde{x}_0 is $(n, \gamma_n(C, \delta))$ -simple.

- Similar estimates to (3.61) hold in the case \tilde{x}_0 has a weak (D, n) -gap at \tilde{x}_k . It provides a lower bound on

$$\prod_{j=0}^{k-1} |\tilde{x}_0 - \tilde{x}_j| \geq \exp \left(-\frac{6C}{N\rho} k n^{\frac{\delta}{1+\delta}} \log_2 n \right). \quad (3.66)$$

It turns out that if $k^{1+\delta} < \frac{\rho \log D}{2C} n$ and \tilde{x}_0 has a weak (D, n) -gap at \tilde{x}_k , then \tilde{x}_0 is $(n, \gamma_k(C, \delta))$ -hyperbolic. Indeed, in this case $|\tilde{x}_0 - \tilde{x}_k| < D^{-n} < \gamma_k^{1/\rho}(C, dt)$ and, therefore, by inductive hypothesis \tilde{x}_0 is $(k, \gamma_k(C, \delta))$ -hyperbolic. Points \tilde{x}_k and \tilde{x}_0 are so close that n is divisible by k we can approximate $df^n(\tilde{x}_0)$ with $(df^k(\tilde{x}_0))^{n/k}$ as we do in the first item of the proof.

Suppose we have weak (D, n) -gaps at \tilde{x}_{k_1} and \tilde{x}_{k_2} for some $k_1 < k_2 \leq n$ and there is no (D, n, r_{2k}) -gap at \tilde{x}_{k_1} . Then \tilde{x}_{k_1} is so close to \tilde{x}_0 that the Euclidean Algorithm from Part II of this paper [K5] implies that k_2 is divisible by k_1 . Denote $p_1 = k_2/k_1$. Moreover, \tilde{x}_{k_1} is so close to \tilde{x}_0 that the following approximation holds true:

$$\prod_{j=1}^{k_2} |\tilde{x}_0 - \tilde{x}_j| \geq \frac{1}{2} \prod_{s=1}^{p_1} |\tilde{x}_0 - \tilde{x}_{sk_1}| \left(\prod_{j=1}^{k_1} |\tilde{x}_0 - \tilde{x}_j| \right)^{p_1} \quad (3.67)$$

Absence of (D, n, r_{2k_1}) -gap at x_{k_1} shows that

$$\begin{aligned} & |\tilde{x}_0 - \tilde{x}_{sk_1}| \geq \\ & \geq \min \left\{ D^{-n \log_2 n} \min_{0 < j \leq k_1-1} |\tilde{x}_0 - \tilde{x}_j|, r_{2k_1}^{4(N+N^2)}, \left(\prod_{j=0}^{k_1-2} |\tilde{x}_{k-1} - \tilde{x}_j| \right)^{4(N+2)} \right\} \end{aligned}$$

Since there is a (D, n) -weak gap at \tilde{x}_{k_2} we have

$$\min_{1 \leq s \leq p_1-1} |\tilde{x}_0 - \tilde{x}_{sk_1}| \geq D^{-n} |\tilde{x}_0 - \tilde{x}_{k_1}| \quad (3.68)$$

Combining (3.66), (3.67), and (3.68) we get a lower bound on $\prod_{j=1}^{k_2-1} |\tilde{x}_0 - \tilde{x}_j|$. To see that such a lower bound has the same form as (3.66) notice that both terms in

the right-hand side of (3.67) are bounded from below by $\exp(-CKkn^{\frac{\delta}{1+\delta}} \log_2 n)$ for some constant $K = K(N, \rho, \delta)$ and $k_2^{1+\delta} > \frac{\rho \log D}{2C} n$. Therefore, $\prod_{j=1}^{k_2-1} |\tilde{x}_0 - \tilde{x}_j|$ has a lower bound of the same form $\exp(-CK'kn^{\frac{\delta}{1+\delta}} \log_2 n)$ for some constant $K' = K'(N, \rho, \delta)$. Iterating this arguments for weak (D, n) -gaps at $\tilde{x}_{k_1}, \tilde{x}_{k_2}, \dots, \tilde{x}_{k_s} = \tilde{x}_0$ which are not (D, n, r_{2k}) -gaps we obtain lower bound on

$$\prod_{j=1}^{n-1} |\tilde{x}_0 - \tilde{x}_j| \geq \exp(-CK''kn^{\frac{\delta}{1+\delta}} \log_2 n) \quad (3.69)$$

for some constant $K'' = K''(N, \rho, \delta)$. This proves $(n, \gamma_n(C))$ -simplicity of \tilde{x}_0 .

- Unfortunately, using this scheme *only for* $\gamma_n(C, \delta) = \exp(-Cn^{1+\delta})$ we can prove that above $\tilde{x}_0 = x_j$ is $(n, \gamma_n(C, \delta))$ -simple. For example, if $\gamma_n(C, \delta) = \exp(-Cn \log n)$, then \tilde{x}_0 as above might be non- $(n, \gamma_n(C, \delta))$ -simple. In Appendix D, we construct examples of trajectories of a full shift on two symbols that have no (n, C, δ, ρ) -leading saddles and are not $(n, \gamma_n(C, \delta))$ -simple for $\gamma_n(C, \delta) = \exp(-Cn \log n)$. This shows that additional ideas are required to improve $\gamma_n(C, \delta) = \exp(-Cn^{1+\delta})$ to $\gamma_n(C, \delta) = \exp(-Cn \log n)$ or better.

3.5.1 Decomposition of nonsimple parameters into groups

With the notations of the general problem (2.42) and (2.43), for any $D > 2$ we introduce the set of parameters associated with an almost periodic point of period n having a gap at the k -th point of its trajectory:

$$\begin{aligned} B_n^{\text{gap}(k)}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}; D) &= \{\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}}) : \tilde{f}_{\vec{\varepsilon}} \in IH(n-1, C, \delta, \rho), \\ \tilde{f}_{\vec{\varepsilon}} &\text{ has an } (n, \gamma_n^{1/\rho}(C, \delta))\text{-periodic, but not } (n, \gamma_n(C, \delta))\text{-hyperbolic} \\ &\text{ point } x_0 \text{ with a } (D, n, r_{2k})\text{-gap at } x_k = \tilde{f}_{\vec{\varepsilon}}^k(x)\} \end{aligned} \quad (3.70)$$

Choose $D = \max\{M_{1+\rho}^{30/\rho}, \exp(C/100)\}$. Theorem 2.4.6 implies existence of inclusions (2.42) and (2.43). Similarly, the Shift Theorem implies that the following inclusions hold:

$$\begin{aligned} B_n^{\text{non}}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}) &\subseteq \cup_{k|n} B_n^{\text{gap}(k)}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}; D); \\ B_n(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}) &\subseteq B_n^{\text{sim}}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}) \cup \left(\cup_{k|n} B_n^{\text{gap}(k)}(C, \delta, \rho, \vec{\mathbf{r}}, \tilde{f}; D) \right). \end{aligned} \quad (3.71)$$

Return to our C^2 -smooth 1-dimensional model, $N = \rho = 1$. To fit the notation of the model, for a sufficiently small γ_n , e.g. $\gamma_n \leq \gamma_n(C, \delta)$, we introduce the set

$$\begin{aligned} B_{n,\tau}^{\text{gap}(k)}(C, \delta, \tilde{f}, \gamma_n; D) &= \{\varepsilon \in HB_{<2n}^{\text{st}}(\tau) : \tilde{f}_\varepsilon \in IH(n-1, C, \delta, 1), \\ \tilde{f}_\varepsilon &\text{ has an } (n, \gamma_n)\text{-periodic, but not } (n, \gamma_n)\text{-hyperbolic} \\ &\text{ point } x_0 \text{ with a } (D, n, \tau)\text{-gap at } x_k = \tilde{f}_\varepsilon^k(x)\} \end{aligned} \quad (3.72)$$

and rewrite inclusion (3.71) in the notation of the 1-dimensional model

$$\begin{aligned} B_{n,\tau}^{\text{non}}(C, \delta, \tilde{f}, \gamma_n) &\subseteq \cup_{k|n} B_{n,\tau}^{\text{gap}(k)}(C, \delta, \tilde{f}, \gamma_n; D); \\ B_{n,\tau}(C, \delta, \tilde{f}, \gamma_n) &\subseteq B_{n,\tau}^{\text{sim}}(C, \delta, \tilde{f}, \gamma_n) \cup \left(\cup_{k|n} B_{n,\tau}^{\text{gap}(k)}(C, \delta, \tilde{f}, \gamma_n; D) \right). \end{aligned} \quad (3.73)$$

This inclusion shows that to prove Proposition 3.5.1, it is sufficient to prove

Proposition 3.5.7. *With the conditions of Proposition 3.5.1, let k be some integer that divides n . Then for $D = \max\{M_2^{30}, \exp(C/100)\}$ and a sufficiently small positive γ_n , e.g., $\gamma_n \leq \gamma_n(C, \delta)$, we have the following estimate on the measure of maps \tilde{f}_ε 's associated with an (n, γ_n) -periodic, but not (n, γ_n) -hyperbolic point that has a (D, n, τ) -gap at the k -th point of its trajectory.*

$$\begin{aligned} \mu_{<2n,\tau}^{\text{st}} \{ B_{n,\tau}^{\text{gap}(k)}(C, \delta, \tilde{f}, \gamma_n; D) \} &\leq \\ &\leq 128n^\delta 6^{2n} M_2^{10n+1} \exp(-Cn \log_2 n / 200). \end{aligned} \quad (3.74)$$

Let us give a name to the right-hand side of the inequality (3.59) for the Brick of parameters $HB_{<2k}^{\text{st}}(\tau)$ of the standard thickness. Let $\mathbf{X}_k = \{x_j\}_{j=0}^{k-1}$ be a k -tuple and $D > 2$ be some number. We call the number

$$\begin{aligned} \Delta_{k,n,\tau}^{\text{st}} \{ \mathbf{X}_k, D \} &= \\ \min \left\{ D^{-n \log_2 n} \min_{0 < j \leq k-1} |x_0 - x_j|, \frac{\tau^{4(N+N^2)}}{((2k)!)^{4(N+N^2)}}, \prod_{j=0}^{k-2} |x_{k-1} - x_j|^{4(N+2)} \right\} \end{aligned} \quad (3.75)$$

the (D, n, τ) -gap number associated to the k -tuple \mathbf{X}_k and the Brick $HB_{<2n}^{\text{st}}(\tau)$ of standard thickness. In the case under current consideration, $N = 1$. Similarly, one can define the (D, n, r_{2k}) -gap number for a nonincreasing sequence $\vec{r} = \{r_m\}_{m=0}^\infty$ replacing $\tau/(2k)!$ by r_{2k} .

Let $\mathbf{X}_k(x_0, g) = \{g^j(x_0)\}_{j=0}^{k-1}$. By the definition of the (D, n, τ) -gap number, if x_0 has a (D, n, τ) -gap at x_k , then x_0 is $(k, \Delta_{k,n,\tau}^{\text{st}} \{ \mathbf{X}_k(x_0, g), D \})$ -periodic. Introduce

the set of “bad” parameters

$$\begin{aligned}
B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D) &= \{\varepsilon \in HB_{<2n}^{\text{st}}(\tau) : \tilde{f}_\varepsilon \in IH(n-1, C, \delta, 1), \\
&\tilde{f}_\varepsilon \text{ has a point } x_0 \text{ that is } (n, \gamma_n(C, \delta))\text{-periodic,} \\
&\text{has a } (D, n, \tau)\text{-gap at } x_k = \tilde{f}_\varepsilon^k(x_0), \text{ is } (k, \gamma_n(C, \delta))\text{-simple,} \\
&\text{and is not } (k, M_2^{3n} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, \tilde{f}_\varepsilon), D\})\text{-hyperbolic}\}.
\end{aligned} \tag{3.76}$$

In Section 3.6 using the Shift Theorem above we shall prove the following lemma.

Lemma 3.5.8. *Let $C > 0$, $\delta > 0$, and $k, n \in \mathbb{Z}_+$ be some positive integer, and let k divide n . Then with the notation above for any $\gamma_n \leq \gamma_n(C, \delta)$ and $D = \max\{M_2^{30}, \exp(C/100)\}$ we have*

$$B_{n,\tau}^{\text{gap}(k)}(C, \delta, \tilde{f}, \gamma_n; D) \subset B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D). \tag{3.77}$$

The proof is postponed until Section 3.6.

Remark 3.5.9. $B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D)$ depends only on properties of trajectories of length k , whereas $B_{n,\tau}^{\text{gap}(k)}(C, \delta, \tilde{f}, \gamma_n; D)$ depends on properties of those of length n . This reduces consideration of sufficiently recurrent, nonsimple, and not hyperbolic, trajectories of length n to studying only their initial “simple” parts of length k , where x_k is a point of the first sufficiently close return to x_0 with n divisible by k . In this case one can effectively perturb such a trajectory at the k -th return using the Newton family $f(x) + u_0 + u_1(x - x_0) + \cdots + u_{2k-1}(x - x_{k-1}) \prod_{j=0}^{k-2} (x - x_j)^2$.

3.5.2 Decomposition into i -th recurrent pseudotrajectories

We now split the set $B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D)$ into a finite union. Let $\tilde{\gamma}_{n,i}(C, \delta, \rho) = M_2^{4ni} \gamma_n^{1/4\rho}(C, \delta)$.

Definition 3.5.10. *Let $g \in \text{Diff}^{1+\rho}(B^N)$ be a $C^{1+\rho}$ -smooth diffeomorphism (respectively $g \in C^{1+\rho}(I, I)$ be a $C^{1+\rho}$ -smooth map), $C > 0$, $\delta > 0$, $\tau > 0$, and $D > 2$ be some constants. We call a point $x_0 \in B^N$ (respectively $x_0 \in I$) i -th recurrent with constants $(C, \delta, \rho, \tau, D)$ if for $\mathbf{X}_k(x_0, g) = \{g^j(x_0)\}_{j=0}^{n-1}$ we have*

$$\tilde{\gamma}_{n,i}(C, \delta, \rho) \leq \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, g), D\} < \tilde{\gamma}_{n,i+1}(C, \delta, \rho). \tag{3.78}$$

Remark 3.5.11. *Recall that the (D, n, r_{2k}) -gap number is defined by replacing in the definition of the (D, n, τ) -gap number $\tau/(2k)!$ by r_{2k} . Similarly, we define an i -th recurrent point with constants $(C, \delta, \rho, r_{2k}, D)$ by replacing in Definition 3.5.10 above the (D, n, τ) -gap number $\Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\}$ by the (D, n, r_{2k}) -gap number.*

For the purpose of this section, $\rho = 1$, and for brevity redenote $\tilde{\gamma}_{n,i} = \tilde{\gamma}_{n,i}(C, \delta, 1)$. Define the set of parameters from $B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D, i)$ associated to i -th order recurrent with respect to $(C, \delta, 1, D)$ trajectories that satisfy conditions (3.76):

$$B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D, i) = \{\varepsilon \in HB_{<2n}^{\text{st}}(\tau) : \tilde{f}_\varepsilon \text{ has a point } x_0 \text{ as in (3.76) and } i\text{-th recurrent with constants } (C, \delta, 1, \tau, D)\}. \quad (3.79)$$

Lemma 3.5.12. *With the notations of Lemma 3.5.8, for $L = Cn^\delta/(4 \log M_2)$ we have*

$$B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D) \subseteq \bigcup_{i=0}^{L-1} B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}; M_2, D, i). \quad (3.80)$$

Lemma 3.5.13. *Let $C > 0$, $\delta > 0$, and $n \in \mathbb{Z}_+$ be some numbers, and let $k \in \mathbb{Z}_+$ divide n . Then with the notation as above, for $D = \max\{M_2^{30}, \exp(C/100)\}$, any $\gamma_n \leq \gamma_n(C, \delta)$, and any $i \in \mathbb{Z}_+$ such that $0 \leq i < L$ we have*

$$\mu_{<2n,\tau}^{\text{st}}\{B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D, i)\} \leq 128 \cdot 6^{2k} M_2^{10n+1} D^{-n \log_2 n/2}. \quad (3.81)$$

We postpone the proof of this Lemma until the next Section.

Proofs of Propositions 3.5.1 and 3.5.7: Since there are at most n numbers $k < n$ dividing n , Proposition 3.5.1 is an easy corollary of Proposition 3.5.7. To prove Proposition 3.5.7, we combine the Shift Theorem, Lemmas 3.5.8, 3.5.12, and 3.5.13 as follows. Let γ_n be sufficiently small, e.g., $\gamma_n \leq \gamma_n(C, \delta)$, and let $D = \max\{M_2^{30}, \exp(C/100)\}$. Then

- By the Shift Theorem, the set of all parameter values $B_{n,\tau}^{\text{non}}(C, \delta, \tilde{f}, \gamma_n)$ associated to maps \tilde{f}_ε with an (n, γ_n) -periodic, essentially non- (n, γ_n) -simple, but not (n, γ_n) -hyperbolic point, is contained in the union (3.73) over all k dividing n of $B_{n,\tau}^{\text{gap}(k)}(C, \delta, \tilde{f}, \gamma_n; D)$ associated to maps \tilde{f}_ε with an (n, γ_n) -periodic, but non- (n, γ_n) -hyperbolic point with a (D, n, τ) -gap at x_k .

- By Lemma 3.5.8 we have that $B_{n,\tau}^{\text{gap}(k)}(C, \delta, \tilde{f}, \gamma_n; D)$ is contained in the set of parameters $B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D)$ associated to maps \tilde{f}_ε which have a non- $(k, \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, \tilde{f}_\varepsilon), D\})$ -hyperbolic point x_0 with a (D, n, τ) -gap at x_k .

- By Lemma 3.5.12, in turn, $B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D)$ is contained in the union of $\{B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D, i)\}_{i=0}^{L-1}$ such that the i -th set $B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D, i)$ is associated to maps \tilde{f}_ε that have a non- $(k, \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, \tilde{f}_\varepsilon), D\})$ -hyperbolic point x_0 with a (D, n, τ) -gap at x_k and such that the k -tuple $\{x_j = \tilde{f}_\varepsilon^j(x_0)\}_{j=0}^{k-1}$ is i -th recurrent.

- Application of Lemma 3.5.13 provides well enough estimates the measures of $B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D, i)$ to prove Proposition 3.5.7 and hence Proposition 3.5.1. Q.E.D.

In the next section, we shall prove Lemma 3.5.13 using the Discretization Method and then we shall prove Lemmas 3.5.8 and 3.5.12 using simple approximation arguments similar to the one given in the proof of Proposition 3.1.2.

3.6 The measure of maps \tilde{f}_ε having i -th recurrent, non sufficiently hyperbolic trajectories with a gap and proofs of auxiliary lemmas

We shall prove Lemma 3.5.13 in three steps.

Step 1. Reduction to polynomial perturbations of degree $2k - 1$.

The measure $\mu_{<2n,\tau}^{\text{st}}$ is Lebesgue product probability measure and each of its component $\mu_{m,\tau}^{\text{st}}$ is Lebesgue probability measure (see (3.3))

$$\mu_{<2n,\tau}^{\text{st}} = \mu_{<2k,\tau}^{\text{st}} \times \left(\times_{m=2k}^{2n-1} \mu_{m,\tau}^{\text{st}} \right). \quad (3.82)$$

Therefore, by Fubini/Tonelli Theorem it is sufficient to prove that

$$\begin{aligned} \mu_{<2k,\tau}^{\text{st}} \left\{ B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D, i) \cap \{(\varepsilon_{2k}, \dots, \varepsilon_{2n-1})\} \right\} \\ \leq 128 \cdot 6^{2k} M_2^{10n+1} D^{-n \log_2 n/2}. \end{aligned}$$

uniformly over $\{(\varepsilon_{2k}, \dots, \varepsilon_{2n-1})\} \in HB_{2k}^{\text{st}}(\tau) \times HB_{2k+1}^{\text{st}}(\tau) \times \dots \times HB_{2n-1}^{\text{st}}(\tau)$. To simplify notations we omit $(\varepsilon_{2k}, \dots, \varepsilon_{2n-1})$ and write as if the set of parameters $B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D, i)$ is a subset of the Brick $HB_{<2k}^{\text{st}}(\tau)$.

From now on we consider the $2k$ -parameter family of polynomial perturbations

$$\left\{ \tilde{f}_{\varepsilon(2k-1)} = \tilde{f}(x) + \sum_{m=0}^{2k-1} \varepsilon_m x^m \right\}, \quad (3.83)$$

where $\varepsilon(2k-1) = (\varepsilon_0, \dots, \varepsilon_{2k-1}) \in HB_{<2k}^{\text{st}}(\tau)$. Recall that $HB_{<2k}^{\text{st}}(\tau)$ is supplied with the Lebesgue product probability measure $\mu_{<2k,\tau}^{\text{st}}$.

Step 2. An estimate of the measure of parameters associated with a trajectory $\{x_j\}_{j=0}^{n-1}$ with a gap at x_k that is i -th recurrent and not sufficiently hyperbolic.

With the notations of Section 3.2 we give the following

Definition 3.6.1. Let $D > 2$. We say that a $\tilde{\gamma}_n$ -pseudotrajectory $\mathbf{X}_k = \{x_j\}_{j=0}^{k-1} \subset I_{\tilde{\gamma}_n}^k$ associated to some map $g : I \hookrightarrow I$ has a (D, n, τ) -gap at a point $x_k = g(x_{k-1})$ if (3.59) is satisfied and there is no $m < k$ such that x_0 has a (D, n, τ) -gap at x_m .

By analogy with (3.15), for each i satisfying $0 \leq i < L$ of Lemma 3.5.12 we define

$$\begin{aligned} B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(\tilde{f}, \tilde{\gamma}_{n,i}, M_2; D, i; x_0, \dots, x_{k-1}) &= \{\varepsilon \in HB_{<2k}^{\text{st}}(\tau) : \\ \mathbf{X}_k = \{x_m\}_{m=0}^{k-1} \subset I_{\tilde{\gamma}_{n,i}}^k \text{ is a } \tilde{\gamma}_{n,i}\text{-pseudotrajectory associated to } \varepsilon & \\ \text{that is } i\text{-th recurrent with constants } (C, \delta, 1, \tau, D), & \\ (k, 2M_2^n \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\})\text{-periodic and not } (k, 2M_2^{3n} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\})\text{-hyperbolic} & \} \end{aligned} \quad (3.84)$$

To show an analogy with the case simple trajectory case consider the following

Problem 3.6.2. Estimate the measure of $\varepsilon \in HB_{<2k}^{\text{st}}(\tau)$ for which the k -tuple $\mathbf{X}_k = \{x_j\}_{j=0}^{k-1}$ is

- A) a $\tilde{\gamma}_{n,i}$ -pseudotrajectory, i.e., $|\tilde{f}_\varepsilon(x_j) - x_{j+1}| \leq \tilde{\gamma}_{n,i}$ for $j = 0, \dots, k-2$;
- B) i -th recurrent with constants $(C, \delta, 1, \tau, D)$, i.e.,
 - $\tilde{\gamma}_{n,i} \leq \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\} < \tilde{\gamma}_{n,i+1}$;
- C) $(k, 2M_2^n \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\})$ -periodic, i.e.,
 - $|\tilde{f}_\varepsilon(x_{k-1}) - x_0| < 2M_2^n \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\}$;
- D) not $(n, 2M_2^{3n} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\})$ -hyperbolic, i.e.,
 - $\left| \prod_{j=0}^{k-1} (\tilde{f}_\varepsilon)'(x_j) - 1 \right| \leq 2M_2^{3n} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\}$;

For a fixed k -tuple of points $\mathbf{X}_k = \{x_j\}_{j=0}^{k-1}$, consider the Newton family of polynomial perturbations

$$\tilde{f}_{u,\mathbf{X}_k}(x) = \tilde{f}(x) + \sum_{m=0}^{2k-1} u_m \prod_{j=0}^{m-1} (x - x_j \pmod{k}). \quad (3.86)$$

Recalling (2.17) and Fig. 2.2, we notice that for any $m < k$ and $s > 0$ the image $\tilde{f}_{u,\mathbf{X}_k}(x_m)$ (respectively derivative $\tilde{f}'_{u,\mathbf{X}_k}(x_m)$) of (respectively at) the point x_m is independent of the Newton coefficients u_{m+s} (respectively u_{m+k+s}) with $s > 0$. This implies that the Newton coefficients u_0, \dots, u_{k-2} determine if property (A) holds.

Fix u_0, \dots, u_{k-2} . Now the Newton coefficient u_{k-1} determines if property (C) of almost periodicity holds. Once u_{k-1} is fixed, the Newton coefficient u_{2k-1} determines if property (D) of almost nonhyperbolicity holds.

Following formulas (3.27), (3.28), and formulas (3.30), (3.33) with n replaced by k and γ_n replaced by $2M^n \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\}$ for periodicity and $2M^{3n} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\}$ for hyperbolicity and, we have that for fixed \mathbf{X}_k , the measure of $(u_0, \dots, u_{k-1}, u_{2k-1})$ with conditions (3.85) is bounded as in (3.34). Then we apply the Distortion Lemma with n replaced by k . This gives an additional factor 6^k . Thus we get a bound

$$\begin{aligned} & \mu_{<2k,\tau}^{\text{st}} \left\{ u(2k-1) \in W_{<2k,1}^{u,\mathbf{X}_n} : \tilde{f}_{u,\mathbf{X}_k} \text{ satisfies conditions (3.85)} \right\} \leq \\ & 4 M_2 6^k \prod_{m=0}^{k-2} \frac{m! \tilde{\gamma}_{n,i}}{\tau \prod_{j=0}^{m-1} |x_m - x_j|} \\ & \frac{(k-1)!}{\tau} \frac{2M_2^n \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\}}{\prod_{j=0}^{k-2} |x_{k-1} - x_j|} \frac{(2k-1)!}{\tau} \frac{2M_2^{3n} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\}}{\prod_{j=0}^{k-2} |x_{k-1} - x_j|^2}. \end{aligned} \quad (3.87)$$

Consider the last two terms in the right-hand side product. Definition of the (D, n, τ) -gap number and the inequality $\min(a, b, c) \leq a^{1/2} b^{1/4} c^{1/4}$ show that these two terms are bounded by

$$M_2^{4n} \frac{k! (2k)!}{\tau \tau} \frac{(\Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\})^2}{(\prod_{j=0}^{k-2} |x_{k-1} - x_j|)^3} \leq 2M_2^{4n} D^{-n \log_2 n/2} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\} \quad (3.88)$$

Therefore,

$$\begin{aligned} & \mu_{<2k,\tau}^{\text{st}} \left\{ u(2k-1) \in W_{<2k,1}^{u,\mathbf{X}_n} : \tilde{f}_{u,\mathbf{X}_k} \text{ satisfies conditions (3.85)} \right\} \\ & \leq 32 M_2^{4n+1} 6^k \prod_{m=0}^{k-2} \frac{m! \tilde{\gamma}_n}{\tau \prod_{j=0}^{m-1} |x_m - x_j|} M_2^{4n} D^{-n \log_2 n/2} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\}. \end{aligned} \quad (3.89)$$

The next step of the proof of Lemma 3.5.13 as in the case of simple trajectories (Section 3.2) is to collect all possible “bad” pseudotrajectories and make sure that those pseudotrajectories indeed approximate sufficiently well all “bad” true trajectories.

Step 3. Collecting of “bad” i -th recurrent, not sufficiently hyperbolic trajectories $\{x_j\}_{j=0}^{k-1}$ with a gap at x_k using grids $\{I_{\tilde{\gamma}_{n,i}}\}_i$ of variable size in i .

In the case of simple trajectories in Section 3.2, we considered only one $\tilde{\gamma}_n$ -grid of a fixed size and collected all simple “bad” trajectories in the Collection Lemma (Section 3.4). In the case of i -th recurrent trajectories with a gap at x_k we define

grids $\{I_{\tilde{\gamma}_{n,i}}\}_i$ of i dependent size $\tilde{\gamma}_{n,i}$. Then we prove that $\tilde{\gamma}_{n,i}$ -pseudotrajectories approximate real i -th recurrent trajectories with a gap at x_k sufficiently well. Finally, we collect all possible i -th recurrent $\tilde{\gamma}_{n,i}$ -pseudotrajectories with a gap at x_k and sum the estimates of the measures of “bad” sets. Let’s realize this program. Recall that $\tilde{\gamma}_{n,i} = M_2^{4ni} \gamma_n(C, \delta)$ and call the i -th grid

$$I_{\tilde{\gamma}_{n,i}} = \{x \in I : \exists k \in \mathbb{Z} \text{ such that } x = (2k + 1)\tilde{\gamma}_{n,i}\} \subset I. \quad (3.90)$$

Definition 3.6.3. Let $\{x_j\}_{j=0}^{k-1} \in I_{\tilde{\gamma}_{n,i}}^k$ be a k -tuple for some $i \in \mathbb{Z}_+$. Then the k -tuple $\mathbf{X}_k = \{x_j\}_{j=0}^{k-1}$ is called i -th recurrent with constants $D > 2$ and $M_2 > 0$ if

$$\frac{1}{2} \tilde{\gamma}_{n,i} \leq \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\} \leq 2 \tilde{\gamma}_{n,i+1}. \quad (3.91)$$

For simplicity if the k -tuple $\mathbf{X}_k = \{x_j\}_{j=0}^{k-1}$ is a $\tilde{\gamma}_{n,i}$ -pseudotrajectory associated to some parameter $\varepsilon \in HB_{<2k}^{\text{st}}(\tau)$ which is i -th recurrent with some constants $D > 2$ and the C^2 -norm M_2 of the family (3.7), then we say that \mathbf{X}_k is an i -th recurrent $\tilde{\gamma}_{n,i}$ -pseudotrajectory.

Remark 3.6.4. Definition of an i -th recurrent pseudotrajectory is reasonable only for a grid of a sufficiently small size. Indeed, for any i -th recurrent trajectory we need to find a $\tilde{\gamma}_{n,i}$ -pseudotrajectory whose periodicity, hyperbolicity, and product of distances along itself approximate well enough those of the trajectory.

Define a discretized version of the set $B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D, i)$, given by (3.79) associated with all i -th recurrent trajectories of length k which has a weak (D, n, τ) -gap at x_k and is not $M_2^{3n} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, \tilde{f}_\varepsilon), D\}$ -hyperbolic.

$$\begin{aligned} & B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2, \tilde{\gamma}_{n,i}; D, i; x_0) = \\ & \cup_{\mathbf{X}_k = \{x_j\}_{j=1}^{k-1} \in I_{\tilde{\gamma}_{n,i}}^{k-1}} B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2, \tilde{\gamma}_{n,i}; D, i; x_0, \dots, x_{k-1}). \end{aligned} \quad (3.92)$$

Similarly to the case of simple trajectories, we prove that after discretization all real “bad” trajectories can be sufficiently well approximated by pseudotrajectories of a certain grid so that quantities of periodicity, hyperbolicity, existence of a gap, and product of distances along the trajectory are almost the same. Namely,

Lemma 3.6.5. In the notations as above we have

$$B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}; D, i) \subset \cup_{x_0 \in I_{\tilde{\gamma}_{n,i}}} B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2, \tilde{\gamma}_{n,i}; D, i; x_0). \quad (3.93)$$

The proof of this Lemma is very similar to the proof of Lemma 3.5.8 and is omitted.

To complete the proof of Lemma 3.5.13 we apply the Collection Lemma to (3.89) to estimate the measure of $B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2, \tilde{\gamma}_{n,i}; D, i; x_0)$. We get

$$\begin{aligned} & \mu_{<2k,\tau}^{\text{st}}\{B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2, \tilde{\gamma}_{n,i}; D, i; x_0)\} \leq \\ & \leq 32 \cdot 6^{2k} M_2^{4n+1} D^{-n \log_2 n/2} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k, D\}. \end{aligned}$$

The number of grid points is $2\gamma_{n,i}^{-1}$. By definition of i -th recurrent pseudotrajectory we have $2\gamma_{n,i}^{-1} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X} - k, D\} \leq 4M_2^{6n}$. This implies (3.81), which in turn implies Lemma 3.5.13. Remarks at the end of the last section also show that this proves Proposition 3.5.7 too. This Proposition implies Proposition 3.5.1.

Proof of Lemma 3.5.8: Fix a parameter $\varepsilon \in B_{n,\tau}^{\text{gap}(k)}(C, \delta, \tilde{f}, \gamma_n; D)$; we wish to show that $\varepsilon \in B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D)$. By definition (3.72) there is an (n, γ_n) -periodic, non- (n, γ_n) -hyperbolic point x_0 with a (D, n, τ) -gap at $x_k = \tilde{f}_\varepsilon^k(x_0)$. By definition (3.76), we want to show that x_0 is not $(k, M_2^{3n} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, \tilde{f}_\varepsilon), D\})$ -hyperbolic.

Since n is divisible by k , we can split the trajectory $\{x_j = \tilde{f}_\varepsilon^j(x_0)\}_{j=0}^{n-1}$ of length n into $p = n/k$ parts of length k each. Consider the linearization

$$(\tilde{f}_\varepsilon^n)'(x_0) = (\tilde{f}_\varepsilon^k)'(x_{(p-1)k}) \cdot \cdots \cdot (\tilde{f}_\varepsilon^k)'(x_k) \cdot (\tilde{f}_\varepsilon^k)'(x_0). \quad (3.94)$$

Definition 3.5.3 of a (D, n, τ) -gap at x_k says that $|x_0 - x_k| \leq \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, \tilde{f}_\varepsilon), D\}$, and hence for every $1 \leq j \leq p-1$ and $0 \leq s < k$ we have

$$|x_s - x_{jk+s}| \leq M_1^n \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, \tilde{f}_\varepsilon), D\}. \quad (3.95)$$

This implies that the trajectory $\{x_j\}_{j=0}^{n-1}$ of length n consists of p almost identical parts of length k each. Thus, for each $0 \leq m < n$ we have

$$\left| (\tilde{f}_\varepsilon)'(x_m) - (\tilde{f}_\varepsilon)'(x_{m \pmod{k}}) \right| \leq M_2^{n+1} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, \tilde{f}_\varepsilon), D\}. \quad (3.96)$$

Considering the product over $m = 0, 1, \dots, n-1$ of each term on the left hand side above, we get that

$$\left| (\tilde{f}_\varepsilon^n)'(x_0) - \left((\tilde{f}_\varepsilon^k)'(x_0) \right)^p \right| \leq n M_2^{2n} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, \tilde{f}_\varepsilon), D\}. \quad (3.97)$$

Therefore, since $\left| \left| (\tilde{f}_\varepsilon^n)'(x_0) \right| - 1 \right| \leq \gamma_n \leq \gamma_n(C, \delta)$, and by the proof of Lemma 3.5.12 below, $\gamma_n(C, \delta) < \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, \tilde{f}_\varepsilon), D\}$, it follows that

$$\left| \left| (\tilde{f}_\varepsilon^k)'(x_0) \right| - 1 \right| \leq \left| \left| \left((\tilde{f}_\varepsilon^k)'(x_0) \right)^p \right| - 1 \right| \leq 2n M_2^{2n} \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, \tilde{f}_\varepsilon), D\}, \quad (3.98)$$

and ε belongs to $B_{n,\tau}^{\text{gap}(k),\Delta^{\text{st}}}(C, \delta, \tilde{f}, M_2; D)$. Q.E.D.

Proof of Lemma 3.5.12: It follows from the Shift Theorem (last sentence) with $N = 1$ that if a point x_0 has no $(n, C, \delta, 1)$ -leading saddles and a (D, n, τ) -gap at x_k , then x_0 is $(k, \gamma_n(C, \delta))$ -simple. Therefore, for the (D, n, τ) -gap number (3.75) we have the following bounds

$$\gamma_n(C, \delta) = \exp(-Cn^{1+\delta}) \leq \Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, \tilde{f}_\varepsilon), D\} \leq 2D^{-n}. \quad (3.99)$$

Thus since $D = M_2^{30} \geq 2$, each such x_0 with a (D, n, τ) -gap at x_k is i -th recurrent with constants $(C, \delta, \rho, \tau, D)$ for some $i \geq 0$. If $i \geq L = Cn^\delta/(4 \log M_2)$, then $\Delta_{k,n,\tau}^{\text{st}}\{\mathbf{X}_k(x_0, \tilde{f}_\varepsilon), D\} \geq M_2^{4ni} \gamma_n(C, \delta) \geq 1 > 2D^{-n}$. This contradicts the above inequality. Q.E.D.

Proof of Lemma 3.6.5: The proof is by approximation arguments (3.94)-(3.98) similar to the proof of Lemma 3.5.8 above. Q.E.D.

Chapter 4

Comparison of the Discretization Method in 1–dimensional and N –dimensional Cases

In this section we discuss changes that one has to make and difficulties that arise while generalizing the Discretization Method for C^2 –smooth 1–dimensional maps to a Discretization Method for the $C^{1+\rho}$ –smooth N –dimensional diffeomorphisms. Many of the arguments, such as the decomposition into pseudotrajectories of Section 3.2, remain essentially the same. We focus primarily on the obstacles to generalizing the estimates of Sections 3.3 and 3.4 for simple trajectories.

4.1 Dependence of the main estimates on N and ρ

Recall that the Inductive Hypothesis (Definition 2.0.1) asserts that points that are $(n, \gamma_n^{1/\rho}(C, \delta))$ –periodic are also $(n, \gamma_n(C, \delta))$ –hyperbolic. The reason for the exponent $1/\rho$ is as follows. In the Discretization Method, we approximate trajectories of length n with $\tilde{\gamma}_n$ –pseudotrajectories on a grid. In order that γ_n –hyperbolicity of the pseudotrajectory imply hyperbolicity of the true trajectory for a $C^{1+\rho}$ map, it requires that $\tilde{\gamma}_n \leq \gamma_n^{1/\rho}$ (up to a factor exponential in n). In our heuristic estimate (3.24) on the measure of “bad” parameters, the number of initial points in the $\tilde{\gamma}_n$ –grid is (again up to an exponential factor) $\tilde{\gamma}_n^{-N} \geq \gamma_n^{-N/\rho}$. The best possible bounds one can get (for nonrecurrent trajectories) on the “measure of periodicity” and the “measure of hyperbolicity” in (3.24) are respectively the N th power of the periodicity and the

hyperbolicity γ_n . Thus, in order for the right side of (3.24) to be small, we need that the periodicity be at most roughly $\gamma_n^{1/\rho}$.

Next, we explain the exponent $1/4N$ in Definition 2.34 of simple trajectories. The actual estimate we obtain on the “measure of periodicity” discussed in the previous paragraph is not the N th power of the periodicity $\gamma_n^{N/\rho}$, but instead $\gamma_n^{N/\rho} r_{n-1}^{-N} (\prod_{j=0}^{n-2} |x_{n-1} - x_j|)^{-N}$, where $\{x_j\}_{j=0}^{n-1}$ is a trajectory and r_{n-1} is the width of an admissible Hilbert Brick in the direction of degree $n-1$ polynomials (see Definition 1.3.1). This estimate reduces to (3.30) in the case $N = \rho = 1$ and $r_{n-1} = \tau/(n-1)!$, and is obtained in a similar fashion, treating each of the N coordinates independently. In Appendix A, we will show (Proposition A.5) that the analogue of the bound (3.33) on the “measure of hyperbolicity” is, up to a factor exponential in n , $\gamma_n r_{2n-1}^{-N^2} (\prod_{j=0}^{n-2} |x_{n-1} - x_j|)^{-2}$. Again ignoring exponential factors, with $\tilde{\gamma}_n = \gamma_n^{1/\rho}$ the number of initial points in (3.24) is $\gamma_n^{-N/\rho}$, making the right side of (3.24)

$$\gamma_n r_{n-1}^{-N} r_{2n-1}^{-N^2} \left(\prod_{j=0}^{n-2} |x_{n-1} - x_j| \right)^{-N-2} \quad (4.1)$$

By Definition 1.3.1, for admissible Hilbert Bricks, γ_n decays faster than any power of r_{2n} . Thus to make the bound (3.24) on the measure of “bad” parameters small, we basically need γ_n to dominate $(\prod_{j=0}^{n-2} |x_{n-1} - x_j|)^{-N-2}$. This is certainly true if $\prod_{j=0}^{n-2} |x_{n-1} - x_j| \geq \gamma_n^{1/4N}$. Though our choice of the particular exponent $1/4N$ is somewhat arbitrary, the factor of N in the exponent is necessary.

4.2 The multidimensional space of divided differences and dynamically essential parameters

In the 1-dimensional case, with a fixed n -tuple of points $\{x_j\}_{j=0}^{n-1} \subset I$, the space of Newton Interpolation Polynomials (the Divided Differences Space) is $2n$ -dimensional. In Section 2.2, in formulas (2.17), we noticed that there is a simple relation between parameters of Newton Interpolation Polynomials and dynamical properties of the trajectory. In terms of the family (3.86), we have that u_0 determines the position of $f_{u(2n-1); \mathbf{x}_n}(x_0)$, that u_1 determines the position of $f_{u(2n-1); \mathbf{x}_n}(x_1)$, provided that u_0 (and hence $f_{u(2n-1); \mathbf{x}_n}(x_0)$) is fixed, and so on — for $k = 2, \dots, n-1$, we have that u_k determines the position of $f_{u(2n-1); \mathbf{x}_n}(x_k)$, provided that $\{u_j\}_{j=0}^{k-1}$ (hence

$\{f_{u(2n-1); \mathbf{x}_n}(x_j)\}_{j=0}^{k-1}$ are fixed. Similarly, for $k = 0, \dots, n-1$, we have that u_{n+k} determines the derivative $(f_{u(2n-1); \mathbf{x}_n})'(x_k)$, provided that $\{u_j\}_{j=0}^{n+k-1}$ (hence the positions $\{f_{u(2n-1); \mathbf{x}_n}(x_j)\}_{j=0}^{n-1}$ and derivatives $\{(f_{u(2n-1); \mathbf{x}_n})'(x_j)\}_{j=0}^{k-1}$) are fixed.

This correspondence makes transparent estimates of the measure of parameters associated with a particular (pseudo-)trajectory having a given property of periodicity and hyperbolicity (see (3.27-3.33)). In the multidimensional case ($N > 1$), such a correspondence between dynamical properties of trajectories and coefficients of Newton Interpolation Polynomials becomes much less transparent.

In Section 2.2, we define the space of Divided Differences $\mathcal{DD}^{1,n}(I, \mathbb{R}) = I^n \times \mathbb{R}^{2n}$ in the 1-dimensional case, where $I = [-1, 1]$. In this case, we estimated the measure of “bad” parameters in Sections 3.3–3.4.

Recall now the notation of Section 1.3. In N dimensions, we define the space of Divided Differences

$$\begin{aligned} \mathcal{DD}^{N,n}(B^N, \mathbb{R}^N) = & \left\{ (x_0, \dots, x_{n-1}; \{\vec{u}_\alpha\}_{|\alpha|=0}, \dots, \{\vec{u}_\alpha\}_{|\alpha|=2n-1}) \in \right. \\ & \left. \underbrace{B^N \times \dots \times B^N}_{n \text{ times}} \times \mathbb{R}^{\nu(0,N)} \times \dots \times \mathbb{R}^{\nu(2n-1,N)} \right\} = \\ & \underbrace{B^N \times \dots \times B^N}_{n \text{ times}} \times W_{0,N}^{u, \mathbf{X}_0} \times W_{1,N}^{u, \mathbf{X}_1} \dots \times W_{n-1,N}^{u, \mathbf{X}_{n-1}} \times W_{n,N}^{u, \mathbf{X}_n} \times \dots \times W_{2n-1,N}^{u, \mathbf{X}_n}, \end{aligned} \quad (4.2)$$

where B^N is the N -dimensional unit ball, $\nu(k, N)$ is N times the number of N -dimensional multiindices α with $|\alpha| = k$, $\mathbf{X}_k = \{x_j\}_{j=0}^{k-1}$, and $W_{k,N}^{u, \mathbf{X}_k(\text{mod } n)}$ is the space of homogeneous polynomials of degree k from \mathbb{R}^N to \mathbb{R}^N with the Newton basis defined below. There are two issues we face that were not a concern for the 1-dimensional Newton basis (2.26).

Nonuniqueness. It turns out that the choice of a basis in the space of Divided Differences $\mathcal{DD}^{N,n}(B^N, \mathbb{R}^N)$ and the definition of the Newton map

$$\mathcal{L}_{\mathbf{X}_n}^N : (\{\vec{\varepsilon}_\alpha\}_{|\alpha| \leq 2n-1}) \rightarrow (\{\vec{u}_\alpha\}_{|\alpha| \leq 2n-1}) \quad (4.3)$$

(defined by (2.24) for $N = 1$) for a multiindex $\alpha \in \mathbb{Z}_+^N$ is far from unique. In the 1-dimensional case, the standard basis is $\{x^k\}_{k=0}^{2n-1}$ and the Newton basis is $\left\{ \prod_{j=0}^{k-1} |x - x_j| \right\}_{k=0}^{2n-1}$. In the N -dimensional case, $(x - x_j) \in \mathbb{R}^N$ is an N -dimensional

vector. For a fixed basis in \mathbb{R}^N , let $(x - x_j)_s$ denote the s -th coordinate of the vector $(x - x_j)$. The number of different monomials of the form

$$\left\{ \prod_{j=0}^{k-1} (x_k - x_j)_{i(j)} \right\}_{\{i(0), \dots, i(k-1)\} \in \{1, \dots, N\}^k} \quad (4.4)$$

is N^k , while the number of homogeneous monomials in N variables of degree k , i.e. $\{x^\alpha\}_{|\alpha|=k}$, is bounded above by k^N , which is much smaller than N^k for $k \gg N$.

Therefore, among the monomials (4.4) we need to choose an appropriate basis and define an appropriate Newton map $\mathcal{L}_{\mathbf{X}_n}^N$. The *standard* way to choose a Newton basis [GY] is as follows. For $\alpha \in \mathbb{Z}_+^N$, let the Newton basis monomial for the multiindex α be

$$\begin{aligned} (x; x_0, \dots, x_{(|\alpha|-1) \pmod n})^\alpha &= \prod_{i_1=0}^{\alpha_1-1} (x - x_{i_1})_1 \times \\ &\prod_{i_2=0}^{\alpha_2-1} (x - x_{\alpha^1+i_2})_2 \times \dots \times \prod_{i_N=0}^{\alpha_N-1} (x - x_{\alpha^{N-1}+i_N})_N, \end{aligned} \quad (4.5)$$

where $\alpha^j = \sum_{i=0}^j \alpha_i$ for $j = 1, \dots, N-1$. The Newton basis for $W_{k,N}^{u, \mathbf{X}_n}$, the space of homogeneous vector-polynomials of degree k , consists of N such monomials (one for each basis vector of \mathbb{R}^N) for each α with $|\alpha| = k$.

Dynamically essential coordinates. After a Newton basis is chosen, one needs to make sure that it is effective for dynamical purposes. In Section 3.3, we noticed in (3.30) and (3.33) that in order to perturb by Newton Interpolation Polynomials in an effective way, we need to make sure that the product of distances $\prod_{j=0}^{n-2} |x_{n-1} - x_j|$ is not too small. Similarly, in the multidimensional case we need at least one Newton monomial $(x; x_0, \dots, x_{n-2})^\alpha$ with $|\alpha| = n-1$ not to be too small. The most natural way to choose a “good” monomial is by taking the maximal coordinates of corresponding vectors. Let $v \in \mathbb{R}^N$ be a nonzero vector and $\|v\| = (\sum_{i=1}^N v_i^2)^{1/2}$. Denote by

$$m(v) = \min\{i : 1 \leq i \leq N, |v_i| = \max_{j=1}^N |v_j|\}$$

the minimal index of one of the largest components v_i of v . Then

$$\left| \prod_{j=0}^{n-2} (x_{n-1} - x_j)_{m(x_{n-1}-x_j)} \right| \geq N^{\frac{n-1}{2}} \prod_{j=0}^{n-2} |x_{n-1} - x_j|. \quad (4.6)$$

This is a satisfactory estimate, because $\gamma_n(C, \delta)$ is a stretched exponential in n , so we can neglect factors that are exponential in n . For given $\mathbf{X}_n = \{x_j\}_{j=0}^{n-1}$, we call the monomials

$$\prod_{j=0}^{k-1} (x - x_j)_{m(x_k - x_j)}, \quad k = 0, \dots, n-1 \quad (4.7)$$

dynamically essential. Using these monomials, we shall imitate estimates (3.27)–(3.34) in the multidimensional Discretization Method.

The dynamically essential monomials do not necessarily belong to the standard Newton basis (4.5), so we will need to define the basis differently depending on which monomials are dynamically essential for the given pseudo-trajectory \mathbf{X}_n . This is not a major obstacle, since we already use a different basis for each \mathbf{X}_n in the Discretization Method, but it does further complicate the argument.

The necessity of altering the basis is illustrated by the following example for $N = 2$: $x_0 = (1, 0)$, $x_1 = (0, 1)$, $x_2 = (1, 1)$. Then for all α with $|\alpha| = 2$, we have $(x_2; x_0, x_1)^\alpha = 0$. Thus, the monomial $(x; x_0, x_1)^\alpha$ is useless to perturb the image of x_2 .

4.3 The multidimensional Distortion Lemma

One of the key ingredients of the Discretization Method is the Distortion Lemma from Section 3.4. Fix an n -tuple of points $\mathbf{X}_n = \{x_j\}_{j=0}^{n-1} \subset B^N$. The Distortion Lemma in dimension 1 shows that the Newton map $\mathcal{L}_{\mathbf{X}_n}^1 : W_{<2n,1} \rightarrow W_{<2n,1}^{u, \mathbf{X}_n}$ expands a brick $HB_{<2n}^{st}(\tau)$ of standard thickness along each direction by at most a factor of 3 in each direction. Since, the space of 1-dimensional divided differences $W_{<2n,1}$ is $2n$ -dimensional, this gives a total volume distortion of at most 3^{2n} . This factor is part of the estimates (3.13) and (3.42), but is ultimately unimportant since it is exponential in n .

In the multidimensional case, this naive approach does not work. One can, indeed, show that the Newton map expands a brick of standard thickness along each direction by at most a factor 3^N . However, the space of divided differences $W_{<2n,N}$ has dimension of the order of $(2n)^N$. So, the naive estimate of distortion is $3^{N(2n)^N}$, which is highly unaffordable. In the multidimensional case, we need a more precise estimate on distortion.

First, we define a Cubic Brick of at most standard thickness, which differs from the brick $HB_{<2n}^{st}(\vec{\mathbf{r}})$ defined by (2.27) in two ways. First, it is a parallelepiped for all

N , whereas $HB_{<2n}^{st}(\vec{\mathbf{r}})$ is a product of balls whose dimension is greater than 1 when $N > 1$. Second, we require that the side lengths of a Cubic Brick decay very rapidly. We will estimate distortion of such Cubic Bricks by the Newton map, then in the next section we will cover a brick with Cubic Bricks,

Definition 4.3.1. Let $\vec{\lambda}_{<k} = (\lambda_0, \dots, \lambda_{k-1}) \in \mathbb{R}_+^k$ be a vector with strictly positive components. If for every $0 \leq m < k$ we have $\lambda_m \geq \lambda_{m+1}(m+1)^{4N}$, we call the rectangular parallelepiped

$$CHB_{<k}^{N,st}(\vec{\lambda}_{<k}, \vec{\delta}) = \left\{ \{\vec{\varepsilon}_\alpha\}_{|\alpha|<k} : \forall 0 \leq m < k, \right. \\ \left. \forall \alpha \in \mathbb{Z}_+^N, |\alpha| = m, \forall j = 1, \dots, N, |\varepsilon_\alpha^j - \delta_\alpha^j| \leq \lambda_m \right\} \quad (4.8)$$

a Cubic Brick of at most standard thickness centered at $\vec{\delta}$ (here the superscript j denotes the j th coordinate). Similarly, for $\lambda_k \in \mathbb{R}_+$ let

$$CHB_k^{N,st}(\vec{\lambda}_k, \vec{\delta}) = \left\{ \{\vec{\varepsilon}_\alpha\}_{|\alpha|=k} : \forall \alpha \in \mathbb{Z}_+^N, \right. \\ \left. \forall |\alpha| = k, \forall j = 1, \dots, N, |\varepsilon_\alpha^j - \delta_\alpha^j| \leq \lambda_k \right\} \quad (4.9)$$

An example of a suitable thickness vector $\vec{\lambda}_{<k}$ is $\{\lambda_m = \frac{\tau}{(m!)^{4N}}\}_{m=0}^{k-1}$.

To formulate a multidimensional version of the Distortion Lemma, we extend the definition of the parameters (3.38) and (3.39) allowed by the family (3.8) in Section 3.4. Consider a $C^{1+\rho}$ -smooth diffeomorphism $f \in \text{Diff}^{1+\rho}(B^N)$ of the unit ball B^N into its interior and some positive integer n . Let $CHB_{<2n}^{N,st}(\vec{\lambda}_{<2n}, \vec{\delta})$ be a Cubic Brick of at most standard thickness, which defines the family of diffeomorphisms of the unit ball B^N into its interior

$$\left\{ f_{\vec{\varepsilon}}(x) = f(x) + \sum_{|\alpha| \leq 2n-1} \vec{\varepsilon}_\alpha x^\alpha \right\}_{\{\vec{\varepsilon}_\alpha\}_{|\alpha| \leq 2n-1} \in CHB_{<2n}^{N,st}(\vec{\lambda}_{<2n}, \vec{\delta})} \quad (4.10)$$

Fix an n -tuple of points $\mathbf{X}_n = \{x_j\}_{j=0}^{n-1} \subset B^N$, and fix the standard Newton basis $\{(x; x_0, \dots, x_{(|\alpha|-1) \pmod n})^\alpha\}_{|\alpha| \leq 2n-1}$ in the multidimensional space of divided differences $\mathcal{DD}^{N,n}(B^N, \mathbb{R}^N)$, defined by (4.5). Denote by

$$\mathcal{L}_{\mathbf{X}_n}^N : (\{\vec{\varepsilon}_\alpha\}_{|\alpha| \leq 2n-1}) \rightarrow (\{\vec{u}_\alpha\}_{|\alpha| \leq 2n-1}) \quad (4.11)$$

the Newton map that corresponds to rewriting an N -vector polynomial of degree $2n - 1$ given in the basis $\{x^\alpha\}_{|\alpha| \leq 2n-1}$ in the Newton basis (4.5).

Consider the reparametrization of family (4.10) by the Newton parameters

$$f_{\vec{u}(2n-1), \mathbf{X}_n}(x) = f(x) + \sum_{|\alpha| \leq 2n-1} \vec{u}_\alpha(x; x_0, \dots, x_{(|\alpha|-1) \pmod n})^\alpha. \quad (4.12)$$

Denote by $\vec{u}(m) = (\{\vec{u}_\alpha\}_{|\alpha| \leq m})$ (respectively $\vec{u}_m = (\{\vec{u}_\alpha\}_{|\alpha|=m})$) the set of all Newton parameters of degree at most m (respectively of degree exactly m).

In order to state the Discretization Lemma in the N -dimensional case, we have to define the set of values of Newton parameters $\vec{u}(2n - 1)$ that are allowed by the family (4.10) and the Cubic Brick (4.8).

Let $\mathbf{X}_k = \{x_j\}_{j=0}^{k-1}$ for $0 \leq k \leq n$, and for convenience below let $\mathbf{X}_k = \mathbf{X}_n$ for $n < k \leq 2n - 1$. Let

$$\pi_{<2n, \leq k}^{N, u, \mathbf{X}_k} : W_{<2n, N}^u \rightarrow W_{\leq k, N}^{u, \mathbf{X}_k} \quad \text{and} \quad \pi_{<2n, k}^{N, u, \mathbf{X}_k} : W_{<2n, N}^u \rightarrow W_{k, N}^{u, \mathbf{X}_k}$$

be the natural projection onto the space $W_{\leq k, N}^{u, \mathbf{X}_k}$ of N -vector polynomials in N variables of degree k and the space $W_{k, N}^{u, \mathbf{X}_k}$ of homogeneous N -vector polynomials in N variables of degree k . As for $N = 1$, this projection depends only on \mathbf{X}_k , not on $\mathbf{X}_n \setminus \mathbf{X}_k$.

Notice that the image of the Cubic Brick $CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$ under the Newton map $\mathcal{L}_{\mathbf{X}_n}^N$ is a parallelepiped

$$\mathcal{P}_{<2n, \mathbf{X}_n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta}) = \mathcal{L}_{\mathbf{X}_n}^N(CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})), \quad (4.13)$$

because the map $\mathcal{L}_{\mathbf{X}_n}^N$ is linear.

We call the parallelepiped $\mathcal{P}_{<2n, \mathbf{X}_n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$ the set of parameters *allowed by the family* (4.10). Notice that the values of $\vec{u}(2n - 1) = (\vec{u}_0, \dots, \vec{u}_{2n-1}) \in W_{<2n, N}^u$ that do not belong to $\mathcal{P}_{<2n, \mathbf{X}_n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$ are of *no interest* for us, because they lie outside of the range of the family (4.10) under consideration. Now we define the range of allowed parameters $\vec{u}(k) \in W_{\leq k, N}^{u, \mathbf{X}_k}$ for each $0 \leq k \leq 2n - 1$.

Denote the images of the Cubic Brick $CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$ under composition of the Newton map $\mathcal{L}_{\mathbf{X}_n}^N$ and the projections $\pi_{<2n, \leq k}^{N, u, \mathbf{X}_k}$ and $\pi_{<2n, k}^{N, u, \mathbf{X}_k}$ by

$$\begin{aligned} \mathcal{P}_{<2n, \leq k, \mathbf{X}_k}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta}) &= \pi_{<2n, \leq k}^{N, u, \mathbf{X}_k} \circ \mathcal{L}_{\mathbf{X}_n}^N(CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})) \subset W_{\leq k, N}^u \\ \mathcal{P}_{<2n, k, \mathbf{X}_k}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta}) &= \pi_{<2n, k}^{N, u, \mathbf{X}_k} \circ \mathcal{L}_{\mathbf{X}_n}^N(CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})) \subset W_{k, N}^u. \end{aligned} \quad (4.14)$$

It follows from the definition of the Newton map $\mathcal{L}_{\mathbf{X}_n}^N$ that for each $0 \leq k \leq 2n - 1$, the sets $\mathcal{P}_{<2n, \leq k, \mathbf{X}_k}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$ and $\mathcal{P}_{<2n, k, \mathbf{X}_k}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$ are polyhedrons depending on \mathbf{X}_k . We call $\mathcal{P}_{<2n, \leq k, \mathbf{X}_k}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$ the set of parameters of order k allowed by the family (4.10).

Similarly to the 1-dimensional Distortion Lemma from Section 3.4, its N -dimensional generalization gives an estimate on the ratio of volumes of the polyhedra $\mathcal{P}_{<2n, \leq k, \mathbf{X}_k}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$ of allowed Newton parameters $\vec{u}(k)$ and the corresponding Cubic Bricks $CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$.

The Preliminary N -dimensional Distortion Lemma. *Let $\mathbf{X}_n = \{x_j\}_{j=0}^{n-1} \subset B^N$ be an n -tuple of points in the unit ball B^N , and let $\mathcal{L}_{\mathbf{X}_n}^N : W_{<2n, N} \rightarrow W_{<2n, N}^{u, \mathbf{X}_n}$ be the Newton map (4.11) corresponding to the basis (4.5) for $W_{<2n, N}^{u, \mathbf{X}_n}$. Let $\vec{\lambda}_{<2n} = (\lambda_0, \dots, \lambda_{2n-1}) \in \mathbb{R}_+^{2n}$ be a vector with strictly positive components that defines a Cubic Brick of $CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$ of at most standard thickness and $Leb_{\leq k, N}$ (resp. $Leb_{k, N}$) be the Lebesgue measure on the space $W_{\leq k, N}$ (respectively $W_{k, N}$) of (respectively of homogeneous) N -vector polynomials in N variable of degree k . Then for all $0 \leq k \leq 2n - 1$, we have the volume ratio estimate*

$$\frac{Leb_{k, N} \left(\mathcal{P}_{<2n, k, \mathbf{X}_k}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta}) \right)}{Leb_{k, N} \left(CHB_k^{N, st}(\vec{\lambda}_k, \vec{\delta}) \right)} \leq 3^{N^2 n}. \quad (4.15)$$

Remark 4.3.2. *As in the 1-dimensional case, the Newton map $\mathcal{L}_{\mathbf{X}_n}^N$ is volume-preserving, and hence the parallelepipeds $\mathcal{P}_{<2n, \mathbf{X}_n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$ and $CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$ have the same volume. However, the estimate (4.15) concerns the volumes of projections of these parallelepipeds. Since $CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$ is a rectangular parallelepiped (aligned with the coordinate axes) and $\mathcal{P}_{<2n, \mathbf{X}_n}^{N, st}(\vec{\lambda}_{<2n}, \vec{\delta})$ is not, the projections of the latter set have larger volume according to the amount of shear in the Newton map. The Distortion Lemma bounds the effect of the shear.*

The lemma above will be proved in Part II of this paper in a more general form required for the proof of the Main Result (Theorem 1.3.7). The other main ingredient of Section 3.4, the Collection Lemma, proceeds in much the same way for general N as it does for $N = 1$.

4.4 From a brick of at most standard thickness to an admissible brick

In order to apply the method which we are developing in this paper to other problems about generic properties of dynamical systems, we need to have a sufficiently rich space of parameters (enough freedom to perturb). Various dynamical phenomena have a “size” that is exponential in the period or number of iterations. Since we perturb trajectories of length n with polynomials whose degree is proportional to n , it seems essential to have a Hilbert Brick of parameters with sides decaying at most exponentially in the period of the polynomials the respective parameters multiply. That is, with the notations of Definition 1.3.1, we would like to have $r_n \geq \exp(-Cn)$ for some $C > 0$. However, even for the 1-dimensional model, if we consider a brick of parameters with exponentially decaying sides, then we cannot control the distortion properties of the Newton map (see Remark 3.4.1).

In order to circumvent this problem, we do the following. Consider a Hilbert Brick $HB^N(\vec{\mathbf{r}})$ of an admissible size $\vec{\mathbf{r}} = \{r_k\}_{k=0}^\infty$ (see Definition 1.3.1). Using Fubini’s Reduction from Section 2.3 at the n th stage of induction over the period, we reduce consideration of an infinite-dimensional Hilbert Brick $HB^N(\vec{\mathbf{r}})$ to a finite-dimensional brick $HB_{<2n}^N(\vec{\mathbf{r}})$.

Recall that $HB_{<2n}^N(\vec{\mathbf{r}})$ belongs to the space of N -vector polynomials of degree $2n - 1$, denoted by $W_{<2n,N}$.

Definition 4.4.1. *We call the Cubic Brick in $W_{<2n,N}$ of at most standard thickness centered at a point $\vec{\delta}_{<2n} \in W_{<2n,N}$ and associated to the brick $HB_{<2n}^N(\vec{\mathbf{r}})$ the one denoted by $CHB_{<2n}^{N,st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), \vec{\delta}_{<2n})$ and defined by (4.8) where $\lambda_m(\vec{\mathbf{r}}) = r_m/(m!)^{4N}$ and $\vec{\lambda}_{<2n}(\vec{\mathbf{r}}) = (\lambda_0(\vec{\mathbf{r}}), \dots, \lambda_{2n-1}(\vec{\mathbf{r}}))$.*

Remark 4.4.2. *It follows from the nonincreasing property of the sequence r_m (see Definition 1.3.1) that*

$$(m+1)^{4N} \lambda_{m+1}(\vec{\mathbf{r}}) = (m+1)^{4N} \frac{r_{m+1}}{((m+1)!)^{4N}} \leq \frac{r_m}{(m!)^{4N}} = \lambda_m(\vec{\mathbf{r}}), \quad (4.16)$$

so that the Cubic Brick $CHB_{<2n}^{N,st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), \vec{\delta}_{<2n})$ is of at most standard thickness.

We wish to cover the brick $HB_{<2n}^N(\vec{\mathbf{r}})$ in $W_{<2n,N}$ by a collection of Cubic Bricks $\{CHB_{<2n}^{N,st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), \vec{\delta}_{<2n})\}_{\vec{\delta}_{<2n} \in G(\vec{\mathbf{r}})}$ of at most standard thickness associated with it. First we define a grid in $W_{<2n,N}$ on which the centers $\vec{\delta}_{<2n}$ should lie in order that the Cubic Bricks fit together without overlap.

Recall that $W_{<2n,N} = \times_{m=0}^{2n-1} W_{m,N}$, $\dim W_{m,N} = \nu(m, N)$, and denote $\eta(k, N) = \sum_{m=0}^k \nu(m, N)$. Let

$$\begin{aligned} \mathbb{Z}_{2\vec{\lambda}_{<2n}(\vec{\mathbf{r}})}^{\eta(2n-1,N)} &= \mathbb{Z}_{2\lambda_0(\vec{\mathbf{r}})}^{\nu(0,N)} \times \mathbb{Z}_{2\lambda_1(\vec{\mathbf{r}})}^{\nu(1,N)} \times \cdots \times \mathbb{Z}_{2\lambda_{2n-1}(\vec{\mathbf{r}})}^{\nu(2n-1,N)} \\ &\subset W_{0,N} \times W_{1,N} \times \cdots \times W_{2n-1,N} = W_{<2n,N}, \end{aligned} \quad (4.17)$$

where $\mathbb{Z}_{2\lambda_m(\vec{\mathbf{r}})}^{\nu(m,N)}$ is the grid in $\mathbb{R}^{\nu(m,N)}$ with spacing $2\lambda_m(\vec{\mathbf{r}})$ in each coordinate. Let

$$G(\vec{\mathbf{r}}) = \left\{ \vec{\delta}_{<2n} \in \mathbb{Z}_{2\vec{\lambda}_{<2n}(\vec{\mathbf{r}})}^{\eta(2n-1,N)} : HB_{<2n}^N(\vec{\mathbf{r}}) \cap CHB_{<2n}^{N,st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), \vec{\delta}_{<2n}) \neq \emptyset \right\}. \quad (4.18)$$

The Cubic Bricks with centers in $G(\vec{\mathbf{r}})$ are the ones needed for our covering:

$$HB_{<2n}^N(\vec{\mathbf{r}}) \subset \cup_{\vec{\delta} \in G(\vec{\mathbf{r}})} CHB_{<2n}^{N,st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), \vec{\delta}_{<2n}). \quad (4.19)$$

The Covering Lemma. *The ratio of the volume of the covering (4.19) by Cubic Bricks and the volume of the brick $HB_{<2n}^N(\vec{\mathbf{r}})$ is bounded by e^{6N} .*

Proof: We claim that the covering (4.19) is contained in the slightly larger brick $HB_{<2n}^N(\{(1 + \frac{2\sqrt{N}}{(m!)^{3N}})r_m\}_{m=0}^{2n-1})$. The amount that the covering extends beyond $HB_{<2n}^N(\vec{\mathbf{r}})$ in the direction of a given multiindex α with $|\alpha| = m$ is at most the diameter in the α direction of one of the Cubic Bricks $CHB_{<2n}^{N,st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), \vec{\delta}_{<2n}^j)$, which by Definition 4.4.1 is $2\sqrt{N}\lambda_m(\vec{\mathbf{r}}) = 2\sqrt{N}r_m/(m!)^{4N}$. Recall the definitions (2.27) of $HB_{<2n}^N(\vec{\mathbf{r}})$ and (1.12) of the norm used therein. In this norm, the diameter of $CHB_{<2n}^{N,st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), \vec{\delta}_{<2n}^j)$ in the directions with $|\alpha| = m$ is $2\sqrt{N}r_m/(m!)^{4N}$ times the square root of $\sum_{|\alpha|=m} \binom{m}{\alpha}^{-1} < (m!)^N$, from which our claim follows. The ratio of volumes that we wish to bound is then at most $\prod_{m=0}^{2n-1} \exp(2\sqrt{N}/(m!)^{3N}) \leq e^{2e\sqrt{N}} < e^{6N}$. Q.E.D.

Recall now that our main goal in the proof of Theorem 1.3.7 is to get an estimate on the measure of the “bad” set of parameters (2.1) inside of the Hilbert Brick $HB^N(\vec{\mathbf{r}})$. Using the Fubini reduction from Section 2.3, we know that it is sufficient to get an estimate on the measure of the “bad” set in a finite-dimensional slice of the form $HB_{<2n}^N(\vec{\mathbf{r}}) \times \{\vec{\varepsilon}_{\geq 2n}\}$ that is uniform over $\vec{\varepsilon}_{\geq 2n} \in HB_{\geq 2n}^N(\vec{\mathbf{r}})$ see (2.31-2.32). Notice now that if we can prove that for each Cubic Brick slice $CHB_{<2n}^{N,st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), \vec{\delta}_{<2n}) \times \{\vec{\varepsilon}_{\geq 2n}\}$ from the covering collection $\vec{\delta}_{<2n} \in G(\vec{\mathbf{r}})$ that the fraction of parameters in the slice

that are “bad” is uniformly small over $\vec{\delta}_{<2n} \in G(\vec{\mathbf{r}})$, then the same fraction is small in the whole slice $HB_{<2n}^N(\vec{\mathbf{r}}) \times \{\vec{\varepsilon}_{\geq 2n}\}$. (By the Covering Lemma, we must increase the bound on the fraction of “bad” parameters only by the factor e^{6N} , which is independent of n .) By the Fubini reduction, this shows that the measure of the “bad” set in $HB^N(\vec{\mathbf{r}})$ is small too. Thus it is sufficient to prove the following estimate.

$$\frac{\mu_{<2n, \vec{\mathbf{r}}}^N \left(B_n(C, \delta, \rho, \vec{\mathbf{r}}, f, \vec{\varepsilon}_{\geq 2n}) \cap CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), \vec{\delta}_{<2n}) \right)}{\mu_{<2n, \vec{\mathbf{r}}}^N \left(CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), \vec{\delta}_{<2n}) \right)} \leq \mu'_n(C, \delta, \rho, M_{1+\rho}). \quad (4.20)$$

uniformly over all $\delta_{<2n} \in J(\vec{\mathbf{r}})$ and so that $\sum_{n=1}^{\infty} \mu'_n(C, \delta, \rho, M_{1+\rho})$ converges for all positive C, δ, ρ , and $M_{1+\rho}$, and tends to zero as C tends to infinity.

4.5 The main estimate on the measure of “bad” parameters

In this subsection we formulate the main theorem of the rest of the paper which implies (the Main) Theorem 1.3.7. It will be proved in Part II of this paper.

Theorem 4.5.1. *For any $\rho > 0$ and any diffeomorphism $f \in \text{Diff}^{1+\rho}(B^N)$, consider a Hilbert Brick $HB^N(\vec{\mathbf{r}})$ of an admissible size $\vec{\mathbf{r}}$ with respect to f and the family of analytic perturbations of f*

$$\{f_{\vec{\varepsilon}}(x) = f(x) + \phi_{\vec{\varepsilon}}(x)\}_{\vec{\varepsilon} \in HB^N(\vec{\mathbf{r}})} \quad (4.21)$$

(see (1.10)) with the Lebesgue product probability measure $\mu_{\vec{\mathbf{r}}}^N$ (see (1.15)) associated to $HB^N(\vec{\mathbf{r}})$.

Then for any positive integer n and any $\vec{\varepsilon}_{\geq 2n} \in HB_{\geq 2n}^N(\vec{\mathbf{r}})$, consider a slice (2.27) of the Hilbert Brick $HB^N(\vec{\mathbf{r}})$ of the form

$$HB_{<2n}^N(\vec{\mathbf{r}}) \times \{\vec{\varepsilon}_{\geq 2n}\} \subset HB_{<2n}^N(\vec{\mathbf{r}}) \times HB_{\geq 2n}^N(\vec{\mathbf{r}}) = HB^N(\vec{\mathbf{r}}). \quad (4.22)$$

Inside of this slice fix a grid point $\vec{\delta}_{<2n} \in G(\vec{\mathbf{r}})$, and consider the Cubic Brick

$$CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), \vec{\delta}_{<2n}) \times \{\vec{\varepsilon}_{\geq 2n}\} \subset HB_{<2n}^N(\vec{\mathbf{r}}) \times \{\vec{\varepsilon}_{\geq 2n}\}. \quad (4.23)$$

from the covering (4.19) associated to the brick $HB_{<2n}^N(\vec{\mathbf{r}})$.

Let $\tilde{f} = f_{(\vec{\delta}_{<2n}, \vec{\varepsilon}_{\geq 2n})}$ be a diffeomorphism corresponding to the center of the Cubic Brick $CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), \vec{\delta}_{<2n}) \times \{\vec{\varepsilon}_{\geq 2n}\}$. Consider the family of diffeomorphisms

$$\begin{aligned} & \{ \tilde{f}_{\vec{\varepsilon}_{<2n}} \}_{\vec{\varepsilon}_{<2n} \in CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), 0)} \supset \\ & \{ f_{(\vec{\varepsilon}_{<2n} + \vec{\delta}_{<2n}, \vec{\varepsilon}_{\geq 2n})} \}_{(\vec{\varepsilon}_{<2n} + \vec{\delta}_{<2n}) \in HB_{<2n}^N(\vec{\mathbf{r}}) \cap CHB_{<2n}^{N, st}(\vec{\lambda}_{<2n}(\vec{\mathbf{r}}), \vec{\delta}_{<2n})}. \end{aligned} \quad (4.24)$$

Then for $C \geq 30/\rho \log M_{1+\rho}$, the fraction of the measure of “bad” parameters $B_n(C, \delta, \rho, \vec{\mathbf{r}}, f, \vec{\varepsilon}_{\geq 2n})$, defined in (2.32), inside $CHB_{<2n}^{N, st}(\vec{\lambda}_{2n-1}(\vec{\mathbf{r}}), \vec{\delta}_{<2n}) \times \{\vec{\varepsilon}_{\geq 2n}\}$ satisfies the bound

$$\begin{aligned} & \frac{\mu_{<2n, \vec{\mathbf{r}}}^N \left(B_n(C, \delta, \rho, \vec{\mathbf{r}}, f, \vec{\varepsilon}_{\geq 2n}) \cap CHB_{<2n}^{N, st}(\vec{\lambda}_{2n-1}(\vec{\mathbf{r}}), \vec{\delta}_{<2n}) \right)}{\mu_{<2n, \vec{\mathbf{r}}}^N \left(CHB_{<2n}^{N, st}(\vec{\lambda}_{2n-1}(\vec{\mathbf{r}}), \vec{\delta}_{<2n}) \right)} \leq \\ & 24C e^N n^{1+\delta} 3^{2nN^2} M_{1+\rho} \max\{M_{1+\rho}, \exp C/100\}^{-n/2} \end{aligned} \quad (4.25)$$

The discussion from the end of the previous section shows that this theorem implies (the Main) Theorem 1.3.7.

Appendix A: Properties of hyperbolicity

In this paper we have quantified the hyperbolicity of periodic points in order to bound from below the distance between periodic points of the same period. Recall the definitions of $\gamma(L)$, $\gamma_n(x, f)$, and $\gamma_n(f)$ from (1.6) and (1.7) and the text between them. In this appendix we will prove Proposition 1.1.6, along with a result that says that the hyperbolicity of a power of a linear operator is no smaller than the hyperbolicity of the operator.

Proposition 1.1.6 follows immediately from the following lemma.

Lemma A.1. *Given the hypotheses of Proposition 1.1.6, for every pair of distinct period n points of f , say $x = f^n(x) \neq y = f^n(y)$, the distance $|x - y|$ between them is at least $(M_{1+\rho}^{-n(1+\rho)} \gamma_n(f))^{1/\rho}$.*

Proof: Let $v = (y - x)/|x - y|$. Then

$$\begin{aligned} v &= \frac{f^n(y) - f^n(x)}{|x - y|} = \frac{1}{|x - y|} \int_0^{|x-y|} \frac{d}{d\lambda} f^n(x + \lambda v) d\lambda \\ &= \frac{1}{|x - y|} \int_0^{|x-y|} df^n(x + \lambda v) v d\lambda. \end{aligned} \tag{.26}$$

It follows that

$$\frac{1}{|x - y|} \int_0^{|x-y|} (df^n(x + \lambda v)v - v) d\lambda = 0. \tag{.27}$$

Let $w = df^n(x)v - v$; by hypothesis $|w| \geq \gamma_n(f)$. Also,

$$\begin{aligned} |df^n(x + \lambda v)v - v - w| &= |(df^n(x + \lambda v) - df^n(x))v| \\ &\leq \|df^n(x + \lambda v) - df^n(x)\|. \end{aligned} \tag{.28}$$

Now

$$\begin{aligned}
df^n(x + \lambda v) - df^n(x) &= [df(f^{n-1}(x + \lambda v)) - df(f^{n-1}(x))]df^{n-1}(x + \lambda v) \\
&\quad + df(f^{n-1}(x))[df(f^{n-2}(x + \lambda v)) - df(f^{n-2}(x))]df^{n-2}(x + \lambda v) \\
&\quad + \cdots + df^{n-1}(f(x))[df(x + \lambda v) - df(x)].
\end{aligned} \tag{.29}$$

Since $M_{1+\rho}$ is an upper bound on the $C^{1+\rho}$ norm of f , it is an upper bound on the norm of $df(z)$ for all $z \in M$. It follows that $|f^k(x + \lambda v) - f^k(x)| \leq M_{1+\rho}^k \lambda$ for $k = 0, 1, \dots, n-1$, and hence

$$\begin{aligned}
\|df^n(x + \lambda v) - df^n(x)\| &\leq \sum_{k=0}^{n-1} M_{1+\rho} (M_{1+\rho}^k \lambda)^\rho M_{1+\rho}^{n-1} \\
&= M_{1+\rho}^n \frac{M_{1+\rho}^{n\rho} - 1}{M_{1+\rho}^\rho - 1} \lambda^\rho < M_{1+\rho}^{n(1+\rho)} \lambda^\rho.
\end{aligned} \tag{.30}$$

(Recall that we assumed $M_{1+\rho} > 2^{1/\rho}$ in the definition of $M_{1+\rho}$.)

By the results above we then have

$$\begin{aligned}
0 &= \frac{1}{|x-y|} \int_0^{|x-y|} (df^n(x + \lambda v)v - v) \cdot \frac{w}{|w|} d\lambda \\
&\geq \frac{1}{|x-y|} \int_0^{|x-y|} (|w| - \|df^n(x + \lambda v) - df^n(x)\|) d\lambda \\
&> \frac{1}{|x-y|} \int_0^{|x-y|} (\gamma_n(f) - M_{1+\rho}^{n(1+\rho)} \lambda^\rho) d\lambda \\
&> (\gamma_n(f) - M_{1+\rho}^{n(1+\rho)}) |x-y|^\rho.
\end{aligned} \tag{.31}$$

From this we get the desired upper bound on $|x-y|$. This completes the proof of the Lemma. Q.E.D.

Notice that the notion of hyperbolicity $\gamma(L)$ of a linear operator L as a lower bound on $|Lv - v|$ for unit vectors v occurs naturally in the proof above. It is not possible to make an analogous estimate with the same power on the period n hyperbolicity of f if the hyperbolicity is defined in the more usual manner, as in [Y], taking the minimum distance of the eigenvalues of L from the unit circle in \mathbb{C} . To see this, consider the following C^2 map $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ for small $\gamma > 0$:

$$\begin{aligned}
f(x_1, x_2, \dots, x_N) &= \\
&((1 + \gamma)x_1 - x_2, (1 + \gamma)x_2 - x_3, \dots, (1 + \gamma)x_n - x_{n-1}, (1 + \gamma)x_n - x_1^2).
\end{aligned} \tag{.32}$$

Notice that f has two nearby fixed points, 0 and $(\gamma^N, \gamma^{N+1}, \dots, \gamma^{2N-1})$, that are within roughly γ^N of each other. Notice also that $df(0)$ is upper triangular and hence all N of its eigenvalues are $1 + \gamma$, so that by the eigenvalue notion of hyperbolicity 0 is an (n, γ) -hyperbolic fixed point of f . (Though df has eigenvalues closer to the unit circle at the other fixed point, they are still much farther away than γ^N for large N .) On the other hand, for $v = (1, \gamma, \dots, \gamma^{N-1})$ we have $|v|$ slightly larger than 1 while

$$|Lv - v| = |(0, 0, \dots, \gamma^N)| = \gamma^N, \quad (.33)$$

so that our notion of hyperbolicity is commensurate with the spacing between the fixed points.

To this point, when using the hyperbolicity of a linear operator L , it has only been important that $|Lv - v|$ not be small for unit vectors v . The reason for estimating from below $|Lv - \exp(2\pi i\phi)v|$ for $\phi \in [0, 1)$ in (1.6) is that the hyperbolicity of L will be a lower bound on the hyperbolicity of L^k for positive integers k . In terms of diffeomorphisms, this estimate gives a bound on how close points of period nk can lie to a hyperbolic point of period n . For the eigenvalue-based notion of hyperbolicity, the estimate is trivial, but for our notion it must be proved.

Lemma A.2. *For every linear operator $L : \mathbb{C}^N \rightarrow \mathbb{C}^N$ and $k \in \mathbb{Z}^+$, we have $\gamma(L^k) \geq \gamma(L)$.*

Proof: Suppose $\gamma(L^k) < \gamma(L)$; then for some $\phi \in [0, 1)$ and unit vector $v \in \mathbb{C}^N$ we have $|L^k v - \exp(2\pi i\phi)v| < \gamma(L)$. Without loss of generality we may assume that $\phi = 0$; otherwise replace L with $\exp(-2\pi i\phi/k)L$, so that the $\gamma(L)$ and $\gamma(L^k)$ are unaffected and $|L^k v - v| < \gamma(L)$. Let $\omega = \exp(2\pi i/k)$, and for $j = 0, 1, \dots, k-1$ let

$$w_j = v + \omega^j Lv + \omega^{2j} L^2 v + \dots + \omega^{(k-1)j} L^{k-1} v. \quad (.34)$$

Notice that $w_0 + w_1 + \dots + w_{k-1} = kv$, and since v is a unit vector we must have $|w_j| \geq 1$ for some j . But

$$\begin{aligned} Lw_j - \omega^{-j}w_j &= Lv - \omega^{-j}v + \omega^j L^2 v - Lv + \dots + \omega^{(k-1)j} L^k v - \omega^{(k-2)j} L^{k-1} v \\ &= \omega^{(k-1)j} L^k v - \omega^{-j}v = \omega^{-j}(Lv - v), \end{aligned} \quad (.35)$$

the last step because ω is a k -th root of unity. This yields $|Lw_j - \omega^{-j}w_j| = |Lv - v| < \gamma(L)$, contradicting the definition of $\gamma(L)$. This completes the proof of the lemma. Q.E.D.

The next lemma is a simple estimate on how much a small perturbation of a linear operator can change its hyperbolicity.

Lemma A.3. *For any pair of linear operators L and Δ of \mathbb{R}^N into itself, hyperbolicity satisfies the estimate*

$$\gamma(L + \Delta) \geq \gamma(L) - \|\Delta\|. \quad (.36)$$

Proof: By the definition of hyperbolicity,

$$\gamma(L + \Delta) = \inf_{\phi \in [0,1]} \inf_{\|v\|=1} |(L + \Delta)v - \exp(2\pi i\phi)v|. \quad (.37)$$

By triangle inequality, for all $v \in \mathbb{R}^N$,

$$|(L + \Delta)v - \exp(2\pi i\phi)v| \geq |Lv - \exp(2\pi i\phi)v| - |\Delta v|. \quad (.38)$$

This implies the statement of the lemma. Q.E.D.

The following lemma generalizes the previous two lemmas.

Lemma A.4. *For all $N \times N$ matrices A, A_1, A_2, \dots, A_m ,*

$$\gamma(A_m A_{m-1} \cdots A_1) \geq \gamma(A) - \sum_{j=1}^m \|A - A_j\|. \quad (.39)$$

Proof: Choose $v_0 \in \mathbb{C}^N$ and $\phi \in \mathbb{R}$ such that

$$|A_m A_{m-1} \cdots A_1 v_0 - e^{i\phi} v_0| = \gamma(A_m A_{m-1} \cdots A_1) |v_0|. \quad (.40)$$

Let $v_1 = A_1 v_0$, $v_2 = A_2 A_1 v_0$, \dots , $v_m = A_m A_{m-1} \cdots A_1 v_0$. For $j = 0, 1, \dots, m-1$ let

$$\omega_j = e^{i(-\phi + 2\pi j)/m} \quad (.41)$$

and

$$u_j = v_0 + \omega_j v_1 + \omega_j^2 v_2 + \cdots + \omega_j^{m-1} v_{m-1}. \quad (.42)$$

Choose ℓ for which $|v_\ell| = \max(|v_0|, |v_1|, \dots, |v_{m-1}|)$, and notice that

$$\sum_{j=0}^{m-1} \omega_j^{-\ell} u_j = m v_\ell. \quad (.43)$$

Thus there exists j such that

$$|u_j| \geq |v_\ell| = \max(|v_0|, |v_1|, \dots, |v_{m-1}|). \quad (.44)$$

Then we have

$$\begin{aligned}
\gamma(A) &\leq |Au_j - \omega_j^{-1}u_j|/|u_j| \\
&= \frac{|-\omega_j^{-1}v_0 + Av_0 - v_1 + A\omega_j v_1 - \omega_j v_2 + \cdots + A\omega_j^{m-1}v_{m-1}|}{|u_j|} \\
&= \frac{|\omega_j^{m-1}(v_m - e^{i\phi}v_0) + (A - A_1)v_0 + (A - A_2)\omega_j v_1 + \cdots + (A - A_m)\omega_j^{m-1}v_{m-1}|}{|u_j|} \\
&\leq \frac{\gamma(A_m A_{m-1} \cdots A_1)|v_0| + \|A - A_1\||v_0| + \|A - A_2\||v_1| + \cdots + \|A - A_m\||v_{m-1}|}{|u_j|} \\
&\leq \gamma(A_m A_{m-1} \cdots A_1) + \|A - A_1\| + \|A - A_2\| + \cdots + \|A - A_m\|,
\end{aligned}$$

which is equivalent to the desired inequality. Q.E.D.

Proposition A.5. *Let $r \leq 1 \leq K$ be positive numbers and A, B be linear operators of \mathbb{R}^N into itself given by $N \times N$ matrices from $M_N(\mathbb{R})$ with real entries. Consider an N^2 -parameter family $\{A_U = A + UB\}_{U \in C^{N^2}(r)}$, where $C^{N^2}(r)$ is the cube in $M_N(\mathbb{R})$ whose entries are bounded in absolute value by r . Suppose that $\|B\|, \|B^{-1}\| \leq K$. Then for the Lebesgue product probability measure μ_{r, N^2} on the cube $C^{N^2}(r)$ and all $0 < \gamma \leq \min(r, 1)$, we have*

$$\mu_{r, N^2} \left\{ U \in C^{N^2}(r) : \gamma(A_U) \leq \gamma \right\} \leq \frac{C(N)K^{2N^2}\gamma}{r^2}, \quad (.45)$$

where the constant $C(N)$ depends only on N .

Proof: For $0 < \gamma \leq 1$ and $\phi \in [0, 1)$, define the sets of non- γ -hyperbolic matrices by

$$\begin{aligned}
NH_N^\gamma(\mathbb{R}) &= \{L \in M_N(\mathbb{R}) : \gamma(L) \leq \gamma\}, \\
NH_N^{\gamma, \phi}(\mathbb{R}) &= \{L \in M_N(\mathbb{R}) : \inf_{|v|=1} |(L - \exp(2\pi i\phi)v)| \leq \gamma\}.
\end{aligned} \quad (.46)$$

Then

$$NH_N^\gamma(\mathbb{R}) = \cup_{\phi \in [0, 1)} NH_N^{\gamma, \phi}(\mathbb{R}). \quad (.47)$$

We claim that

$$NH_N^\gamma(\mathbb{R}) \subset \cup_{j=0, \dots, [\frac{5}{\gamma}] - 1} NH_N^{2\gamma, j/[\frac{5}{\gamma}]}(\mathbb{R}). \quad (.48)$$

Indeed, suppose that $L \in NH_N^\gamma(\mathbb{R})$. Then for some number $\phi \in [0, 1)$ and vector $v \in \mathbb{R}^N$ with $|v| = 1$, we have $|(L - \exp(2\pi i\phi))v| \leq \gamma$. Let j be the nearest integer to $[5/\gamma]\phi$ and let $\phi_\gamma = j/[5/\gamma]$; then $\phi - \phi_\gamma \leq 1/(2(5/\gamma - 1)) < \gamma/(2\pi)$. Thus

$$\begin{aligned} |(L - \exp(2\pi i\phi_\gamma))v| &\leq \\ |(L - \exp(2\pi i\phi))v| + |\exp(2\pi i\phi) - \exp(2\pi i\phi_\gamma)| &\leq 2\gamma \end{aligned} \tag{.49}$$

and $L \in NH_N^{2\gamma, j/[5/\gamma]}(\mathbb{R})$ as claimed.

Next, we claim that every matrix in $NH_N^{2\gamma, j/[5/\gamma]}(\mathbb{R})$ lies within 2γ of a matrix in $NH_N^{0, j/[5/\gamma]}(\mathbb{R})$, where we use the Euclidean (\mathbb{R}^{N^2}) norm on $M_N(\mathbb{R})$ (not the matrix norm). Consider $L \in NH_N^{2\gamma, j/[5/\gamma]}(\mathbb{R})$, $\phi \in [0, 1)$, and $v \in \mathbb{R}^N$ with $|v| = 1$ and $|(L - \exp(2\pi ij/[5/\gamma]))v| \leq 2\gamma$. Let $w = (L - \exp(2\pi ij/[5/\gamma]))v$ and let $M \in M_N(\mathbb{R})$ be the matrix whose k th row is $w_k v$, where w_k is the k th coordinate of w . Then the Euclidean norm of M is $|w| \leq 2\gamma$ and $Mv = w$, so that $(L - M - \exp(2\pi ij/[5/\gamma]))v = 0$ and hence $L - M \in NH_N^{0, j/[5/\gamma]}(\mathbb{R})$.

We complete the estimate (.45) by estimating for each j the number of γ -balls needed to cover $NH_N^{0, j/[5/\gamma]}(\mathbb{R})$ within an appropriate bounded domain. It then follows from the previous paragraph that if we inflate each of these balls to the concentric ball of radius 3γ , the collection of larger balls will cover $NH_N^{2\gamma, j/[5/\gamma]}(\mathbb{R})$, and from the paragraph before that the union over j of these covers will then cover $NH_N^\gamma(\mathbb{R})$. To this end, we show that each $NH_N^{0, j/[5/\gamma]}(\mathbb{R})$ is a real algebraic set and compute its codimension¹. Then we will apply an estimate of Yomdin [Y] on the number of γ -balls necessary to cover a given algebraic set by polynomials of known degree.

Notice that

$$NH_N^{0, \phi}(\mathbb{R}) = \{L \in M_N(\mathbb{R}) : \det(L - \exp(2\pi i\phi)Id) = 0\}. \tag{.50}$$

We split into the two cases $\exp(2\pi i\phi) \in \mathbb{R}$ (that is, $\phi = 0$ or $1/2$) and $\exp(2\pi i\phi) \notin \mathbb{R}$. In the first case, the equation $\det(L \pm Id) = 0$ is a polynomial of degree N in the entries of L , so $NH_N^{0, 0}(\mathbb{R})$ and $NH_N^{0, 1/2}(\mathbb{R})$ are real algebraic sets defined by a single polynomial of degree N .

In the second case, decompose the equation $\det(L - \exp(2\pi i\phi)Id) = 0$ into two parts: $Re[\det(L - \exp(2\pi i\phi)Id)] = 0$ and $Im[\det(L - \exp(2\pi i\phi)Id)] = 0$. Each part is given by a real polynomial of degree N . Furthermore, these two polynomials are algebraically independent, since otherwise $Re[\det(L - \exp(2\pi i\phi)Id)]$ and $Im[\det(L - \exp(2\pi i\phi)Id)]$ would satisfy some polynomial relationship and, thus,

¹Unfortunately $NH_N^0(\mathbb{R})$, in contrast to $NH_N^{0, j/[5/\gamma]}(\mathbb{R})$, is not algebraic

$\det(L - \exp(2\pi i\phi)Id)$ would take on values only in some real algebraic subset of the complex plane. However, for $N \geq 2$ (which is necessary for complex eigenvalues), by considering real diagonal matrices L we see that the values of $\det(L - \exp(2\pi i\phi)Id)$ contain an open set in \mathbb{C} . Therefore, $NH_N^{0,\phi}(\mathbb{R})$ is a real algebraic set given by two algebraically independent polynomials of degree N .

Covering Lemma for Algebraic Sets ([Y], Lemma 4.6) *Let $V \subset \mathbb{R}^m$ be an algebraic set given by k algebraically independent polynomials p_1, \dots, p_k of degrees d_1, \dots, d_k respectively, i.e. $V = \{x \in \mathbb{R}^m : p_1(x) = 0, \dots, p_k(x) = 0\}$. Let $C_A^m(s)$ be the cube in \mathbb{R}^m with side $2s$ centered at some point A . Then for $\gamma \leq s$, the number of γ -balls necessary to cover $V \cap C_A^m(s)$ does not exceed $C(D, m)(2s/\gamma)^{m-k}$, where the constant $C(D, m)$ depends only on the dimension m and product of degrees $D = \prod_i d_i$.*

Remark A.6. *Some additional arguments based on Bezout's Theorem give an upper estimate of $C(D, m)$ by $2^m D$ for γ sufficiently small.*

To complete the proof of Proposition A.5, we apply the Covering Lemma for Algebraic Sets to each $NH^{0,j/[5/\gamma]}(\mathbb{R})$, where $j = 0, \dots, [5/\gamma] - 1$, with $m = N^2$, $s = Kr$, and A as in the statement of the proposition. (Notice that if $U \in C^{N^2}(r)$ then $A + UB \in C_A^{N^2}(Kr)$, so we need only cover the part of $NH^{0,j/[5/\gamma]}(\mathbb{R})$ lying in the latter set.) In the case that $j/[5/\gamma] = 0$ or $1/2$, we have $k = 1$, $d_1 = N$, and $D = N$, so the number of covering γ -balls is bounded by $C(N, N^2)(2Kr/\gamma)^{N^2-1}$. In the case of other j , we have $k = 2$, $d_1 = d_2 = N$, and $D = N^2$, so the number of covering γ -balls is bounded by $C(N^2, N^2)(2Kr/\gamma)^{N^2-2}$. The number of j 's of the second type is less than $5/\gamma$. Combining all these estimates along with (.48) we get that $NH^\gamma(\mathbb{R}) \cap C_A^{N^2}(Kr)$ can be covered by $C(N^2, N^2)(2 + 5/(2Kr))(2Kr/\gamma)^{N^2-1}$ balls of radius 3γ .

Finally, notice that the preimage of a ball of radius 3γ under the map $U \mapsto A + UB$ is contained in a ball of radius $3K\gamma$, whose μ_{r, N^2} -measure is less than $(3K\gamma/r)^{N^2}$. Therefore the total measure of $3K\gamma$ -balls needed to cover the set $\{U \in C^{N^2}(r) : \gamma(A + UB) \leq \gamma\}$ is at most $C(N)K^{2N^2}\gamma/r^2$, where the constant $C(N)$ depends only on N . Q.E.D.

Appendix B: Orthogonal transformations of \mathbb{R}^N and the spaces of homogeneous polynomials

In this appendix, we prove that the scalar product (1.12) in the space $W_{k,N}$ of homogeneous N -vector polynomials of degree k in N variables is invariant with respect to orthogonal transformations of \mathbb{R}^N .

Lemma B.1. *Let $x \in \mathbb{R}^N$ be given by N coordinates $x = (x_1, \dots, x_N)$. For $k \in \mathbb{Z}_+$, consider homogeneous polynomials $p_k(x) = \sum_{|\alpha|=k} \vec{\varepsilon}_\alpha x^\alpha \in W_{k,N}$ in N variables, where $x^\alpha = x_1^{\alpha_1} \dots x_N^{\alpha_N}$. Let $O \in SO(N)$ be an orthogonal transformation of \mathbb{R}^N . Denote by $x' = (x'_1, \dots, x'_N)$ the new coordinate system induced by the relation $x = Ox'$. Write $p'_k(x') = p_k(Ox') = \sum_{|\alpha|=k} \vec{\varepsilon}'_\alpha (x')^\alpha$ in the new coordinate system. Then for all $\{\vec{\varepsilon}_\alpha\}_{|\alpha|=k}$ and $\{\vec{\nu}_\alpha\}_{|\alpha|=k}$,*

$$\sum_{|\alpha|=k} \binom{k}{\alpha}^{-1} \langle \vec{\varepsilon}_\alpha, \vec{\nu}_\alpha \rangle = \sum_{|\alpha|=k} \binom{k}{\alpha}^{-1} \langle \vec{\varepsilon}'_\alpha, \vec{\nu}'_\alpha \rangle. \quad (.51)$$

Proof: For this lemma it will be helpful to use a different notation for monomials. Given a k -tuple $\beta = (\beta_1, \dots, \beta_k) \in \{1, 2, \dots, N\}^k$, define $x^{(\beta)} = x_{\beta_1} x_{\beta_2} \dots x_{\beta_k}$. Notice that $x^{(\beta)} = x^\alpha$ where α_i is the number of indices j for which $\beta_j = i$. Write $\alpha(\beta)$ for the multiindex corresponding in this manner to the k -tuple β , and notice that for each multiindex α there are $\binom{k}{\alpha}$ different k -tuples β for which $\alpha(\beta) = \alpha$. Let $|\beta| = k$.

Given a polynomial p_k as in the statement of the lemma, we can write $p_k(x) = \sum_{|\beta|=k} \vec{\varepsilon}_{(\beta)} x^{(\beta)}$, where $\vec{\varepsilon}_{(\beta)} = \binom{k}{\alpha}^{-1} \vec{\varepsilon}_{\alpha(\beta)}$. We can also rewrite the scalar product as follows:

$$\sum_{|\alpha|=k} \binom{k}{\alpha}^{-1} \langle \vec{\varepsilon}_\alpha, \vec{\nu}_\alpha \rangle = \sum_{|\beta|=k} \langle \vec{\varepsilon}_{(\beta)}, \vec{\nu}_{(\beta)} \rangle. \quad (.52)$$

(Remember that for each α , there are $\binom{k}{\alpha}$ corresponding terms on the right-hand side.)

Our goal is then to show that

$$\sum_{|\beta|=k} \langle \vec{\varepsilon}_{(\beta)}, \vec{\nu}_{(\beta)} \rangle = \sum_{|\beta|=k} \langle \vec{\varepsilon}'_{(\beta)}, \vec{\nu}'_{(\beta)} \rangle. \quad (.53)$$

We prove this by induction on k . For $k = 0$, the identity is trivial. Assume that the identity now holds for some $k \geq 0$. Given β with $|\beta| = k$ and $i \in \{1, 2, \dots, N\}$, let (β, i) be the $(k+1)$ -tuple $(\beta_1, \dots, \beta_k, i)$. Also, let $\vec{\varepsilon}_{(\beta)} = \vec{\varepsilon}_{(\beta, i)}$. The reason for this alternate notation is that we will mean something different below by $\vec{\varepsilon}'_{(\beta)}$ and $\vec{\varepsilon}'_{(\beta, i)}$. In the former case, we fix i and apply the coordinate transformation O to the polynomial $\sum_{|\beta|=k} \vec{\varepsilon}_{(\beta)} x^{(\beta)}$ to get the coefficients $\vec{\varepsilon}'_{(\beta)}$. In the latter case, we transform the polynomial $\sum_{|(\beta, i)|=k+1} \vec{\varepsilon}_{(\beta, i)} x^{(\beta, i)}$ to get the coefficients $\vec{\varepsilon}'_{(\beta, i)}$.

Next, notice that

$$\sum_{|(\beta, i)|=k+1} \vec{\varepsilon}_{(\beta, i)} x^{(\beta, i)} = \sum_{i=1}^N x_i \sum_{|\beta|=k} \vec{\varepsilon}_{(\beta)} x^{(\beta)}. \quad (.54)$$

Applying the coordinate change $x = Ox'$ to both sides, we get

$$\sum_{|(\beta, i)|=k+1} \vec{\varepsilon}'_{(\beta, i)} (x')^{(\beta, i)} = \sum_{i=1}^N \sum_{j=1}^N O_{ij} x'_j \sum_{|\beta|=k} \vec{\varepsilon}'_{(\beta)} (x')^{(\beta)}. \quad (.55)$$

It follows that

$$\vec{\varepsilon}'_{(\beta, j)} = \sum_{i=1}^N O_{ij} \vec{\varepsilon}'_{(\beta)}. \quad (.56)$$

A similar identity holds with $\vec{\varepsilon}$ replaced by $\vec{\nu}$, whereupon

$$\sum_{|(\beta, j)|=k+1} \langle \vec{\varepsilon}'_{(\beta, j)}, \vec{\nu}'_{(\beta, j)} \rangle = \sum_{j=1}^N \sum_{|\beta|=k} \sum_{i=1}^N \sum_{\ell=1}^N \langle O_{ij} \vec{\varepsilon}'_{(\beta)}, O_{\ell j} \vec{\nu}'_{(\beta)} \rangle. \quad (.57)$$

Since O is an orthogonal matrix, $\sum_{j=1}^N O_{ij} O_{\ell j} = \delta_{i\ell}$. Exchanging the order of summation on the right-hand side above, we then have

$$\sum_{|(\beta, j)|=k+1} \langle \vec{\varepsilon}'_{(\beta, j)}, \vec{\nu}'_{(\beta, j)} \rangle = \sum_{i=1}^N \sum_{|\beta|=k} \langle \vec{\varepsilon}'_{(\beta)}, \vec{\nu}'_{(\beta)} \rangle = \sum_{|(\beta, i)|=k+1} \langle \vec{\varepsilon}'_{(\beta)}, \vec{\nu}'_{(\beta)} \rangle \quad (.58)$$

by the inductive hypothesis. Q.E.D.

Appendix C: Embedding of the space of diffeomorphisms of a compact manifold $\text{Diff}^r(M)$ into the one of the ball $\text{Diff}^r(B^N)$

In this Appendix we describe how to extend a diffeomorphism of a compact manifold embedded into a Euclidean space and what conditions we need to ensure that the results of Sacker [Sac] and Fenichel [F] about persistence of invariant manifolds apply.

Recall that M is a smooth (C^∞) compact manifold. First we consider a manifold $\tilde{M} = M \times [0, 1] / \sim$, where the equivalence relation is defined by $(x, 0) \sim (f(x), 1)$ for all $x \in M$. \tilde{M} is as smooth as f is and carries a naturally defined vector field X_f whose time one map, restricted to $M \times \{0\}$ coincides with f . Such a construction is usually called *suspension*. Now embed \tilde{M} into the interior of the closed unit ball $B^N \subset \mathbb{R}^N$. Recall that given a smooth (C^∞) compact manifold \tilde{M} of dimension D , for $N > 2D$ the Whitney Embedding Theorem says that a generic smooth function from \tilde{M} to \mathbb{R}^N is a diffeomorphism between \tilde{M} and its image. For a moment denote this image by $E(\tilde{M})$, where $E : \tilde{M} \rightarrow \mathbb{R}^N$ is the embedding. Suppose also that the image $E(\tilde{M})$ is contained in the unit ball $B^N(O)$ whose center has distance 2 from the origin. Now embed \tilde{M} into \mathbb{R}^{N+1} as follows: Points on \tilde{M} have naturally defined coordinates $\{(x, \phi) \in M \times [0, 1]\}$ once $M \times \{0\}$ is fixed. Embed $B^N(O) \times \mathbb{S}^1 = \{(y, \phi) \in B^N(O) \times [0, 2\pi) \subset \mathbb{R}^{N+1}\}$ by

$$(y, \phi) \mapsto (\sin \phi \|E(y)\|, \cos \phi E(y)) \in \mathbb{R}^{N+1}.$$

Using this formula embed \tilde{M} . By construction $M \times \{0\}$ belongs to the hyperplane, where the first component equals zero. Now to simplify notation, we identify \tilde{M} and M with their images, so that \tilde{M} and $M \times \{0\}$ become submanifolds of \mathbb{R}^{N+1} .

Let $\tilde{T} \subset \mathbb{R}^{N+1}$ be a closed neighborhood of \tilde{M} , chosen sufficiently small that there is a well-defined projection $\tilde{\pi} : \tilde{T} \rightarrow \tilde{M}$ for which $\tilde{\pi}(\tilde{x})$ is the closest point in \tilde{M} to \tilde{x} . Then for each $\tilde{y} \in \tilde{M}$, $\tilde{\pi}^{-1}(\tilde{y})$ is an $(N - D)$ -dimensional disk. Then we can extend each vector field X_f on \tilde{M} to a vector field \mathcal{X}_f on \tilde{T} so that the component tangent to $\tilde{\pi}^{-1}(\tilde{y})$ is strongly dominated by the orthogonal one and is directed toward \tilde{y} . Consider now the Poincaré return map of \mathcal{X}_f from $T = \tilde{T} \cap \{B^N(O) \times \{0\}\}$ into $B^N(O) \times \{0\}$, which is well-defined by the construction. Denote this map by F . The vector field \mathcal{X}_f is directed so that F maps T strictly inside itself.

Now the closed neighborhood T of M can be considered as a subset of \mathbb{R}^N and can be chosen sufficiently small that there is a well-defined projection $\pi : T \rightarrow M$ for which $\pi(x)$ is the closest point in M to x . Then for each $y \in M$, $\pi^{-1}(y)$ is an $(N - D)$ -dimensional disk. For $0 < \rho < 1$ and $y, z \in M$ choose a linear function $g_{\rho, y, z} : \pi^{-1}(y) \rightarrow \pi^{-1}(z)$ that maps y to z and contracts distances by a factor of ρ , and such that the dependence of $g_{\rho, y, z}$ on y and z is C^r . Then we can extend each $F \in \text{Diff}^r(M)$ to a function $f \in C^r(T)$ that is a diffeomorphism from T to a subset of its interior by letting

$$f(x) = g_{\rho, \pi(x), F(\pi(x))}(x)$$

where $\rho = \min(\|F^{-1}\|_{C^1}^{-r}, 1)/2$. Then by Fenichel's Theorem [F] every sufficiently small perturbation $f_\varepsilon \in C^r(T)$ of such an f has an invariant manifold $M_\varepsilon \subset T$ for which $\pi|_{M_\varepsilon}$ is a C^r diffeomorphism from M_ε to M . Then to such an f_ε we can associate a diffeomorphism $F_\varepsilon \in \text{Diff}^r(M)$ by letting

$$F_\varepsilon(y) = \pi(f_\varepsilon(\pi|_{M_\varepsilon}^{-1}(y))).$$

Notice that the periodic points of f_ε all lie on M_ε and are in one-to-one correspondence with the periodic points of f_ε . Furthermore, because F_ε and $f_\varepsilon|_{M_\varepsilon}$ are conjugate, the hyperbolicity of each periodic orbit is the same for either map. Thus any estimate on $P_n(f_\varepsilon)$ or $\gamma_n(f_\varepsilon)$ applies also to F_ε .

Appendix D: Pathological examples of decay of product of distances of recurrent trajectories

In this Appendix we present several examples of horseshoe diffeomorphisms which show that with the ideas we have in this paper the estimate $\exp(Cn^{1+\delta})$ on the growth of the number of periodic points (the Main Theorem from section 1.3) for any real positive number δ is asymptotically the optimal one. More exactly, the Shift Theorem, stated in 3.5, is crucial to split all almost periodic trajectories into classes as in (3.12). In section 3.5 we outline the proof of this Theorem and it might be helpful to review the strategy presented there, especially, the last remark right before subsection 3.5.1. Here, for $\gamma_n(C, \delta) = \exp(-Cn \ln n)$, which is the "roughly" inverse of the number of periodic points, we give an example of a periodic orbit which is nonsimple, have no leading saddles, and no close returns (gaps), defined in section 3.5. Thus we can't deal with these kind of trajectories with our methods.

Example 1 Consider the sequence of periodic orbits of a horseshoe map with symbolic dynamics

$$\begin{aligned} S_0 &= 1 \\ S_1 &= 0 \\ S_2 &= 01 \\ S_3 &= 010 \\ S_4 &= 01001 \\ S_5 &= 01001010 \\ S_6 &= 0100101001001 \\ &\vdots \end{aligned}$$

Each sequence is the concatenation of the previous two sequences; it can also be generated from the previous sequence by the substitution rules $0 \rightarrow 01$ and $1 \rightarrow 0$. The number of symbols in S_n is the n th Fibonacci number F_n .

Notice also that

$$\begin{aligned}
S_n &= S_{n-1}S_{n-2} \\
&= S_{n-2}S_{n-3}S_{n-2} \\
&= S_{n-3}S_{n-4}S_{n-3}S_{n-3}S_{n-4} \\
&= S_{n-4}S_{n-5}S_{n-4}S_{n-4}S_{n-5}S_{n-4}S_{n-5}S_{n-4} \\
&= \dots
\end{aligned}$$

More formally, the sequence S_n can be generated from S_k for any $0 \leq k \leq n$ by replacing each 0 in S_k by S_{n-k+1} and each 1 by S_{n-k} . We refer below to this decomposition of S_n into copies of S_{n-k} and S_{n-k+1} as “decomposition k ”

Every three symbol subsequence of S_n is either 010, 100, 001, or 101. Furthermore, regarding each S_n as a cyclic sequence, each of the four triplets above occurs at least once in S_4 , at least once in S_5 , at least twice in S_6 , and in general at least F_{n-4} times in S_n for $n \geq 4$. The importance of this observation below will be that in decomposition k for $4 \leq k \leq n$, each of the substrings $S_{n-k+1}S_{n-k}S_{n-k+1}$, $S_{n-k}S_{n-k+1}S_{n-k+1}$, $S_{n-k+1}S_{n-k+1}S_{n-k}$, and $S_{n-k}S_{n-k+1}S_{n-k}$ occurs at least F_{k-4} times.

Now let x_0, x_1, \dots, x_{p-1} be points in the periodic orbit with symbolic dynamics S_n , where $p = F_n$ is the length of S_n . No matter where the symbolic sequence of x_0 starts within S_n , we claim that for n sufficiently large,

$$\prod_{j=1}^{p-1} |x_0 - x_j| \leq e^{(c_1 - c_2 n)p} \leq e^{c_1 p - c_3 p \log p}$$

for some positive constants c_1, c_2, c_3 independent of n . The latter inequality follows from the fact that $p \leq 2^n$, so it remains to prove the first inequality.

Assume that the distance between any two points in the nonwandering set is at most 1. Say the symbolic sequence of x_0 starts at the m th symbol of S_n . If for some positive integer q , the block of $2q - 1$ symbols centered at the m th symbol is repeated centered at the ℓ th symbol, then the distance between the points x_0 and $x_{\ell-m}$ is bounded above by e^{-cq} for an appropriate positive constants c . Here the index $\ell - m$ is taken modulo p .

Now for $4 \leq k \leq n$, in decomposition k the m th symbol in S_n lies in a copy of either S_{n-k} or S_{n-k+1} , which in turn lies in the middle of one of the four substrings $S_{n-k+1}S_{n-k}S_{n-k+1}$, $S_{n-k}S_{n-k+1}S_{n-k+1}$, $S_{n-k+1}S_{n-k+1}S_{n-k}$, and $S_{n-k}S_{n-k+1}S_{n-k}$ described above. Each such substring occurs at least F_{k-4} times in S_n , and all but one of these occurrences contributes a factor of at most $e^{-cF_{n-k}}$ to the product of distances $|x_0 - x_j|$. Therefore for $n \geq 6$,

$$\begin{aligned} \prod_{j=1}^{p-1} |x_0 - x_j| &\leq \prod_{k=6}^n (e^{-cF_{n-k}})^{F_{k-4}-F_{k-5}} \\ &= e^{-c \sum_{k=6}^n F_{n-k} F_{k-6}} \\ &\leq e^{-c(n-5)F_n/F_8} \\ &= e^{c(5-n)p/34}. \end{aligned}$$

(The estimate $F_{n-k}F_{k-6} \geq F_n/F_8$ can be proved by induction, but heuristically this type of estimate follows from the fact that F_n is approximately an exponential function of n .)

Example 2 Consider now the aperiodic nonwandering orbit of the horseshoe map whose symbolic dynamics are given as follows. Given a sequence of positive integers k_1, k_2, \dots , let $S_0 = 0$ and $S_n = 1S_{n-1}S_{n-1} \cdots S_{n-1}1$ where S_{n-1} occurs $2k_n + 1$ consecutive times. For example, if $k_n = n$ then

$$\begin{aligned} S_0 &= 0 \\ S_1 &= 10001 \\ S_2 &= 110001100011000110001100011 \\ &\vdots \end{aligned}$$

Each sequence is symmetric, and for $n \geq 1$, each S_n contains a copy of S_{n-1} at its center. Let L_n be the length of S_n ; then $L_0 = 1$ and $L_n = (2k_n + 1)L_{n-1} + 2$ for $n \geq 1$. One can easily check that $k_1 k_2 \cdots k_n \leq L_n \leq 5^n k_1 k_2 \cdots k_n$.

Let x_0 be the point on the nonwandering set whose symbolic sequence has middle L_n symbols S_n for each $n \geq 0$. By symmetry, to estimate the product of distances $|x_0 - x_j|$ as j goes from 1 to $L_n - 1$, we can estimate the product as j goes from $-(L_n - 1)$ to $L_n - 1$, excluding $j = 0$, and take the square root of the latter estimate.

As in the previous example, let c be a positive constant such that $|x_0 - x_j| \leq e^{-cq}$, where q is the largest positive integer for which the middle $2q - 1$ symbols of the sequences for x_0 and x_j agree, or $q = 0$ if their middle symbols do not agree. Then

for all $n \geq 1$ and $-k_n \leq m \leq k_n$ we have $|x_0 - x_{mL_{n-1}}| \leq e^{-c(k_n - |m| + 1/2)L_{n-1}}$. The square root of the product of these upper bounds, excluding $m = 0$, is

$$\prod_{m=1}^{k_n} e^{-c(k_n - m + 1/2)L_{n-1}} = e^{-ck_n^2 L_{n-1}/2} \leq e^{-ck_n L_n/10}.$$

Here we used the inequality $5k_n L_{n-1} \geq (2k_n + 1)L_{n-1} + 2 = L_n$.

In addition, for n and m as above and all $-k_{n-1} \leq p \leq k_{n-1}$ we have $|x_0 - x_{mL_{n-1} + pL_{n-2}}| \leq e^{-c(k_{n-1} - |p| + 1/2)L_{n-2}}$. The square root of the product of these upper bounds, excluding $p = 0$, is

$$\begin{aligned} \prod_{m=-k_n}^{k_n} \prod_{p=1}^{k_{n-1}} e^{-c(k_{n-1} - |p| + 1/2)L_{n-2}} &= e^{-c(2k_n + 1)k_{n-1}^2 L_{n-2}/2} \\ &\leq e^{-c(2k_n + 1)k_{n-1}(L_{n-1} + 1)/12}. \end{aligned}$$

Here we used the inequality $6k_{n-1}L_{n-2} \geq (2k_{n-2} + 1)L_{n-2} + 3 = L_{n-1} + 1$. Then in turn we can say $(2k_n + 1)(L_{n-1} + 1) \geq (2k_n + 1)L_{n-1} + 3 = L_n + 1$, so that the bound on the product above can be replaced by $e^{-ck_{n-1}L_n/12}$.

In a similar manner, we can bound above another set of terms contributing to the product of distances $|x_0 - x_j|$ by $e^{-ck_{n-\ell}L_n/12}$ for $\ell = 2, 3, \dots, n-1$. Multiplying all these bounds together we conclude that

$$\prod_{j=1}^{L_n-1} |x_0 - x_j| \leq e^{-c(k_1 + k_2 + \dots + k_n)L_n/12}.$$

Notice that if $k_n = k$ independent of n , then $L_n \sim (2k + 1)^n$ and $k_1 + k_2 + \dots + k_n = nk \sim \log L_n$. Thus we get an estimate similar to Example 1.

If $k_n = n^\alpha$, then $\log L_n \sim n \log n$ and hence $k_1 + k_2 + \dots + k_n \sim n^{\alpha+1} \sim (\log L_n)^{\alpha+1}$, loosely speaking. The closest returns to x_0 are of the form $-\log|x_0 - x_{L_n}| \sim k_{n+1}L_n \sim L_n(\log L_n)^\alpha$, loosely speaking again. Thus if we attempt to apply the Inductive Hypothesis with $\gamma_j = C^{-j(\log j)^\beta}$, this example with $\alpha = \beta - 1/2$ shows that the product of distances along a hyperbolic trajectory can be smaller than any fixed power of γ_j for arbitrarily large j , despite the fact that the closest return over j iterations is larger than any fixed power of γ_j for j sufficiently large.

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