Remote Periodic and Quasiperiodic Motions in the Planar Circular Restricted 3-Body Problem of KAM and Aubry-Mather Type

Vadim Kaloshin & Timothy Nguyen

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Abstract

The planar circular restricted three-body problem (PCR3BP) is a standard example of a system with two degrees of freedom that is not integrable. Despite the efforts of many great mathematicians for over the past two centuries, our understanding of the solutions of PCR3BP is still far from complete. One way to understand the behavior of such solutions is through the investigation of invariant sets, which in this case, consists of periodic and quasiperiodic solutions. Up to now, such investigations have relied upon the highly successful KAM theory, developed only about fourty years ago. However, such applications of KAM theory are limited since KAM theory only describes systems that are close to integrable. This places strong constraints on the Jacobi constant and mass ratio of PCR3BP. Furthermore, KAM theory can prove only the existence of quasiperiodic solutions.

In this paper, we prove the existence of both quasiperiodic and periodic solutions for PCR3BP. Moreover, our results hold for a much larger, more practical range of Jacobi constant and mass ratio than that required for KAM theory. In fact, we show that periodic and quasiperiodic solutions exist for parameters of PCR3BP that approximate a "Sun–Jupiter–planar Pluto" system. Our approach, however, is based upon the more recent and less well-known Aubry-Mather theory. Aubry-Mather theory is a mathematical theory that provides a framework for constructing periodic and quasiperiodic solutions for Hamiltonian systems with 1.5 or 2 degrees of freedom. Moreover, these systems need not be near-integrable. By constructing an appropriate Poincaré return map on a surface of constant energy, we are able to examine the dynamics of PCR3BP on a two-dimensional section, thereby giving us an appropriate setting for the application of Aubry-Mather theory and enabling us to prove the existence of a rich variety of periodic and quasiperiodic motions. We shall also apply KAM theory to prove the existence of invariant curves.

Introduction

Consider the planar circular restricted 3-body problem (PCR3BP). Namely, consider two massive bodies, the *primaries*, performing uniform circular motion about their center of mass. Normalizing the masses of the primaries so that their masses sum to unity, we obtain primaries of mass μ and $1-\mu$ respectively, where $0 < \mu < 1$ is the *mass ratio*. In addition, we chose coordinates so that the center of mass of the system is located at the origin, and we normalize the period of the circular motion to 2π. By entering into a frame which rotates with the primaries, we can choose rectangular coordinates (x, y) so that the primaries are fixed at $(1-\mu, 0)$ and $(-\mu, 0)$, respectively. Finally, we introduce a third massless body P into the system, so that it does not effect the primaries. PCR3BP investigates how P moves.

The distance of P to the primaries is given by $d_1(x,y) = [(x-(1-\mu))^2 + y^2]^{1/2}$ and $d_2(x,y) = [(x-\mu)^2 + y^2]^{1/2}$. The standard formula for the *Jacobi constant* C, the only integral for PCR3BP, is given by

$$C = x^2 + y^2 + \frac{2\mu}{d_1} + \frac{2(1-\mu)}{d_2} - (\dot{x}^2 + \dot{y}^2). \tag{1}$$

The standard physical interpretation for PCR3BP is to consider the massive body the Sun, the less massive body Jupiter, and the massless body a planet (see Figure 1).

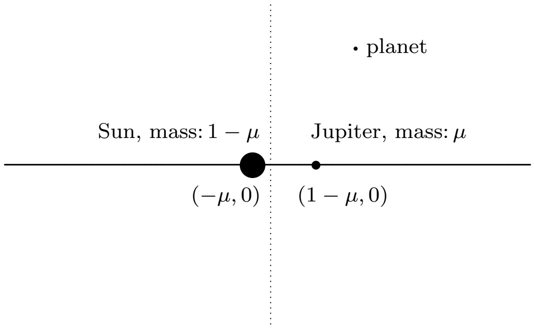

Figure 1. The rotating frame

In this paper we shall consider values of the Jacobi constant $C > 4$. Easy arguments show that there are three regions in the (x, y)-plane where P can be located at any time. These regions are usually called *Hill regions*. Namely, there are two bounded regions in a neighborhood of the primaries (either d_1^{-1} or d_2^{-1} is large with the remaining terms appearing in the Jacobi constant small), and then an unbounded region that contains $x^2 + y^2 \geq C$. In the first region, Birkhoff [Bi] found periodic orbits and Moser [SM] proved the existence of KAM type quasiperiodic motions. Chenciner and Llibre [CL] extended existence of KAM type quasiperiodic motions to the second region. In this paper we investigate motion in the third *outer region* and prove existence of variety of periodic and quasiperiodic motions using KAM and Aubry-Mather theory (for twist maps).

Recall that Aubry-Mather theory deals with exact area-preserving twist (EAPT) maps of the cylinder $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$. EAPT maps of $\mathbb{A}$ naturally arise as Poincaré return maps of either autonomous Hamiltonian systems with 2 degrees of freedom or time-periodic systems with 1 degree of freedom. PCR3BP is an autonomous Hamiltonian system with 2 degrees of freedom, i.e. $(x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4$ and the phase space is four-dimensional. Since Aubry-Mather theory can be applied to Hamiltonian systems which are not necessarily close to integrable, we apply AM theory for a range of (μ, C) which is significantly larger than the range of applications of KAM theory. For example, in the case $\mu = 1/1000$, which is the approximately mass ratio for Sun and Jupiter, we prove existence of periodic and quasiperiodic Aubry-Mather type motions for when $C \gtrapprox 5$ and P is roughly at least 4 times further away from the heavy body of mass $1 - \mu$ than the light body of mass μ. Application of KAM theory, in contrast, begins when P is significantly further away from the primaries than a factor of 4, and the mass ratio has to be extremely small even compared to 1/1000.

The phase space $\mathbb{R}^4$ is foliated by invariant generically 3-dimensional energy surfaces[1] $\Pi_E = \{(x, y, \dot{x}, \dot{y}) : C(x, y, \dot{x}, \dot{y}) = 2E\}$. For convenience, we will call E the *energy*, and we will often refer to E instead of C to avoid unnecessary factors of 2. In order to investigate trajectories on an energy hypersuface Π_E and apply Aubry-Mather theory[2] we need to find:

• A 2-dimensional surface S_E inside Π_E, where a Poincaré return map $F : U \subset S_E \to S_E$ is defined.

• A coordinate system in $\mathbb{R}^4$ such that we could verify that F is an EAPT map. Here the most important condition to check is the so-called "twist condition", which is an analog of convexity.

The most convenient coordinate system for PCR3BP is the action-angle coordinate system for the corresponding two-body problem (i.e. the rotating Kepler problem), in which we set $\mu = 0$. For the

[1]An autonomous Hamiltonian system always has for a first integral its Hamiltonian function. We shall always call the hypersurface obtained by setting the Hamiltonian equal to a constant as an energy surface, even if the Hamiltonian does not truly represent the actual, physical energy of the system. This is the case for PCR3BP.

[2]To apply KAM theory we shall use the same approach.

rotating Kepler problem, there is a well-defined action-angle coordinate system $(\theta, \theta_r, J, J_r) \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}$, where the Hamiltonian becomes

$$H_0(\theta, \theta_r, J, J_r) = -\frac{1}{2J^2} - (J - J_r).$$

After adding the massless body P, we let $\mu > 0$. This has the effect of adding a small perturbation term ΔH to H_0, since instead of a fixed center of attraction, we now have two attracting bodies. In the polar coordinates (r, φ) of the rotating frame, it is easy to verify that the distances to the primaries is given by

$$\begin{aligned} d_1 &= \left(r^2 - 2(1-\mu) r \cos\varphi + (1-\mu)^2\right)^{1/2} \\ d_2 &= (r^2 + 2\mu r \cos\varphi + \mu^2)^{1/2}. \end{aligned}$$

Thus, the perturbation hamiltonian ΔH in polar coordinates is given by

$$\Delta H = \frac{1}{r} - \frac{\mu}{d_1} - \frac{1-\mu}{d_2}.$$

Since we can express polar coordinates in terms of action-angle variables, we thus arrive at the Hamiltonian H of PCR3BP expressed (implicitly) in terms of action-angle variables:

$$\begin{aligned} H &= H_0 + \Delta H \\ &= -\frac{1}{2J^2} - (J - J_r) + \left(\frac{1}{r} - \frac{\mu}{d_1} - \frac{1-\mu}{d_2}\right). \end{aligned} \tag{2}$$

The equations of motion for PCR3BP in action-angle coordinates are thus

$$\begin{cases} \dot{\theta} &= \dfrac{1}{J^3} - 1 + \dfrac{\partial \Delta H}{\partial J} \\ \dot{\theta}_r &= 1 + \dfrac{\partial \Delta H}{\partial J_r} \\ \dot{J} &= -\dfrac{\partial \Delta H}{\partial \theta} \\ \dot{J}_r &= -\dfrac{\partial \Delta H}{\partial \theta_r}. \end{cases} \tag{3}$$

By fixing $E := C/2 \equiv -H$, we can eliminate J_r by expressing J_r implicitly as a function of E and the remaining three action-angle variables (see §2). Thus, in the outer region, we can take (θ, θ_r, J) as local coordinates on Π_E. By setting $\theta_r = 0$ and placing a certain lower bound on J (see §2), we obtain a 2-dimensional surface $S_E \subset \Pi_E$ homeomorphic to a half-infinite cylinder. When the perturbation term $\frac{\partial \Delta H}{\partial J_r}$ is sufficiently small, (3) shows that $\dot{\theta}_r \approx 1$, and therefore we obtain a (locally) well-defined Poincaré return map F on S_E.

The twist condition says that every "vertical" line $l_\phi = \{\theta = \phi\} \subset S_E$, $\phi \in \mathbb{T}$, after application of F is monotonically twisted, i.e., the angle component $\theta'(\theta, J)$ of $F(\theta, J)$ is a monotonic function of J. The unperturbed rotating Kepler problem, with $\Delta H = 0$, clearly satisfies this condition, since $d(J^{-3})/dJ < 0$. Establishing the twist condition for PCR3BP, however, is complicated by the following degeneracies:

- the unperturbed twist $d(J^{-3})/dJ$ tends to zero while upper bounds for the derivatives of ΔH tend to infinity as J tends to infinity.
- since we are interested in bound motion, we need the eccentricity $e = \sqrt{1 - (J - J_r)^2/J^2}$ of the orbits to satisfy $0 \le e < 1$. However, it turns out that $e = 0$ is a degeneracy, and for fixed C, $e \to 1$ as J tends to infinity.

Another well-known difficulty is that

- the relationship between action-angle variables and polar coordinates can only be written implicitly.

This complicates estimates on ΔH and its derivatives, and at some point, we will need a computer to provide numerical data for our results.

Nevertheless, we will be able to show in this paper that F is an EAPT map (see §3) on an open subset of S_E that is homeomorphic to a cylinder. Usually Aubry-Mather theory is done inside of a *Birkhoff Region of Instability* (BRI), i.e. inside a region of S_E that is homeomorphic to a cylinder, bounded by invariant curves on both ends, and has no invariant curves in the interior. A priori, we do not know if there exists a BRI, and so we will need to derive additional estimates using so called "minimal configurations" to prove the existence of Aubry-Mather sets in §6. From this, we will be able to deduce our following two main theorems:

Consider a solution to PCR3BP in the outer region. By fixing C and eliminating J_r, we can express this solution as $(\theta, \theta_r, J)(t)$, $t \in \mathbb{R}$.

Definition 0.1 *A solution has rotation number ω if the lifts $\hat{\theta}(t)$ and $\hat{\theta}_r(t)$ of $\theta(t)$ and $\theta_r(t)$, respectively, to the universal cover $\mathbb{R}$ are such that both $\hat{\theta}(t)$ and $\hat{\theta}_r(t)$ are monotone and*

$$\lim_{|t|\to+\infty} \frac{\hat{\theta}(t)}{\hat{\theta}_r(t)} = \omega.$$

It is easy to see that this definition does not depend on a choice of a lift to $\mathbb{R}$. Given a rotation number ω, we define its *inertial rotation number* $\bar{\omega} = \omega + 2\pi$.

Roughly speaking, if a solution has rotation number ω, its orbit will make one revolution in the (x, y)-plane in time $2\pi/\omega$. This is because $\dot{\theta}_r(t) \approx 1$. We define the inertial rotation number to undo the rotation of the rotating frame, so that the inertial rotation number is related to the orbit of the massless body in the non-rotating frame.

We have the following as our main result:

Theorem AM. *Let PCR3BP(μ, C) be the set of solutions to PCR3BP with mass ratio μ and Jacobi constant C. For any pair of initial conditions (μ, C) lying in some open set $U_{\mathrm{AM}} \subset (0, 1/2] \times \mathbb{R}$, there exists nonempty intervals $\mathcal{J}_{\mathrm{AM}}(\mu, C)$ and $\Omega_{\mathrm{AM}}(\mu, C)$ with the following property: for any $\omega \in \Omega_{\mathrm{AM}}$, there exists a solution $(\theta(t), \theta_r(t), J(t))$ in PCR3BP(μ, C) with rotation number ω that stays inside $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathcal{J}_{\mathrm{AM}}$. Moreover, for each rational number $\omega = p/q \in \Omega_{\mathrm{AM}}(\mu, C)$ there is a periodic solution in PCR3BP(μ, C) with rotation number p/q.*

Remark 0.2 *In terms of actual numbers, we will be interested in computing the intervals $\mathcal{J}_{\mathrm{AM}}(\mu, C)$ and $\Omega_{\mathrm{AM}}(\mu, C)$ for large C and small μ, and we will disregard bounds on U_{AM}.*

Let $\bar{\Omega}_{\mathrm{AM}}(\mu, C) := \Omega(\mu, C) + 2\pi$ be the corresponding interval of inertial rotation numbers in Theorem AM. Below are some numerical data for this theorem:

μ	C	$\bar{\Omega}(\mu, C)$	$\mathcal{J}_{\mathrm{AM}}(\mu, C)$
1/1000	5	$[0.32, 0.37]$	$[2.57, 2.71]$
1/1000	5.4	$[0.18, 0.30]$	$[2.76, 3.29]$
1/1000	7	$[0.04, 0.14]$	$[3.55, 5.61]$
1/1000	10	$[0.01, 0.05]$	$[5.03, 10.3]$

So for instance, for $\mu = 1/000$ and $C = 5.4$, our numerics show that there exists solutions to PCR3BP with inertial rotation number $\in [0.18, 0.30]$ and which lie inside the set $S^1 \times S^1 \times [2.76, 3.29]$ in (θ, θ_r, J) variables. Intuitively, what this means is that the massless body, in the non-rotating frame, takes 3.3–5.6 times longer to make a revolution than the primaries, and that this body stays a factor of $3.6 = C^2/8$ to $7.3 = C^2/4$ times further away from the primaries than each other. Moreover, we get periodic solutions (in the rotating frame) for every rotation number $\in [-2\pi + 0.18, -2\pi + 0.30]$.

Application to the Solar System. The case $\mu = 1/1000$ and $C \approx 5.4$ is interesting because this choice of (μ, C) roughly corresponds to a "Sun-Jupiter-planar Pluto" regime. That is, $\mu = 1/1000$ approximates the mass ratio of Jupiter and Sun, while $C = 5.4$, in these units, roughly corresponds to a light object approximately the distance of Pluto. While in actuality, Pluto does not lie in the plane of the solar system, and in fact, the planet most responsible for perturbing its orbit is Neptune, if we ignore Neptune and pretend Pluto were in the plane (hence, "planar Pluto"), our numerics come close to describing periodic and quasiperiodic motions in a neighborhood close to Pluto's orbit. It turns out we do not quite get Pluto even in the planar case, since the value of its eccentricity is actually a bit low (see Section §6.2 for further details). Nevertheless, we relate our numerics to such a Sun-Jupiter-planar Pluto system to give a sense of how applicable our numbers are. For $C \approx 5$, this corresponds to the Jacobi constant of Neptune. However, Neptune, being only 20 times less massive than Jupiter, is somewhat of an unfair planet to consider for PCR3BP. The higher values of C listed would apply for objects further out in the solar system.

As a final part of this paper, we also prove that quasiperiodic solutions (in fact, invariant curves) exist in PCR3BP via the application of a KAM theorem.

Definition 0.3 *A solution to PCR3BP is a KAM quasiperiodic solution (of rotation number ω) if it has irrational rotation number ω, the trajectory is dense in an analytic 2-dimensional torus $\mathbb{T}^2_\omega$ embedded into $\mathbb{R}^4$, and there is an analytic coordinate system $\phi = (\phi_1, \phi_2) \in \mathbb{T}^2_\omega$ such that the flow (3) becomes $\dot{\phi} = (\omega, 1)$.*

Definition 0.4 *A number ω is said to be diophantine if there exist $K > 0$ and $\sigma > 0$ such that for any pair of integers $p, q \neq 0$, we have*

$$\left| \omega - \frac{p}{q} \right| \geq \frac{K}{q^{2+\sigma}}.$$

Theorem KAM. *With the notations above for any Jacobi constant $C > 4$ and any diophantine number $0 < \omega < \omega_0$, there is a function $\mu_0 = \mu_0(C, \omega, K, \sigma)$ such that if the mass ratio $0 < \mu < \mu_0$, then PCR3BP(μ, C) has a KAM quasiperiodic solution of rotation number ω. Moreover, there is $C_0 = C_0(\mu)$ such that for any Jacobi constant $C > C_0$, PCR3BP(μ, C) has a KAM quasiperiodic motion.*

The paper is organized as follows. In Section 1 we present our choice of mathematical formalism for PCR3BP, including a derivation of the Hamiltonian (2), an introduction to action-angle coordinates we use, and a discussion of the term ΔH. In Section 2, we state explicitly what standard assumptions we hold in this paper. In Section 3, we introduce the Poincaré return map F, explain the conditions needed to verify that F is an exact-area preserving twist map, and derive a sufficient condition guaranteeing the twist condition. Section 4 is devoted to placing upper bounds on the partial derivatives of ΔH with respect to the action-angle variables. In Section 5, we prove, via Theorem 5.7, that the twist condition holds on an open set $\mathcal{J}_{\text{twist}}(\mu, C)$ of J's and discuss numerics of $\mathcal{J}_{\text{twist}}(\mu, C)$. In Section 6 we introduce Aubry-Mather theory, complete the proof of the applicability of Aubry-Mather theory, namely, that F is an $EAPT$ map, and then apply Aubry-Mather theory to prove Theorem AM. Numerical results occupy the remainder of the section. In Section 7, we apply KAM theory and prove Theorem KAM. In Appendix A we establish necessary bounds to justify numerics and appliy the implicit function theorem to prove that the elimination of the J_r-variable is possible once we fix E (see Section 2).

1 Derivation of Hamiltonian

Here we shall derive the Hamiltonian for PCR3BP in terms of action-angle variables. We proceed in a few steps.

1.1 The Non-Rotating Frame

We adopt the notation from the introduction. In order to write the Hamiltonian for PCR3BP, first suppose that massless body P had mass m. (We still ignore the gravitational attraction of the third body on the two massive bodies. In the end, we will let $m \to 0$). Then the Hamiltonian for the system, being the sum of the kinetic and potential energy is

$$H_m = \left(\frac{p_r^2}{2m} + \frac{p_\psi^2}{2mr^2} \right) + \left(-\frac{m\mu}{d_1(t)} - \frac{m(1-\mu)}{d_2(t)} \right),$$

where p_r and p_ψ are the momenta conjugate to the coordinates r and ψ, respectively. Here, the Hamiltonian is time-dependent due to the motion of the primaries in the non-rotating frame. If we make the following change of momentum coordinates

$$\begin{aligned} p_r &= mP_r \\ p_\psi &= mP_\psi, \end{aligned}$$

i.e., we switch to "velocity" coordinates, then we get a symplectic change of coordinates with multiplier $1/m$, which means that Hamiltonian in the new coordinates is given by

$$\begin{aligned} H_m(r, \varphi, P_r, P_\psi) &= \frac{1}{m} H_m(r, \psi, p_r, p_\psi) \\ &= \left(\frac{P_r^2}{2} + \frac{P_\psi^2}{2r^2} \right) + \left(-\frac{\mu}{d_1(t)} - \frac{(1-\mu)}{d_2(t)} \right). \end{aligned}$$

This is independent of m, and so letting $m \to 0$, the Hamiltonian for PC3BPR is simply

$$H = \left(\frac{P_r^2}{2} + \frac{P_\psi^2}{2r^2} - \frac{1}{r} \right) + \left(\frac{1}{r} - \frac{\mu}{d_1(t)} - \frac{(1-\mu)}{d_2(t)} \right).$$

Observe that the first term is the Hamiltonian for the Kepler problem with the reduced mass in the Kepler problem set equal to unity.[3] The second term is the perturbation Hamiltonian, and we will be interested in this term, since it measures the deviation of H from being an integrable two body Hamiltonian.

1.2 The Rotating Frame

The next step is to enter a frame that rotates with the primaries in order to eliminate the time periodicity of H. This accomplishes the change of variable

$$\varphi = \psi - t,$$

where φ is the polar angle in the rotating frame. To make the change of variables from $(q_1, q_2, p_1, p_2) = (r, \psi, p_r, p_\psi)$ to $(Q_1, Q_2, P_1, P_2) = (r, \varphi, P_r, P_\psi)$, canonical, we use the following generating function:

$$\begin{aligned} S_1(q_1, q_2, P_1, P_2, t) &= (q_2 - t)P_2 + q_1 P_1 \\ &= (\psi - t)P_\varphi + rP_r \end{aligned}$$

The canonical transformation for this type of generating function is

$$p_i = \frac{\partial S_1}{\partial q_i} \qquad Q_i = \frac{\partial S_1}{\partial P_i}.$$

[3] For the Kepler problem, the Hamiltonian (with the appropriate normalizations) is $H_2 = \left(\frac{P_r^2}{2\tilde{m}} + \frac{P_\psi^2}{2\tilde{m}r^2} - \frac{1}{r} \right)$, where $\tilde{m}$ is the reduced mass of the system. Here, we have $\tilde{m} = 1$, even though $\tilde{m}$ does not represent the mass of anything in PCR3BP. It should be thought of as just a formal parameter.

Thus,

$$\begin{aligned} p_r &= \frac{\partial S_1}{\partial r} = P_r \\ p_\psi &= \frac{\partial S_1}{\partial \psi} = P_\varphi \\ r &= \frac{\partial S_1}{\partial P_r} = r \\ \varphi &= \frac{\partial S_1}{\partial P_\varphi} = \psi - t. \end{aligned}$$

Hence, the Hamiltonian for PCR3BP in rotating polar coordinates (r, φ) is

$$\begin{aligned} H(r, \varphi, P_r, P_\varphi) &= H(r, \psi, p_r, p_\psi) + \frac{\partial S_1}{\partial t} \\ &= \left(\frac{P_r^2}{2} + \frac{P_\varphi^2}{2r^2} - P_\varphi - \frac{1}{r}\right) + \left(\frac{1}{r} - \frac{\mu}{d_1} - \frac{(1-\mu)}{d_2}\right) \\ &=: H_0 + \Delta H. \end{aligned}$$

We now have an autonomous Hamiltonian with 2 degrees of freedom.

1.3 Action-Angle Variables

From the above, we see that

$$H_0 = \left(\frac{P_r^2}{2} + \frac{P_\varphi^2}{2r^2} - P_\varphi - \frac{1}{r}\right)$$

is the Hamiltonian for the rotating Kepler problem. For the non-rotating Kepler problem, there exists a set of action-angle coordinates $(\theta_1, \theta_2, J_1, J_2)$, such that the Hamiltonian becomes ([GPS], p. 466)

$$\left(\frac{P_r^2}{2} + \frac{P_\psi^2}{2r^2} - \frac{1}{r}\right) = -\frac{1}{2J_1^2}.$$

Moreover, $J_2 = P_\psi$, where P_ψ is the constant angular momentum. By construction, the change from polar variables to action-angle variables is canonical. Thus, the Hamiltonian for the rotating Kepler problem in action-angle coordinates is

$$-\frac{1}{2J_1^2} - J_2.$$

Our final step is to change to a new set of action-angle coordinates $(\theta, \theta_r, J, J_r)$. This is accomplished by the generating function

$$S_2(\theta_1, \theta_2, J, J_r) = (\theta_1 + \theta_2)J - \theta_2 J_r,$$

which gives us the following canonical transformation:

$$\begin{aligned} J_1 &= \frac{\partial S_2}{\partial \theta_1} = J \\ J_2 &= \frac{\partial S_2}{\partial \theta_2} = J - J_r \\ \theta &= \frac{\partial S_2}{\partial J} = \theta_1 + \theta_2 \\ \theta_r &= \frac{\partial S_2}{\partial J_r} = -\theta_2. \end{aligned}$$

Thus, the Hamiltonian for the rotating Kepler problem in $(\theta, \theta_r, J, J_r)$ coordinates is

$$H_0 = -\frac{1}{2J^2} - (J - J_r).$$

By the way in which we derived our action-angle coordinates, we always have $J \geq J_r \geq 0$.

In summary, the change of coordinates proceeded as $(r, \psi, P_r, P_\psi) \to (\theta_1, \theta_2, J_1, J_2) \to (\theta, \theta_r, J, J_r)$. We can thus express the polar coordinates in terms of the action-angle coordinates and thus express the perturbation Hamiltonian implicitly in terms of action-angle variables.

In the non-rotating frame, when one solves the Kepler problem, one finds the following formulas for the polar variables (variations on these formulas can be found in any text which solves the Kepler problem, see e.g. [GPS]):

$$\begin{aligned} r &= a(1 - e\cos\xi) \\ \psi &= \cos^{-1}\left(\frac{\cos\xi - e}{1 - e\cos\xi}\right), \end{aligned}$$

where the eccentricity e, eccentric anomaly ξ, semi-major axis a, can be given in terms of the Keplerian energy E_K and angular momentum P_ψ as follows:

$$\begin{aligned} e &= \sqrt{1 + 2E_K P_\psi^2} \\ \xi - e\sin\xi &= \frac{2\pi}{a^{3/2}} t \\ a &= \frac{1}{2|E_K|}. \end{aligned}$$

But from the previous derivations, we have $E_K = -1/2J^2$ and $P_\psi = J - J_r$. Hence, the three equations above can be expressed in terms of action-angle coordinates. From this, the fact that the polar variable r does not change in the rotating frame, and that the rotating polar angle φ satisfies $\dot{\varphi} = \dot{\psi} - 1$, we have the following formulas for the polar variables (r, φ) in the rotating-frame in terms of action-angle variables:

$$\begin{aligned} r &= J^2(1 - e\cos\xi) \\ \varphi &= \cos^{-1}\left(\frac{\cos\xi - e}{1 - e\cos\xi}\right) - \theta_r, \end{aligned}$$

where

$$\begin{aligned} e &= \sqrt{1 - \frac{(J - J_r)^2}{J^2}} \\ \xi - e\sin\xi &= \theta + \theta_r. \end{aligned}$$

These formulas require a suitable choice for the arbitrary initial conditions of the angles θ and θ_r.

2 Standard Assumptions

Define the *energy*

$$E := C/2 \equiv -H = 1/2J^2 + (J - J_r) - \Delta H. \tag{4}$$

It will be more convenient to think in terms of E rather than C. Define the "unperturbed energy" by

$$E_0(J, J_r) = 1/2J^2 + (J - J_r) = \frac{1}{2a} + \sqrt{a(1 - e^2)}, \tag{5}$$

which as we can observe, is the negative of the energy (i.e. the negative of the Hamiltonian) of the rotating Kepler problem. Finally, let us define the quantity $J^-(E)$ to be the unique solution > 1 to

$$J^-(E) + \frac{1}{2J^-(E)^2} = E. \tag{6}$$

Thus, $J^-(E)$ gives approximately the minimum admissible value of J having fixed E (it is approximate since we ignore ΔH), since $J_r \geq 0$. When we do numerics later, we will set $E = E_0(J^*, J_r^*)$ for various values J^* and J_r^*, so that we can think of our situation as a perturbation of the corresponding two body problem with the same energy, semi-major axis $a = {J^*}^2$, and eccentricity $e = \sqrt{1 - (J^* - J_r^*)^2/{J^*}^2}$.

2.1 Some Constraints

Below are some of the assumptions we hold. They are not strict requirements for our results, but they are convenient and make our estimates more explicit. The reader interested in refining our numerics is free to tweak the below assumptions.

(A1) $C > 4$ and $r > 2$
(A2) μ is small (we take $\mu = 1/1000$) [4]
(A3) $J > J^-(E)$.

These conditions serve three purposes:

1. $C > 4$ guarantees that there is an outer region, separate from the two inner regions, in the (x, y)-plane in which P can be located. Being in the outer region guarantees that we are bounded away from the primaries, and thus bounded from the singularities in the Hamiltonian. For definitiveness, we take $r > 2$ as our lower bound.

2. From the formula $E = 1/2J^2 + (J - J_r) - \Delta H$, since $J \geq J_r \geq 0$, the range of admissible J for sufficiently large E is the union of a finite interval (the inner Hill regions) and a half-infinite interval $[J_E, \infty)$ (the outer Hill region), where $J_E \approx J^-(E)$. As we shall see later, we need to avoid the singularity around $J \approx J^-(E)$, and hence, (A3).

3. (A1) and (A3) guarantee that there are no restrictions on the values that the angles θ and θ_r can assume. Indeed, from looking at the Jacobi constant in the rectangular coordinates of the rotating-frame (1) we see that the set $r \geq \sqrt{C}$ is a subset of the outer region where P can be located. That is, there are no constraints on φ if $r \geq \sqrt{C}$, and hence, no constraints on θ and θ_r. We will show in the below subsection that (A1) and (A3) imply that for every fixed C, the region of phase space so considered will be a subset of the unbounded region $r \geq \sqrt{C}$.

2.2 Elimination of J_r

By fixing the energy E, we obtain a 3-dimensional surface Π_E. To simplify matters, we hope to express one of the action-angle coordinates in terms of E and the remaining three. Eliminating the action J_r is a natural choice, since in the unperturbed rotating Kepler problem, we can use the unperturbed energy E_0 to solve for J_r exactly:

$$J_r = J - E_0 + \frac{1}{2J^2} \tag{7}$$

[4] In fact, our theoretical results work for all values of μ. However, since we are interested in Sun-Jupiter systems, we take $\mu = 1/1000$ in our numerics. Moreover, since we want to think of PCR3BP as a perturbation of a two body system, it is appropriate to think of μ as being small. In reality, μ can be as large as we want, so long as the massless body is sufficiently far away from the primaries. This is why we mention U_{AM} in the introduction, even though we will not be interested in computing it.

For PCR3BP, we have

$$E = E_0 - \Delta H,$$

where E, not E_0, is constant in time. Nevertheless, we can still write (7) for PCR3BP, the difference being that J_r, J, and E_0 are now all varying in time. It turns out that we can still (locally) eliminate the coordinate J_r on Π_E and take (θ, θ_r, J) as local-coordinates on Π_E. In other words, we can express $J_r = J_r(E, \theta, \theta_r, J)$. The reason is that $\dfrac{\partial \Delta H}{\partial J_r}$ is small, and so $\dfrac{\partial H}{\partial J_r} \approx 1 > 0$, and so we can apply the implicit function theorem (actually, we use only the fact that H is monotone in J_r). Since we will not have estimates on the derivatives of ΔH until later, we relegate the simple proof that J_r is a well-defined function of E and the remaining action-angle variables to Proposition A of the Appendix.

Nevertheless, we would like to know approximately what is the form of $J_r(E, \theta, \theta_r, J)$. From the above two equations, we have

$$J_r - \left(J - E + \frac{1}{2J^2}\right) =: \delta = \delta(E, \theta, \theta_r, J), \tag{8}$$

with

$$|\delta| \leq \sup |\Delta H|,$$

the supremum being taken over the outer region of Π_E. As expected, δ should be small, on the order of μ. Nevertheless, we place shall place a crude upper bounds on δ. If we write $d_1^{-1} =: r^{-1}(1+\delta_1)^{-1/2}$ and $d_2^{-1} =: r^{-1}(1+\delta_2)^{-1/2}$, then by the mean value theorem, we have

$$\begin{aligned}
\Delta H &= \frac{1}{r} - \frac{\mu}{r}\left(1 - \frac{1}{2}(1+\delta_1^*)^{-3/2}\delta_1\right) - \frac{1-\mu}{r}\left(1 - \frac{1}{2}(1+\delta_2^*)^{-3/2}\delta_2\right) \\
&= \frac{1}{2r}\left(\mu(1+\delta_1^*)^{-3/2}\delta_1 + (1-\mu)(1+\delta_2^*)^{-3/2}\delta_2\right),
\end{aligned}$$

for some δ_1^* and δ_2^*. But since

$$\begin{aligned}
1+\delta_1^* &\geq 1 - \frac{2(1-\mu)}{r} + \frac{(1-\mu)^2}{r^2} = \left(1 - \frac{1-\mu}{r}\right)^2 \\
1+\delta_2^* &\geq 1 - \frac{2\mu}{r} + \frac{\mu^2}{r^2} = \left(1 - \frac{\mu}{r}\right)^2 \\
\delta_1 &\leq \frac{2(1-\mu)}{r} + \frac{1-\mu}{r^2} \leq \frac{3}{r} \\
\delta_2 &\leq \frac{2\mu}{r} + \frac{\mu^2}{r^2} \leq \frac{3\mu}{r},
\end{aligned}$$

it follows that

$$\begin{aligned}
\delta &\leq \frac{1}{2r}\left(\mu\left(1 - \frac{1}{r}\right)^{-3}\frac{3}{r} + (1-\mu)\left(1 - \frac{\mu}{r}\right)^{-3}\frac{3\mu}{r}\right) \\
&\leq \frac{3\mu}{r^2}\left(1 - \frac{1}{r}\right)^{-3} \\
&< 6\mu.
\end{aligned} \tag{9}$$

In the last line, we used $r > 2$ from (A1). Thus, the above steps show that we can eliminate J_r from all expressions by writing

$$J_r = J - E + \frac{1}{2J^2} + \delta, \tag{10}$$

where the remainder δ is less than 6μ in absolute value. However, the above bound (9) for δ is only an initial approximation. We can improve it in just a bit. But first:

From the above analysis, we can see the justifications for (A1) and (A2) in further detail. First, since $E = 1/2J^2 + (J - J_r) - \Delta H$ and ΔH and $1/2J^2$ are small, it is easy to see that for J near $E > 2$, we have $J_r \approx 0$ and $e \approx 0$. Thus, $r \approx E^2$. As J increases, however, J_r must increase so that E stays fixed. This causes e to increase and minimal radial distance $J^2(1-e)$ to decrease. In the limit that $J \to \infty$, $e \to 1$ and $J^2(1-e)$ stays bounded from below by some fixed constant. Indeed, we have

$$\lim_{J\to\infty} J^2\left(1 - \sqrt{1 - \frac{(J-J_r)^2}{J^2}}\right) = \lim_{J\to\infty} J^2\left(1 - \sqrt{1 - \frac{(E - 1/2J^2 - \delta)^2}{J^2}}\right) \geq \frac{(E-6\mu)^2}{2} \approx \frac{C^2}{8}. \quad (11)$$

By the above remarks, we want $C^2/8 > \sqrt{C}$, since this implies that there are no restrictions on θ and θ_r. This occurs for $C > 4$.

From (9) and (11), we see that minimum distance is $(E-6\mu)^2/2$. Then using $r > (E-6\mu)^2/2$ in the line above (9), the estimate for δ improves to

$$\delta < \frac{12\mu}{(E-6\mu)^4}\left(1 - \frac{2}{(E-6\mu)^2}\right)^{-3}. \quad (12)$$

We want δ as small as possible because, in the future, when we take E small, the error δ in (10) will matter.

3 Setting for Aubry-Mather Theory

Aubry-Mather theory is suitable for Hamiltonian systems for 1.5 or 2 degrees of freedom, because such systems often admit Poincaré return maps on two-dimensional sections. A discussion of Aubry-Mather theory will be provided in §6. Our Poincaré map for PCR3BP is defined as follows:

First, we fix μ and E to obtain an energy hypersurface Π_E. Given our standard assumptions, in the outer region of phase space, we can eliminate J_r and use (θ, θ_r, J) as local coordinates on Π_E. To reduce the dimension one further, we take the section $\theta_r = 0$. This gives us a half-infinite cylinder $S_E = \{(\theta, J) : J > J^-(\mu, E)\} \subset \Pi_E$. Since we expect $\frac{\partial \Delta H}{\partial J_r}$ to be small, which means that $\dot{\theta}_r \approx 1$, we expect a Poincaré return map that is approximately a time 2π map to be well-defined on S_E. As we shall see later, this is true, but only locally.

We thus obtain a Poincaré map $F : U \subset S_E \to S_E$ given by

$$F = (F_\theta, F_J) : \begin{pmatrix} \theta \\ J \end{pmatrix} \mapsto \begin{pmatrix} \theta + \frac{2\pi}{J^3} - 2\pi + f(\theta, J) \\ J + g(\theta, J) \end{pmatrix}, \quad (13)$$

where f and g arise from $\Delta H \not\equiv 0$. We shall omit dependence on E for brievity.

Although we now have a return map F on a 2-dimensional section, the application of Aubry-Mather theory also requires that the map be an *exact area-preserving twist* (EAPT) map. This means that

(i) F preserves an area form $A(\theta, J) d\theta \wedge dJ$ on S_E with $A(\theta, J)$ smooth and nowhere vanishing;

(ii) F preserves orientation;

(iii) (twist condition): $\dfrac{\partial F_\theta}{\partial J} < 0$;

(iv) (exactness): given any loop γ which goes once around S_E, the area between γ and its image $\gamma' = F(\gamma)$ is zero, i.e.,

$$\int_{\gamma' - \gamma} J d\theta = 0.$$

Once we can prove that F is an EAPT map, we can use Aubry-Mather theory to prove rigorously the existence of wide variety of periodic and quasiperiodic motions for PCR3BP.

First, (ii) follows from the fact that F is a Poincaré map for a Hamiltonian system. We prove (i) at the end of this section. We make first steps towards proving (iii) here, but the final proofs for (iii) and (iv) will be provided in Sections 5 and 6.2, after we perform some requisite calculations in Section 4. As pointed out, it turns out that F will satisfy the twist condition only locally. Thus, fixing E and any interval $\mathcal{J}$, we say that F is a *twist on* $\mathcal{J}$ if F satisfies the twist condition $\frac{\partial F_\theta}{\partial J} < 0$ on the annulus $\mathbb{S}^1 \times \mathcal{J} \subset S_E$. Next, we consider the norm

$$\|\cdot\|^* := \sup_{(\theta,\theta_r)} |\cdot|.$$

We will apply $\|\cdot\|^*$ to functions of (θ, θ_r, J), and thus, the result will be a nonnegative function of J.

Proposition 3.1 *Fix μ, E and an interval $\mathcal{J}_1 = [J_1^-, J_1^+] \subset [J^-(\mu, E), \infty)$. Let L be the Lipschitz constant of the Hamiltonian vector field (3) restricted to $\{(\theta, \theta_r, J) : J \in \mathcal{J}_1\} \subset \Pi_E$. Define*

$$\begin{aligned}
b &= \sup_{J\in\mathcal{J}_1} \left\| \frac{\partial \Delta H}{\partial \theta} \right\|^* \\
\tau &= 2\pi \left(1 - \sup_{J\in\mathcal{J}_1} \left\| \frac{\partial \Delta H}{\partial J_r} \right\|^* \right)^{-1} \\
C_0(\bar{J}) &= \sup_{J\in[\bar{J}-b\tau,\bar{J}+b\tau]} \left(\left\| \frac{\partial^2 \Delta H}{\partial \theta^2} \right\|^* + \left\| \frac{\partial^2 \Delta H}{\partial \theta \partial \theta_r} \right\|^* + \left\| \frac{\partial^2 \Delta H}{\partial \theta \partial J} \right\|^* \right)
\end{aligned}$$

$$\begin{aligned}
\eta(J) &= \left(1 - (e^{-Lt} - C_0(J)\tau)^2\right)^{1/2} \\
C_1(\bar{J}) &= \sup_{J\in[\bar{J}-b\tau,\bar{J}+b\tau]} \left(\eta(\bar{J}) \left\| \frac{\partial^2 \Delta H}{\partial J \partial \theta} \right\|^* + \eta(\bar{J}) \left\| \frac{\partial^2 \Delta H}{\partial J \partial \theta_r} \right\|^* + \left\| \frac{\partial^2 \Delta H}{\partial J^2} \right\|^* \right) \\
C_2(\bar{J}) &= \sup_{J\in[\bar{J}-b\tau,\bar{J}+b\tau]} \left(\eta(\bar{J}) \left\| \frac{\partial^2 \Delta H}{\partial J_r \partial \theta} \right\|^* + \eta(\bar{J}) \left\| \frac{\partial^2 \Delta H}{\partial J_r \partial \theta_r} \right\|^* + \left\| \frac{\partial^2 \Delta H}{\partial J_r \partial J} \right\|^* \right) \\
C_3(\bar{J}) &= \sup_{J\in[\bar{J}-b\tau,\bar{J}+b\tau]} \left\| \frac{\partial \Delta H}{\partial J} \right\|^* \\
\lambda(J) &= 3(1 - C_0(J)\tau)e^{-L\tau}.
\end{aligned}$$

Then if

$$\frac{-\lambda(J)}{(J + b\tau)^4} + C_1(J) + \frac{\tau}{2\pi} C_2(J) \left(1 + C_3(J) + \frac{1}{(J + b\tau)^3} \right) < 0 \tag{14}$$

for all $J \in \mathcal{J} := (J_1^- + b\tau, J_1^+ - b\tau)$, F is a twist on $\mathcal{J}$.

Remark. The statement of this proposition is complicated by the fact that we cannot precisely evaluate anything, and so we have to place many upper bounds and keep track of them everywhere. We shall see later that for sufficiently large E or sufficiently small μ, we can find a nonempty interval $\mathcal{J}_1$ such that for all practical purposes, $b \approx 0$ and $\tau \approx 2\pi$. This greatly simplifies the verification of (14) and this also gives us the twist interval $\mathcal{J}_1$. But for the sake of a rigorous proof, we retain all the complexity in the proof of this proposition.

Proof Our proof considers Hamilton's equations in variation. Pick $J_0, J_0 + \Delta J \in \mathcal{J}$. Without loss of generality, suppose $\Delta J > 0$. Fix an initial condition $(\theta_0, 0, J_0)$ for (θ, θ_r, J) on S_E, and let the corresponding solution for the Hamiltonian system be

$$u_0(t) = \begin{pmatrix} \theta_0(t) \\ \theta_{r,0}(t) \\ J(t) \end{pmatrix}.$$

Next, pick new initial conditions $(\theta_0, 0, J_0 + \Delta J)$, and let the perturbed solution be

$$u(t) = \begin{pmatrix} \theta_{\Delta J}(t) \\ \theta_{r,\Delta J}(t) \\ J_{\Delta J}(t) \end{pmatrix} = \begin{pmatrix} \theta(t) \\ \theta_r(t) \\ J(t) \end{pmatrix} = \begin{pmatrix} \theta_0(t) + \Delta\theta(t) \\ \theta_{r,0}(t) + \Delta\theta_r(t) \\ J_0(t) + \Delta J(t) \end{pmatrix}.$$

Let T_0 and $T = T(\Delta J)$ be the period of first return of θ_r for the unperturbed and perturbed solutions, respectively, i.e. $\theta_{r,0}(T_0) = \theta_{r,\Delta J}(T) = 2\pi$. Since $J_0 + \Delta J \in \mathcal{J} \subset \mathcal{J}_1$, we have $|\dot{J}(0)| \leq b$. Thus,

$$J(t) \in \mathcal{J}_1, \qquad t \in [0, \tau], \tag{15}$$

since $[J_0 + \Delta J - b\tau, J_0 + \Delta J + b\tau] \subseteq \mathcal{J}_1$. Furthermore, any trajectory which stays inside $\{(\theta, \theta_r, J) : J \in \mathcal{J}_1\}$ for at least one period of θ_r has maximal period

$$2\pi \Big(\inf \dot{\theta}_r \Big)^{-1} = 2\pi \Big(1 - \sup_{J \in \mathcal{J}_1} \left|\left| \frac{\partial \Delta H}{\partial J_r} \right|\right|^* \Big)^{-1} = \tau.$$

It follows that $T_0, T \leq \tau$.

Having a twist condition on $\mathcal{J}$ means that

$$\lim_{\Delta J \to 0} \frac{\theta_{\Delta J}(T(\Delta J)) - \theta_0(T_0)}{\Delta J} < 0. \tag{16}$$

for all $J_0 \in \mathcal{J}$. To show that we indeed have a twist, since $\theta_{\Delta J}(0) = \theta_0(0) = \theta_0$, we first rewrite the left-hand side of the above as

$$\lim_{\Delta J \to 0} \frac{\theta_{\Delta J}(T) - \theta_0(T_0)}{\Delta J} = \lim_{\Delta J \to 0} \frac{1}{\Delta J} \int_{T_0}^{T(\Delta J)} \dot{\theta}_{\Delta J}(t) dt + \int_0^{T_0} \lim_{\Delta J \to 0} \frac{\dot{\theta}_{\Delta J}(t) - \dot{\theta}_0(t)}{\Delta J} dt. \tag{17}$$

In order to evaluate the above expression, we will need to place bounds on $\Delta\theta(t)$, $\Delta\theta_r(t)$, and $\Delta J(t)$ over a period, so that we may see how much $\theta(t)$, $\theta_r(t)$, and $J(t)$ deviate from their unperturbed counterparts. From a general result in differential equations, if φ_1 and φ_2 are solutions to $\dot{x} = f(x, t)$ that remain inside a region Ω, then $|\varphi_1(t) - \varphi_2(t)| \leq |\varphi_1(0) - \varphi_2(0)| e^{K|t|}$, where K is the Lipschitz constant of f on $\Omega \times \mathbb{R}$. In our situation, we can take $\Omega = \{(\theta, \theta_r, J) : J \in \mathcal{J}_1\}$ and $K = L$ for $t \in [0, \tau]$ by (15). This implies that

$$|u(t) - u_0(t)| \leq \Delta J e^{Lt}, \qquad t \in [0, \tau]. \tag{18}$$

Thus,

$$\begin{aligned} |\Delta \dot{J}(t)| &= |\dot{J}(t) - \dot{J}_0(t)| \\ &= \left| \Delta\theta(t) \frac{\partial^2 \Delta H}{\partial \theta^2}(u_0(t)) + \Delta\theta_r(t) \frac{\partial^2 \Delta H}{\partial \theta \partial \theta_r}(u_0(t)) + \Delta J(t) \frac{\partial^2 \Delta H}{\partial \theta \partial J}(u_0(t)) + \dots \right| \\ &\leq \Delta J e^{L\tau} C_0(J_0) + O(\Delta J)^2, \qquad t \in [0, \tau] \end{aligned}$$

Here, we overestimated $|\Delta\theta(t)|$, $|\Delta\theta_r(t)|$, $|\Delta J(t)| \leq \Delta J e^{L\tau}$ from (18). However, we expect that $|\Delta\theta(t)|$ and $|\Delta\theta_r(t)|$ should be smaller than $|\Delta J(t)|$, since the initial displacement $|u(0) - u_0(0)|$ occurs only through $\Delta J(0) = \Delta J$. From the above bound on $\Delta J(t)$, we have

$$\Delta J(t) \geq \Delta J \Big(1 - e^{L\tau} C_0(J_0) \tau \Big) + O(\Delta J)^2, \qquad t \in [0, \tau]. \tag{19}$$

Plugging this estimate into in (18), we find that

$$|\Delta\theta(t)|,\ |\Delta\theta_r(t)| \leq \Delta J\Big(e^{2L\tau} - (1-e^{Lt}C_0(J_0)\tau)^2\Big)^{1/2} + O(\Delta J)^2 = e^{L\tau}\eta(J_0)\Delta J + O(\Delta J)^2 \qquad (20)$$

Indeed, for L and C_0 small, the angular displacements will be small compared to ΔJ, since η will be small. Thus, (20) together with $\Delta J(t) \leq \Delta J e^{L\tau}$ give us bounds for the components of $|u(t)-u_0(t)|$ over a period of motion.

Let us focus on first term of the right-hand side of (17). We will need to estimate dT/dJ. Writing the equation of motion for θ_r in variation, we have

$$\begin{aligned}\dot\theta_r(t) &= \dot\theta_{r,0}(t) + \Delta\theta(t)\frac{\partial^2\Delta H}{\partial J_r\partial\theta}(u_0(t)) + \Delta\theta_r(t)\frac{\partial^2\Delta H}{\partial J_r\partial\theta_r}(u_0(t)) + \Delta J(t)\frac{\partial^2\Delta H}{\partial J_r\partial J}(u_0(t)) + \dots\\ &\geq \dot\theta_{r,0}(t) - (e^{L\tau}\Delta J)C_2(J_0) + O(\Delta J)^2.\end{aligned}$$

Thus,

$$\begin{aligned}0 &= -2\pi + \int_0^T \dot\theta_{r,\Delta J}(t)dt\\ &\geq -2\pi + \int_0^T \dot\theta_{r,0}(t)dt - \int_0^T e^{L\tau}\bar C_2(J_0)\Delta J dt + O(\Delta J)^2\\ &\geq \int_{T_0}^T \dot\theta_{r,0}(t)dt - \tau e^{L\tau}C_2(J_0)\Delta J + O(\Delta J)^2\\ &\geq (T-T_0)\frac{2\pi}{\tau} - \tau e^{L\tau}C_2(J_0)\Delta J + O(\Delta J)^2.\end{aligned}$$

Repeating the above steps in a similar fashion to obtain bounds in the opposite direction, we can conclude

$$\left|\frac{dT}{dJ}(J_0)\right| \leq \frac{\tau^2}{2\pi}e^{L\tau}C_2(J_0),$$

from which we get

$$\begin{aligned}\left|\lim_{\Delta J\to 0}\frac{1}{\Delta J}\int_{T_0}^{T(\Delta J)}\dot\theta_{\Delta J}(t)dt\right| &= \left|\frac{dT}{dJ}(J_0)\dot\theta_{\Delta J}(T_0)\right|\\ &\leq \frac{\tau^2}{2\pi}e^{L\tau}C_2(J_0)\left|\frac{1}{J(T_0)^3} - 1 + \frac{\partial\Delta H}{\partial J}(u(T_0))\right|\\ &\leq \frac{\tau^2}{2\pi}e^{L\tau}C_2(J_0)\left(1 + C_3(J_0) + \frac{1}{(J_0+b\tau)^3}\right). \qquad (21)\end{aligned}$$

As for the second term in (17), writing the equation of motion for $\theta(t)$ in variation and using (19), we have

$$\begin{aligned}\dot\theta(t) &= \frac{1}{J(t)^3} - 1 + \frac{\partial\Delta H}{\partial J}(\theta(t),\theta_r(t),J(t))\\ &= \frac{1}{J_0(t)^3} - 1 + \frac{\partial\Delta H}{\partial J}(u_0(t)) - \frac{3\Delta J(t)}{J_0(t)^4}\\ &\quad + \Big(\Delta\theta(t)\frac{\partial^2\Delta H}{\partial\theta\partial J}(u_0(t)) + \Delta\theta_r(t)\frac{\partial^2\Delta H}{\partial\theta_r\partial J}(u_0(t)) + \Delta J(t)\frac{\partial^2\Delta H}{\partial J\partial J}(u_0(t))\Big) + O(\Delta J)^2\\ &< \dot\theta_0(t) + e^{L\tau}\Delta J\left(-\frac{\lambda(J_0)}{J_0(t)^4} + C_1(J_0)\right) + O(\Delta J)^2.\end{aligned}$$

It follows that

$$\int_0^{T_0} \lim_{\Delta J \to 0} \frac{\dot\theta_{\Delta J}(t) - \dot\theta_0(t)}{\Delta J} dt \le \tau e^{L\tau} \Bigg(- \frac{\lambda(J_0)}{(J_0 + b\tau)^4} + C_1(J_0) \Bigg). \tag{22}$$

Thus, from (21) and (22) it follows that a sufficient condition for there to be a twist on $\mathcal{J}$ is that

$$\frac{-\lambda(J)}{(J + b\tau)^4} + C_1(J) + \frac{\tau}{2\pi} C_2(J) \Bigg(1 + C_3(J) + \frac{1}{(J + b\tau)^3} \Bigg) < 0$$

for all $J \in \mathcal{J}$. ■

Remark 3.2 *One might slightly improve the condition (14) by making* $\lambda(J) = 3(1 - C_0(J)\tau)(1 + L\tau)^{-1}$. *To prove it one needs to replace the proof of the upper bound (20), based on (18), by a more involved estimate. The latter is obtained by analyzing the variational equation associated to (3), where the variation is with respect to J-th component of initial conditions. For further reading on variational equations, see e.g. [AA].*

As stated, this proposition is not very useful, since it not easy to evaluate precisely the suprema of all those quantities. It is therefore more feasible to place upper bounds on these suprema. In the next section, we will do this for all the important quantities in this paper, namely, the derivatives of the polar variables and the perturbation Hamiltonian with respect to the action-angle variables. We will then use these upper bounds in place of $\|\cdot\|^*$ in verifying the twist condition. The next section will therefore consist of a lengthy, albeit necessary, set of calculations and expressions for future use. But first, we establish the first of the EAPT conditions, namely, that F preserves area.

3.1 Area-preservation

Consider the Hamiltonian $H = H_0 + \Delta H$ of PCR3BP(μ, C) expressed in the action-angle coordinates by (2). We will see later, in Corollary 5.5, that all first order partial derivative of ΔH are ϵ-small when $J \in \mathcal{J}_0(\mu, C, \epsilon)$, where $\mathcal{J}_0$ is some interval depending on μ, C, and ϵ. It implies that θ_r changes monotonically and $\dot\theta_r(t) > 1 - \epsilon$. By the implicit function theorem one can eliminate the J_r variable, and consider only evolution of (θ, θ_r, J) (see Appendix A). More exactly, fix an energy surface $\{H = E\}$. Then there is a function $J_r = J_r(\theta, \theta_r, J)$ so that trajectories of the first three equations of motion (3) for $H_E(\theta, \theta_r, J) = H(\theta, J, \theta_r, J_r(\theta, \theta_r, J))$ coincide with trajectories of the all equations of (3) projected along J_r. There is a standard way of defining a time-periodic Hamiltonian $\tilde H_E(\theta, J, t)$ of t-period 2π so that after the identification $\theta_r = t$, trajectories of H_E and $\tilde H_E$ locally coincide up to time reparametrization (see e.g. [BK] Sect. 4.1). It implies that the Poincaré return map F of H_E, as defined in (13), coincides (in its domain of definition) with the time 2π map of H_E. Since the time 2π map of $\tilde H_E$ is symplectic, it preserves the canonical 2-form $\omega^2 = d\theta \wedge dJ + d\theta_r \wedge dJ_r$. Restriction of ω^2 onto S_E is an area form, which implies that F preserves a smooth area form on S_E in its local domain of definition (we had to assume $J \in \mathcal{J}_0$). Here the construction of $\tilde H_E$.

Notice that if we divide each component of the vector field (3) by $\partial_{J_r} H = 1 + \partial_{J_r} \Delta H$, i.e. rescale time, then for the new vector field $\dot\theta_r \equiv 1$. Therefore, if the differential of a Hamiltonian $H'_E(\theta, J, \theta_r, J_r)$ satisfies the following equation

$$dH'_E(\theta, J, \theta_r, J_r) = \frac{\partial_\theta H}{\partial_{J_r} H}\, d\theta + \frac{\partial_J H}{\partial_{J_r} H}\, dJ + \frac{\partial_{\theta_r} H}{\partial_{J_r} H}\, d\theta_r + dJ_r, \tag{23}$$

then the trajectories of H'_E and those of H on the energy surface E coincide up to time reparametrization provided that $\partial_{J_r} H$ is bounded away from zero. By the construction $\partial_{J_r} H'_E \equiv 1$. Consider the implicit function $J_r = J_r(\theta, J, \theta_r)$ and $\tilde H_E(\theta, J, t) = H'_E(\theta, J, \theta_r, J_r(\theta, J, \theta_r)) - J_r(\theta, J, \theta_r)$. Since $\partial_{J_r} \tilde H_E(\theta, J, t) \equiv 0$, it justifies that $\tilde H_E$ does not depend on J_r. This transformation is well-defined as long as $\partial_{J_r} H$ is bounded away from zero.

4 Partial Derivatives and Upper Bounds

4.1 Derivatives of e, r, and φ

We first place bounds on the derivatives of e. The formula $e = \sqrt{1-(J-J_r)^2/J^2}$ is exact, and so we can find exact formulas for the derivatives of e. We then use (10) to eliminate J_r (indeed, expressions involving e are the only situations in which we need to eliminate J_r) in these expressions, where δ is an error term defined in (8) and bounded by (12). In forming upper bounds, we consistently use the fact that $J \geq J_r \geq 0$.

$$
\begin{aligned}
e &= \sqrt{1-\frac{(J-J_r)^2}{J^2}} = \sqrt{1-J^{-2}\left(E-\frac{1}{2J^2}+\delta\right)^2} \\
\frac{\partial e}{\partial \theta} = \frac{\partial e}{\partial \theta_r} &= 0 \\
\left|\frac{\partial e}{\partial J}\right| &= \left|-\frac{J-J_r}{J^2}\sqrt{\frac{J_r}{2J-J_r}}\right| \\
&\leq \frac{|E-1/2J^2+\delta|}{J^2}\sqrt{\frac{J-E+1/2J^2+\delta}{J+E-1/2J^2-\delta}} \\
&\leq \boxed{\frac{E-1/2J^2+\delta}{J^2}}
\end{aligned}
$$

$$
\begin{aligned}
\left|\frac{\partial e}{\partial J_r}\right| &= \left|\frac{J-J_r}{J\sqrt{(2J-J_r)J_r}}\right| \\
&\leq \boxed{\frac{|E-1/2J^2+\delta|}{J\sqrt{J(J-E+1/2J^2-\delta)}}} \\
\left|\frac{\partial^2 e}{\partial J^2}\right| &= \left|\frac{3J^2-6JJ_r+2J_r^2}{J^3}\sqrt{\frac{J_r}{(2J-J_r)^3}}\right| \\
&= \left|-\frac{1}{J}+\frac{2(J-J_r)}{J^2}+\frac{2(J-J_r)^2}{J^3}\right|\cdot\frac{1}{2J-J_r}\cdot\sqrt{\frac{J_r}{2J-J_r}} \\
&\leq \frac{3}{J}\cdot\frac{1}{2J-J_r} \\
&\leq \boxed{\frac{3}{J^2}} \\
\left|\frac{\partial^2 e}{\partial J \partial J_r}\right| &= \left|\frac{-J^2+3JJ_r-J_r^2}{J^2\sqrt{(2J-J_r)^3 J_r}}\right| \\
&= \left|-1+3\left(\frac{J_r}{J}\right)-\left(\frac{J_r}{J}\right)^2\right|\frac{1}{\sqrt{(2J-J_r)^3(J-E+1/2J^2-\delta)}} \\
&\leq \boxed{\frac{1}{\sqrt{J^3(J-E+1/2J^2-\delta)}}}.
\end{aligned}
$$

As we can see, the derivatives with respect to J_r contain a singularity at around $J \approx J^-(E)$. This justifies (A2) (see §2). Also, notice that although for the sake of having precise upper bounds, we kept track of all the small terms $1/2J^2$ and δ in eliminating J_r, for all practical purposes we have $J_r \approx J - E$

in these expressions. Consequently, to get a better feel for the above upper bounds, the reader should freely ignore all the occurrences of δ and $1/2J^2$.

Next, we place bounds on the derivatives of the polar variables r and φ. To do this, we must know the derivatives of ξ, which can be obtained by implicitly differentiating Kepler's equation $\xi - e\sin\xi = \theta + \theta_r$. We thus obtain

$$\frac{\partial \xi}{\partial u} = \begin{cases} \frac{1}{1-e\cos\xi} & u \in \{\theta, \theta_r\} \\ \frac{\sin\xi}{1-e\cos\xi}\left(\frac{\partial e}{\partial u}\right) & u \in \{J, J_r\} \end{cases}$$

$$\frac{\partial^2 \xi}{\partial u \partial v} = \begin{cases} \frac{-e\sin\xi}{(1-e\cos\xi)^3} & u, v \in \{\theta, \theta_r\} \\ \frac{\cos\xi - e}{(1-e\cos\xi)^3}\left(\frac{\partial e}{\partial v}\right) & u \in \{\theta, \theta_r\},\ v \in \{J, J_r\} \\ \left(\frac{\cos\xi - e}{(1-e\cos\xi)^2}\left(\frac{\partial \xi}{\partial v}\right) + \frac{\cos\xi\sin\xi}{(1-e\cos\xi)^2}\left(\frac{\partial e}{\partial v}\right)\right)\left(\frac{\partial e}{\partial u}\right) + \frac{\sin\xi}{1-e\cos\xi}\left(\frac{\partial^2 e}{\partial u \partial v}\right) & u, v \in \{J, J_r\}. \end{cases}$$

Remark 4.1 *We see altogether that are two degeneracies. The degeneracy near $J \approx E$ is more precisely a degeneracy when $e = J_r = 0$, as can be seen from the exact formulas for the derivatives of e with respect to J_r. The second degeneracy occurs at $e = 1$ when we take derivatives of ξ (and hence of r and φ), as can be seen from the factors of $(1 - e)$ in the denominator. Proving the twist will require us to avoid these two degeneracies.*

Using the above formulas for the derivatives of ξ, we now proceed to bound the derivatives of the polar variables in terms of J, e, and the derivatives of e. By fixing E and using the previous bounds on the derivatives of e, this gives us a bound in terms of just E, J, and μ using (12). Moreover, since the derivatives of the polar variables only depend on θ_r implicitly through ξ, we have

$$\|\cdot\|^* = \sup_{(\theta,\theta_r)} |\cdot| \leq \sup_{(\theta,\xi)} |\cdot|,$$

since $\xi - e\sin\xi = \theta + \theta_r$, so that $\xi = \xi(\theta, \theta_r, J)$.

We use the following straightforward facts in placing our subsequent bounds:

$$\sup_\xi \left|\frac{\sin\xi}{1-e\cos\xi}\right| \leq \frac{1}{(1-e^2)^{1/2}} \qquad \sup_\xi \left|\frac{-2+e\cos\xi+e^2}{1-e\cos\xi}\right| \leq 2+e$$

$$\sup_\xi \left|\frac{\sin^2\xi}{1-e\cos\xi}\right| \leq 2 \qquad \sup_\xi \left|\frac{\cos\xi - e}{1-e\cos\xi}\right| \leq 1.$$

This gives us the following bounds:

$$
\begin{aligned}
\left\|\frac{\partial r}{\partial u}\right\|^* &= \left\|\frac{J^2 e\sin\xi}{1-e\cos\xi}\right\|^* \\
&\leq \boxed{\frac{J^2 e}{(1-e^2)^{1/2}}} && u\in\{\theta,\theta_r\} \\
\left\|\frac{\partial r}{\partial J}\right\|^* &= \left\|2J(1-e\cos\xi) - J^2\left(\frac{\cos\xi - e}{1-e\cos\xi}\right)\left(\frac{\partial e}{\partial J}\right)\right\|^* \\
&\leq \boxed{2(1+e)J + J^2\left(\frac{\partial e}{\partial J}\right)} \\
\left\|\frac{\partial r}{\partial J_r}\right\|^* &= \left\|J^2\left(\frac{\cos\xi - e}{1-e\cos\xi}\right)\left(\frac{\partial e}{\partial J_r}\right)\right\|^* \\
&\leq \boxed{J^2\left(\frac{\partial e}{\partial J_r}\right)} \\
\left\|\frac{\partial^2 r}{\partial u\partial v}\right\|^* &= \left\|J^2 e\left(\frac{\cos\xi - e}{(1-e\cos\xi)^3}\right)\right\|^* \\
&\leq \boxed{\frac{J^2 e}{(1-e)^2}} && u,v\in\{\theta,\theta_r\} \\
\left\|\frac{\partial^2 r}{\partial u\partial J}\right\|^* &= \left\|\frac{2Je\sin\xi}{1-e\cos\xi} + \frac{J^2(1-e^2)\sin\xi}{(1-e\cos\xi)^3}\left(\frac{\partial e}{\partial J}\right)\right\|^* \\
&\leq \boxed{2J\left(\frac{e}{(1-e^2)^{1/2}}\right) + J^2\left(\frac{\sqrt{1+e}}{(1-e)^{3/2}}\right)\left|\frac{\partial e}{\partial J}\right|} && u\in\{\theta,\theta_r\} \\
\left\|\frac{\partial^2 r}{\partial u\partial J_r}\right\|^* &= \left\|\frac{J^2(1-e^2)\sin\xi}{(1-e\cos\xi)^3}\left(\frac{\partial e}{\partial J}\right)\right\|^* \\
&\leq \boxed{\frac{J^2\sqrt{1+e}}{(1-e)^{3/2}}\left|\frac{\partial e}{\partial J}\right|} && u\in\{\theta,\theta_r\} \\
\left\|\frac{\partial^2 r}{\partial J^2}\right\|^* &= \Bigg\| - J^2\left(\frac{\cos\xi - e}{1-e\cos\xi}\right)\left(\frac{\partial^2 e}{\partial J^2}\right) - 4J\left(\frac{\cos\xi - e}{1-e\cos\xi}\right)\left(\frac{\partial e}{\partial J}\right) \\
&\qquad -J^2\left(\frac{(-2+e^2+e\cos\xi)\sin^2\xi}{(1-e\cos\xi)^3}\right)\left(\frac{\partial e}{\partial J}\right)^2 + 2(1-e\cos\xi)\Bigg\|^* \\
&\leq \boxed{J^2\left|\frac{\partial^2 e}{\partial J^2}\right| + 4J\left|\frac{\partial e}{\partial J}\right| + \frac{2J^2(2+e)}{1-e}\left|\frac{\partial e}{\partial J}\right|^2 + 2(1+e)} \\
\left\|\frac{\partial^2 r}{\partial J\partial J_r}\right\|^* &= \Bigg\| - J^2\left(\frac{\cos\xi - e}{1-e\cos\xi}\right)\left(\frac{\partial^2 e}{\partial J\partial J_r}\right) - 2J\left(\frac{\cos\xi - e}{1-e\cos\xi}\right)\left(\frac{\partial e}{\partial J_r}\right) \\
&\qquad -J^2\left(\frac{(-2+e^2+e\cos\xi)\sin^2\xi}{(1-e\cos\xi)^3}\right)\left(\frac{\partial e}{\partial J}\right)\left(\frac{\partial e}{\partial J_r}\right)\Bigg\|^* \\
&\leq \boxed{J^2\left|\frac{\partial^2 e}{\partial J\partial J_r}\right| + 2J\left|\frac{\partial e}{\partial J_r}\right| + \frac{2J^2(2+e)}{1-e}\left|\frac{\partial e}{\partial J}\right|\left|\frac{\partial e}{\partial J_r}\right|}
\end{aligned}
$$

$$
\begin{aligned}
\left\|\frac{\partial^2 r}{\partial {J_r}^2}\right\|^* &= \left\| -J^2\left(\tfrac{\cos\xi - e}{1-e\cos\xi}\right)\left(\tfrac{\partial^2 e}{\partial {J_r}^2}\right) - J^2\left(\tfrac{(-2+e^2+e\cos\xi)\sin^2\xi}{(1-e\cos\xi)^3}\right)\left(\tfrac{\partial^2 e}{\partial {J_r}^2}\right)\right\|^* \\
&\leq \boxed{J^2\left|\tfrac{\partial^2 e}{\partial {J_r}^2}\right| + \tfrac{2J^2(2+e)}{1-e}\left|\tfrac{\partial^2 e}{\partial {J_r}^2}\right|} \\
\left\|\frac{\partial \varphi}{\partial \theta}\right\|^* &= \left\|\tfrac{\sqrt{1-e^2}}{(1-e\cos\xi)^2}\right\|^* \\
&\leq \boxed{\tfrac{\sqrt{1+e}}{(1-e)^{3/2}}} \\
\left\|\frac{\partial \varphi}{\partial \theta_r}\right\|^* &= \left\|\tfrac{\sqrt{1-e^2}}{(1-e\cos\xi)^2} - 1\right\|^* \\
&\leq \boxed{\tfrac{\sqrt{1+e}}{(1-e)^{3/2}} - 1} \\
\left\|\frac{\partial \varphi}{\partial u}\right\|^* &= \left\|\tfrac{\sqrt{1-e^2}}{1-e\cos\xi}\left(\tfrac{\partial \xi}{\partial u}\right) + \tfrac{\sin\xi}{\sqrt{1-e^2}(1-e\cos\xi)}\left(\tfrac{\partial e}{\partial u}\right)\right\|^* \\
&= \left\|\tfrac{1}{\sqrt{1-e^2}}\left(\tfrac{\sin\xi}{1-e\cos\xi}\right)\left(\tfrac{2-e\cos\xi-e^2}{1-e\cos\xi}\right)\left(\tfrac{\partial e}{\partial u}\right)\right\|^* \\
&\leq \boxed{\tfrac{2+e}{1-e^2}\left|\tfrac{\partial e}{\partial u}\right|} && u\in\{J, J_r\} \\
\left\|\frac{\partial^2 \varphi}{\partial u\partial v}\right\|^* &= \left\| -\tfrac{2e\sqrt{1-e^2}\sin\xi}{(1-e\cos\xi)^4}\right\|^* \\
&\leq \boxed{\tfrac{2e}{(1-e)^3}} && u,v\in\{\theta,\theta_r\} \\
\left\|\frac{\partial^2 \varphi}{\partial u\partial v}\right\|^* &= \left\| -\tfrac{e}{\sqrt{1-e^2}(1-e\cos\xi)^2}\left(\tfrac{\partial e}{\partial v}\right) - \tfrac{2\sqrt{1-e^2}(\cos\xi-e)}{(1-e\cos\xi)^4}\left(\tfrac{\partial e}{\partial v}\right)\right\|^* \\
&\leq \left(\tfrac{e}{\sqrt{1-e^2}(1-e)^2} + \tfrac{2\sqrt{1-e^2}}{(1-e)^3}\right)\left|\tfrac{\partial e}{\partial v}\right| \\
&\leq \boxed{\tfrac{2+3e}{\sqrt{1+e}(1-e)^{5/2}}\left|\tfrac{\partial e}{\partial v}\right|} && u\in\{\theta,\theta_r\},\ v\in\{J, J_r\} \\
\left\|\frac{\partial^2 \varphi}{\partial u\partial v}\right\|^* &= \Bigg\|\Bigg\{\left[\tfrac{e}{(1-e^2)^{3/2}}\right]\left(\tfrac{\sin\xi}{1-e\cos\xi}\cdot\tfrac{2-e\cos\xi-e^2}{1-e\cos\xi}\right) \\
&\quad + \tfrac{1}{\sqrt{1-e^2}}\left[\tfrac{\cos\xi\sin\xi}{(1-e\cos\xi)^2} + \tfrac{(\cos\xi-e)\sin\xi}{(1-e\cos\xi)^3}\right]\left(\tfrac{-2+e\cos\xi+e^2}{1-e\cos\xi}\right) \\
&\quad + \tfrac{1}{\sqrt{1-e^2}}\tfrac{\sin\xi}{1-e\cos\xi}\Bigg[\tfrac{2e}{1-e\cos\xi} + \tfrac{\cos\xi-e}{(1-e\cos\xi)^2} + \\
&\quad + \tfrac{(-2+e\cos\xi+e^2)(\cos\xi-e)}{(1-e\cos\xi)^3}\Bigg]\Bigg\}\left(\tfrac{\partial e}{\partial u}\right)\left(\tfrac{\partial e}{\partial v}\right) \\
&\quad + \tfrac{1}{\sqrt{1-e^2}}\left(\tfrac{\sin\xi}{1-e\cos\xi}\right)\left(\tfrac{2-e\cos\xi-e^2}{1-e\cos\xi}\right)\left(\tfrac{\partial^2 e}{\partial u\partial v}\right)\Bigg\|^*
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left\{ \tfrac{e}{(1-e^2)^{3/2}} \left(\tfrac{1}{(1-e^2)^{1/2}} \right)(2+e) + \tfrac{1}{(1-e^2)^{1/2}} \left(\tfrac{1}{(1-e)(1-e^2)^{1/2}} + \tfrac{1}{(1-e)(1-e^2)^{1/2}} \right)(2+e) \right. \\
&\qquad \left. + \tfrac{1}{(1-e^2)^{1/2}} \cdot \tfrac{1}{(1-e^2)^{1/2}} \left(\tfrac{2e}{1-e} + \tfrac{1}{1-e} + \tfrac{2+e}{1-e} \right) \right\} \left| \tfrac{\partial e}{\partial u} \right| \left| \tfrac{\partial e}{\partial v} \right| + \tfrac{2+e}{1-e^2} \left| \tfrac{\partial^2 e}{\partial u \partial v} \right| \\
&\leq \boxed{ \left(\tfrac{e^2+2e}{(1-e^2)^2} + \tfrac{5e+7}{(1-e^2)(1-e)} \right) \left| \tfrac{\partial e}{\partial u} \right| \left| \tfrac{\partial e}{\partial v} \right| + \tfrac{2+e}{1-e^2} \left| \tfrac{\partial^2 e}{\partial u \partial v} \right| } \qquad u, v \in \{J, J_r\}.
\end{aligned}
$$

Remark 4.2 *To prove that there is a twist, we do not actually need the precision of the above bounds. Only when we do numerics to examine where there is a twist will we use the precise upper bounds above. These bounds were intended to minimize the blowing up of negative powers of $(1-e)$. This allows us to optimize the range of J for which there is a twist. However to prove the existence of a twist, all we need to do is essentially keep track of the order in J of each term, since the twist condition in Proposition 3.1 involves comparing derivatives of quantities with the leading term $-\lambda/J^4$. By keeping track of orders in J, we will be able to show that the remaining terms of Proposition 3.1 are of higher order in J, and thus that the twist condition (14) will be satisfied for large enough J. The only caveat is the presence of the degeneracies as pointed out by Remark 4.1. It turns out these degeneracies become manageable after a very simple change of variables, and we will explore this in Section 5 when we prove that there is a twist.*

4.2 Derivatives of ΔH

We now examine the size of ΔH in terms of order of magnitudes in r (and hence J). This turns out to be good enough for establishing the *existence* of a twist by the above remarks. However, in order to do precise numerics, we will need to replace these crude order of magnitude estimates with more precise upper bounds, like the bounds we placed on the polar variables. We describe these upper bounds later.

From the formulas for d_1 and d_2, we can write

$$
\begin{aligned}
d_1 &= r \left(1 - \frac{2(1-\mu)\cos\varphi}{r} + \frac{(1-\mu)^2}{r^2} \right)^{1/2} \\
&= r - (1-\mu)\cos\varphi + \frac{(1-\mu)^2 \sin^2\varphi}{2r} + \ldots \\
&= r + O(1) + O(r^{-1}) \\
d_2 &= r \left(1 + \frac{2\mu\cos\varphi}{r} + \frac{\mu^2}{r^2} \right)^{1/2} \\
&= r + \mu\cos\varphi + \frac{\mu^2 \sin^2\varphi}{2r} + \ldots \\
&= r + O(\mu) + O(\mu r^{-1}).
\end{aligned}
$$

Using the binomial formula again, we can also write

$$
\begin{aligned}
\frac{1}{d_1^n} &= \frac{1}{r^n} + \frac{n(1-\mu)\cos\varphi}{r^{n+1}} + \frac{n(1-\mu)^2(1+(n-2)\cos^2\varphi)}{2r^{n+2}} + O\left(\frac{1}{r^{n+3}}\right) \\
\frac{1}{d_2^n} &= \frac{1}{r^n} - \frac{n\mu\cos\varphi}{r^{n+1}} + \frac{n\mu^2(1+(n-2)\cos^2\varphi)}{2r^{n+2}} + O\left(\frac{\mu^2}{r^{n+3}}\right)
\end{aligned}
$$

for $n \geq 1$. From the above formulas, we obtain the following order of magnitude estimate for $\frac{\partial \Delta H_1}{\partial u}$:

$$
\begin{aligned}
\frac{\partial \Delta H}{\partial u} &= -\frac{1}{r^2}\frac{\partial r}{\partial u} + \frac{1-\mu}{d_2^2}\left(\frac{\partial d_2}{\partial u}\right) + \frac{\mu}{d_1^2}\left(\frac{\partial d_1}{\partial u}\right) \\
&= -\frac{1}{r^2}\frac{\partial r}{\partial u} + (1-\mu)\bigg(\frac{1}{r^2} - \frac{2\mu\cos\varphi}{r^3} + O(\mu r^{-4})\bigg)\left(\frac{\partial r}{\partial u}\left(1 + O(\mu r^{-2})\right) + \left(-\mu\sin\varphi + O(\mu r^{-1})\right)\frac{\partial \varphi}{\partial u}\right) \\
&\quad + \mu\bigg(\frac{1}{r^2} + \frac{2(1-\mu)\cos\varphi}{r^3} + O(r^{-4})\bigg)\left(\frac{\partial r}{\partial u}\left(1 + O(r^{-2})\right) + \left((1-\mu)\sin\varphi + O(r^{-1})\right)\frac{\partial \varphi}{\partial u}\right) \\
&= O\left(\frac{\mu}{r^4}\right)\frac{\partial r}{\partial u} + O\left(\frac{\mu}{r^3}\right)\frac{\partial \varphi}{\partial u}.
\end{aligned}
\tag{24}
$$

To see this, let us first look at the coefficient of $\frac{\partial r}{\partial u}$. It cannot be of order r^{-2}, because the sum of all the leading terms $-\frac{1}{r^2} + \frac{\mu}{r^2} + \frac{1-\mu}{r^2}$ cancel. What is surprising is that we even get cancellation to one higher order in r, due to the fact that

$$\mu\left(\frac{n(1-\mu)\cos\varphi}{r^{n+1}}\right) + (1-\mu)\left(\frac{-n\mu\cos\varphi}{r^{n+1}}\right) = 0.$$

for all n. So for $n = 2$ in the above, we get cancellation of terms of order r^{-3} in the coefficient for $\frac{\partial r}{\partial u}$. This cancellation, as we can see, arises due to the center of mass being fixed at the origin. Thus, we have only have terms of order r^{-4} or higher left. A similar cancellation allows the leading coefficients of order r^{-2} for $\frac{\partial \varphi}{\partial u}$ to cancel. By keeping track of μ's as we did in the above, we also see that $\frac{\partial \Delta H}{\partial u}$ is proportional to μ.

If we differentiate the above estimate for $\frac{\partial \Delta H}{\partial u}$, we obtain an estimate for $\frac{\partial^2 \Delta H}{\partial u \partial v}$:

$$\frac{\partial^2 \Delta H}{\partial u \partial v} = O\left(\frac{\mu}{r^4}\right)\frac{\partial^2 r}{\partial u \partial v} + O\left(\frac{\mu}{r^5}\right)\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} + O\left(\frac{\mu}{r^4}\right)\left(\frac{\partial r}{\partial u}\frac{\partial \varphi}{\partial v} + \frac{\partial r}{\partial v}\frac{\partial \varphi}{\partial u}\right) + O\left(\frac{\mu}{r^3}\right)\frac{\partial \varphi}{\partial u}\frac{\partial \varphi}{\partial v} + O\left(\frac{\mu}{r^3}\right)\frac{\partial^2 \varphi}{\partial u \partial v}. \tag{25}$$

As one can see, the order of magnitude estimates for $\frac{\partial \Delta H}{\partial u}$ and $\frac{\partial^2 \Delta H}{\partial u \partial v}$ consist of a sum of terms, each term being the product of an expression involving just the derivatives of the polar variables and an expression free from derivatives of the polar variables and of the form $O(r^{-n})$. We shall refer to an expression of the latter type as being a *coefficient* of the appropriate derivative of ΔH.

By replacing, in the above estimates, the partial derivatives of r and φ with their upper bounds placed earlier in this section, we obtain bounds on the derivatives of ΔH.

5 Proof of the Twist

From all the work spent bounding quantities in the previous section, we are now in the position to establish the twist condition (14) with not too much effort. We want to find a range of J on which there is a twist, but there are degeneracies in e. What we do is switch to a new set of variables, namely (ρ, E), where $E = C/2$ is the energy as usual, and the new variable ρ, is defined by

$$\rho = J/E. \tag{26}$$

By (A2), we need $\rho \gtrapprox 1$. By fixing E, we get a range of J as we vary ρ. We want to show that for large enough E, we can find a range of ρ, and thus a range of J, on which there is a twist. It turns out that when we do this, we no longer have a problem with the degeneracies in e. Why this is so will be provided in the proof of the twist. First, a definition and some lemmas.

Recall that by eliminating J_r, we can express any function of the action-angle variables in terms (θ, θ_r, J, E) instead.

Definition 5.1 *Let $X = X(\theta, \theta_r, J, E)$. Then we say X is of order k in J or X is of order J^k if for any fixed $\rho > 1$,*

$$X(\theta, \theta_r, \rho E, E) = O(E^k) \text{ as } E \to \infty. \tag{27}$$

So for instance, r is of order J^2 since it is proportional to J^2, and φ is of order J^0. It is also straightforward to see that $\frac{\partial e}{\partial J}$, $\frac{\partial e}{\partial J_r}$, $\frac{\partial^2 e}{\partial J^2}$, and $\frac{\partial^2 e}{\partial J \partial J_r}$ are of order J^{-1}, J^{-1}, J^{-2}, and J^{-2}, respectively. Since for any fixed $\rho > 1$, we have

$$\lim_{E\to\infty} e = \lim_{E\to\infty} \sqrt{1 - \frac{(E - 1/2(\rho E)^2 + \delta)^2}{(\rho E)^2}} = \sqrt{1 - \rho^{-2}} < 1, \tag{28}$$

the terms that contribute to the order in J of the derivatives of r and φ are powers of J and the derivatives of e. It is thus easy to see, from the bounds placed on the derivatives in the previous section, that

Lemma 5.2 *The orders of the derivatives of r and φ in J can be computed as follows. First, r and φ are of order J^2 and J^0, respectively. Next, for each derivative with respect to the action variables J, J_r, lower the order by a power of J. Derivatives with respect to the angles θ, θ_r, do not change the order.*

So for instance $\frac{\partial r}{\partial J_r}$, $\frac{\partial \varphi}{\partial \theta}$, and $\frac{\partial \varphi}{\partial J}$ are of order 1, 0, and -1 in J, respectively. Intuitively, the order in J gives an order of magnitude estimate, and it is exactly the one we need to prove that there is a twist, notwithstanding degeneracies. Observe that, by definition, if a quantity X has order J^k, then $X/J^k = X/(\rho E)^k$ is bounded as $E \to \infty$ for any fixed ρ. Thus, for any fixed length J_0, X is bounded by some fixed constant times J^k on $[J^-(\mu, E), J^-(\mu, E) + J_0]$, for all sufficiently large E.

Lemma 5.3 *Let $X = O(r^{-n})$ be any one of the coefficients of $\frac{\partial \Delta H}{\partial u}$ or $\frac{\partial^2 \Delta H}{\partial u \partial v}$ as given by (24) and (25), respectively. Then X is of order J^{-2n}. Furthermore, X/J^{2n} is a sum of terms, each term being bounded by an expression that is*

(i) proportional to a positive power of μ,

(ii) contains as its only other non-constant factors nonnegative powers of $(1-e)^{-1}$ and J^{-1}.

It follows that $\|X\|/J^{2n}$ is bounded by a sum of terms satisfying (i) and (ii) as well. Moreover, $\|X\|/J^k|_{J=\rho E}$ is bounded as $E \to \infty$ for any fixed $\rho > 1$.

Proof We observe that $X = O\left(\frac{\mu}{r^n}\right)$ is an infinite series arising from performing a binomial expansion for d_2/r and/or d_1/r and retaining terms of the appropriate order. Thus, the terms are proportional to μ, some power of r no greater than $-n$, and positive powers of $\cos\varphi$. Only the largest power of r matters for the order in J, and so we get $-2n$ for order in J. Since $r^{-1} \leq J^{-2}(1-e)^{-1}$ and $|\cos\varphi| \leq 1$, once we factor out J^{-2n}, since all terms are of order no greater than $-n$ in r, (ii) follows immediately. The final statements are now clear. ■

Corollary 5.4 *The order in J of the derivatives of ΔH can be computed as follows. First, ΔH is of order J^{-6}. Next, for each derivative with respect to the action variables J, J_r, lower the order by a power of J. Derivatives with respect to the angles θ, θ_r do not change the order.*

Proof Using the two previous lemmas and formulas (24) and (25), add the orders from the derivatives of the polar variables to the orders of their respective coefficients. This gives the stated prescription for finding the order in J of the derivatives of ΔH. ■

Corollary 5.5 *Pick any $\epsilon > 0$. Then for sufficiently large E or sufficiently small μ, we can find an interval $\mathcal{J}_0(\mu, E, \epsilon)$ on which $\left.\left\| \frac{\partial \Delta H}{\partial u} \right\|^*\right|_{\mathcal{J}_0} < \epsilon$ and $\left.\left\| \frac{\partial^2 \Delta H}{\partial u \partial v} \right\|^*\right|_{\mathcal{J}_0} < \epsilon$, $u, v \in \{\theta, \theta_r, J, J_r\}$. Moreover, $|\mathcal{J}_0|$ is monotonically increasing in E and μ^{-1}. Moreover, we can take the left end-point of $\mathcal{J}_0$ as close as we like to $J^-(E) + \delta$ for sufficiently large E or μ^{-1} and δ id defined in (8).*

Proof Both $\frac{\partial H}{\partial u}$ and $\frac{\partial^2 H}{\partial u \partial v}$ are of the form $J^k F(\mu, e, J^{-1})$, where k is the order in J of the derivative, and F is a function of the form described in Lemma 5.3, with $k \in \{-6, -7, -8\}$ depending on the choices for $u, v \in \{\theta, \theta_r, J, J_r\}$. Replace J with ρE in the above bound. When we fix ρ and let $E \to \infty$, F stays bounded, since e stays bounded away from 1 from our previous calculation. Thus, for large E and large J, we can make $J^k F(\mu, e, E^{-1})$ smaller than ϵ, as long as ρ is sufficiently bounded away from ∞. Since J^k decreases with increasing E, the larger we choose E, the larger we may fix ρ, since we can let e get closer to 1 and still be less than ϵ. This determines our finite interval $J_0 = \mathcal{J}_0(\mu, E, \epsilon)$ for which $J^k F(\mu, e, J^{-1}) < \epsilon$. Since F is proportional to μ, the lemma is also true if we make μ small.

We only need to worry about the degeneracy of $\rho \approx 1$ when one of the derivatives is with respect to J_r, since such derivatives contain J_r in the denominator, and in the large E limit, one sees that $J_r^{-1} \to E^{-1}(\rho - 1)^{-1}$. However, the larger we take E, the closer we can take ρ to 1 and still get a small bounded quantity. ■

Corollary 5.6 *Fix $\epsilon > 0$. For sufficiently large E or sufficiently small μ, one can find an interval $\mathcal{J}_0(\mu, C, \epsilon)$ such that the quantities L, b, η, and τ, as defined in Proposition 3.1, satisfy*

$$0 < L,\, b,\, \eta < \epsilon, \qquad |\tau - 2\pi| < \epsilon.$$

Proof Since L is bounded by the norm of the Jacobian of the Hamiltonian flow (3), L is proportional to large reciprocal powers of J. Hence, for large E, L will be as small as desired. Everything else follows similarly. ■

We now have the tools to establish the twist condition:

Theorem 5.7 *For sufficiently large E or sufficiently small μ, there exists an interval $\mathcal{J}_{\text{twist}}(\mu, E) = [\mathcal{J}^-_{\text{twist}}(\mu, E), \mathcal{J}^+_{\text{twist}}(\mu, E)]$ on which the return map F is a twist map. Moreover, $|\mathcal{J}_{\text{twist}}|$ is monotonically increasing in E and μ^{-1}.*

Proof We will make use of Proposition 3.1. Essentially, $-\lambda/J^4$ dominates C_1 and C_2 in (14), since $-\lambda/J^4$ is of order J^{-4} whereas the remaining terms are of order J^{-7}. The fact that we need to perturb J by $b\tau$ and worry about all the suprema is inconsequential. This is because once we fix any $\epsilon > 0$, we can choose an interval $\mathcal{J}_0(\mu, E, \epsilon)$ as in the previous corollary and increase E as needed. The small term $b\tau$, along with L and η, are all uniformly bounded by ϵ, which also implies that λ is uniformly bounded from below by a positive constant ≈ 3 on $\mathcal{J}_0(\mu, E, \epsilon)$. This remains true even while we increase E. When we do this, $J > E$ increases, and by holding $\rho = J/E$ fixed, we have, as in previous calculations, $J^7\left(C_1 + C_2(1 + C_0 + J^{-3})\right)$ stays bounded, since we have factored out the appropriate order in J and $(1 - e)^{-1}$ is uniformly bounded away from one. Thus, for E large enough,

$$-\lambda(\rho E)^3 + (\rho E)^7\left(C_1 + C_2(1 + C_0 + J^{-3})\right) < 0$$

for fixed ρ, since the first term goes to $-\infty$ while the second term remains bounded. This implies the twist condition (14). Moreover, the larger we take E, the larger the range of ρ we can take for the twist interval. In particular, we can also take the left-hand point of the twist interval as close to E as we like for large enough E, although for definitiveness, we always take to be at least $J^-(\mu, E) + 10\mu$ as given by the standard assumption (A3). ■

5.1 Numerics for the Size of the Twist

While Theorem 5.7 finally establishes that there exists a twist, it would be useful to get an idea as to how large E needs to be and how large the twist interval is. Given the complexity of our expressions and (14), we resort to a computer to find the intervals over which there is a twist for fixed μ and E. We do this as follows:

• *Upper bounds:* The twist condition (14) contains many upper bounds on the derivatives of ΔH. The derivatives of ΔH consists of a sum of coefficients times derivatives of the polar variables. We bound the derivatives of the polar variables via the bounds placed in Section 4. These bounds are in terms of only μ, E, and J (e is bounded by a function of μ, E, and J). The coefficients can also be bounded in terms of μ, E, and J via Lemma 5.3. More explicitly, however, we use the bounds placed in the Appendix. Altogether, this allows us to evaluate the upper bounds on the derivatives of ΔH, and hence (14) itself, as a function of μ, E, and J. For $\mu = 1/1000$ and various fixed values of E, we can thus examine over what J there is a twist.

• *Simplifications:* We simplify (14) in a very reasonable way. Namely, we take $\tau = 2\pi$ and $b = 0$. Again, this is because we expect the derivatives of ΔH to be small and have very little effect. Indeed, we carry out these verifications in the Appendix, where we place explicit bounds on the derivatives of ΔH and show that they are indeed small.

To determine the Lipschitz constant L in Proposition 3.1, we note that L is bounded by the norm of the Jacobian of (3). Using our estimates in Section 4 and our bounds for the second derivatives of ΔH, we can thus bound this norm and thus L, as a function of μ, E, and J. For fixed μ and E, we can choose an interval of J, find the numerical maximum of L on this range of J, and then use this value of L in the twist. This is how we compute L in the below numerics. It turns out that in all cases except for the last energy plotted below, we can take $L = -3/E^4$. Indeed, this is to be expected since the main term of the Jacobian of 3 is the $-3/J^4$ term, which decreases in J, and so a maximum of L occurs at $J = J^-(E) \approx E$. On the other hand, for large enough J, the second derivatives of ΔH will dominate and L will increase. This is the case for $E = E_0(5, 0)$ plotted below and we must readjust L accordingly (only slightly in fact).

In the below table, we plot for fixed $\mu = 1/1000$ and energy E the twist intervals as computed using the assumptions above. Recall that $E_0(J, J_r) = 1/2J^2 + (J - J_r)$ is the "unperturbed energy", i.e., the negative of the Hamiltonian for the rotating Kepler problem. When we take $E = E_0(J^*, 0)$, then $E \approx J^*$, and by the standard assumption, we need $J^-_{\text{twist}} \geq J^-(E) + 12\mu \approx J^*$. We find that in our numerical calculations, we can always take J^-_{twist} very close to J^* and still get a twist ($J^-_{\text{twist}} \approx J^* + 0.01$). Thus, we will be a bit sloppy and denote J^-_{twist}, the left endpoint of the twist interval, by simply J^*.

Table 1. Twist Intervals

μ	E	$\mathcal{J}_{\text{twist}}$
1/1000	$E_0(2.4, 0)$	$[2.4, 2.9]$
1/1000	$E_0(2.65, 0)$	$[2.65, 3.5]$
1/1000	$E_0(3.5, 0)$	$[3.5, 5.8]$
1/1000	$E_0(5, 0)$	$[5, 10.4]$

In terms of distances, recall that roughly $r = J^2(1-e)$, where we can bound e in terms of by μ, E, J. Thus, we have a twist (we arbitrarily decided that $|\mathcal{J}_{\text{twist}}(E)| > 1/2$ should count as a twist) even when the massless planet is at least $r = 2.9^2(1 - e(1/1000, E_0(2.4, 0), 2.9)) \approx 3.8$ times further away from the primaries than each other. As seen from the above data, the size of the twist interval grows with E, as expected.

6 Aubry-Mather Theory

We begin with a brief introduction to Aubry-Mather theory. This will give us the tools to verify the exactness condition for F in §6.2, the final condition needed to verify that F is an EAPT map. After this, we can finally apply Aubry-Mather in §6.3 to prove the existence of Aubry-Mather sets, and thus periodic and quasiperiodic solutions, for PCR3BP.

6.1 Introduction

Each C^1 smooth map $\Phi : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ satisfying conditions (i)-(iii) of §3 can be described by a so-called *generating function* $h : \mathbb{R}^2 \to \mathbb{R}$ in the following way: Let $\tilde{\Phi} : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of Φ, given by $\tilde{\Phi}(\tilde{\theta} + 2\pi, \tilde{\theta}') = \tilde{\Phi}(\tilde{\theta}, \tilde{\theta}') + (2\pi, 0)$. Then $\tilde{\Phi}(\tilde{\theta}_0, I_0) = (\tilde{\theta}_1, I_1)$ can be implicitly defined by the following equations:

$$\begin{cases} I_0 &= -\partial_1 h(\tilde{\theta}_0, \tilde{\theta}_1) \\ I_1 &= \partial_2 h(\tilde{\theta}_0, \tilde{\theta}_1) \end{cases} \tag{29}$$

Here $\partial_i,\ i = 1, 2$ is the partial derivative with respect to the i-th component, $h \in C^2$, $h(\tilde{\theta}_0 + 2\pi, \tilde{\theta}_1 + 2\pi) = h(\tilde{\theta}_0, \tilde{\theta}_1)$, and $\partial_2\partial_1 h \leq -\gamma < 0$ for some $\gamma > 0$.

The value of $h(\tilde{\theta}, \tilde{\theta}')$ equals the minimal action to get from $\tilde{\theta}$ to $\tilde{\theta}'$ in time 2π, where action arises from time-periodic Lagrangian system associated to Φ by Moser's Theorem [Mo].

Since Aubry-Mather theory studies EAPT maps, we need to make sure that in addition, Φ satisfies the exactness condition (iv) of §3. To do so, we use the following:

Exactness Theorem. ([LM], p. 1) *Let Φ be as above, and h its generating function. Then Φ is exact if and and only if*[5] *(i) (periodicity condition): $h(\theta + 2\pi, \theta' + 2\pi) = h(\theta, \theta')$; (ii) $h_{12} > 0$.*

We shall apply this theorem to our twist map F in 6.2 to show that it is exact.

Aubry-Mather theory studies the orbit structure of EAPT maps by projecting orbits into their first components, which form *configuration space*. Consider the space of configurations $\mathbb{R}^{\mathbb{Z}} = \{\Theta |\ \Theta : \mathbb{Z} \to \mathbb{R}\}$, that is, the space of bi-infinite sequences of real numbers with the product topology. Given a function $h : \mathbb{R}^2 \to \mathbb{R}$, extend h to arbitrary finite segments $(\tilde{\theta}_j, \ldots, \tilde{\theta}_k)$, $j < k$, of configurations $\Theta \in \mathbb{R}^{\mathbb{Z}}$ by

$$h(\tilde{\theta}_j, \ldots, \tilde{\theta}_k) = \sum_{i=j}^{k-1} h(\tilde{\theta}_i, \tilde{\theta}_{i+1}).$$

Say that segment is *minimal* or *action-minimizing* with respect to h if

$$h(\tilde{\theta}_j, \ldots, \tilde{\theta}_k) \leq h(\tilde{\theta}^*_j, \ldots, \tilde{\theta}^*_k)$$

for all $(\tilde{\theta}^*_j, \ldots, \tilde{\theta}^*_k)$ with $\tilde{\theta}_j = \tilde{\theta}^*_j$ and $\tilde{\theta}_k = \tilde{\theta}^*_k$.

A configuration $\tilde{\theta} \in \mathbb{R}^{\mathbb{Z}}$ is called *minimal* or *action-minimizing* with respect to h if every finite segment of $\tilde{\theta}$ is minimal or action-minimizing with respect to h. The set of all action-minimizing trajectories is denoted by $\widetilde{\Sigma} = \widetilde{\Sigma}(h) \subset \mathbb{R}^{\mathbb{Z}}$.

A configuration $\tilde{\theta} \in \mathbb{R}^{\mathbb{Z}}$ is called *stationary* if

$$\partial_2 h(\tilde{\theta}_{k-1}, \tilde{\theta}_k) + \partial_1 h(\tilde{\theta}_k, \tilde{\theta}_{k+1}) = 0 \quad \text{for all} \quad k \in \mathbb{Z} \tag{30}$$

[5] In the literature, one considers twist maps with a positive twist, i.e. $d\theta'/dJ > 0$. In our case, we have a negative twist since $d\theta'/dJ < 0$. Thus, instead of the usual $h_{12} < 0$ for a positive twist, we have $h_{12} > 0$.

This equation is an analog of the Euler-Lagrange equation in this case. Indeed, this equation says that the sum $\sum_k h(\tilde{\theta}_k, \tilde{\theta}_{k+1})$ is extremized with respect to each $\tilde{\theta}_k$, because formal derivative of the sum with respect to each $\tilde{\theta}_k$ is zero. In particular, each minimal configuration is stationary. The set of all stationary trajectories will be denoted by $\widetilde{St} = \widetilde{St}(h) \subset \mathbb{R}^{\mathbb{Z}}$. We have $\tilde{\Sigma} \subset \widetilde{St}$.

Lemma. *Suppose $h : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 smooth function. Then there is one-to-one correspondence between stationary configurations and orbits of an* EAPT $\Phi : \mathbb{A} \to \mathbb{A}$, *given by the following relation: let $0 \le \tilde{\theta}_0 = \theta_0 < 2\pi$, then*

$$\begin{aligned} \{\tilde{\theta}_k\}_{k\in\mathbb{Z}}, &\longrightarrow \Phi^k(\theta_0, I_0) = (\tilde{\theta}_k \bmod 2\pi, \partial_2 h(\tilde{\theta}_{k-1}, \tilde{\theta}_k)), \\ \{\Phi^k(\theta_0, I_0)\}_{k\in\mathbb{Z}} &\longrightarrow \tilde{\Phi}^k(\tilde{\theta}_0, I_0) = (\tilde{\theta}_k, I_k), \ \{\tilde{\theta}_k\}_{k\in\mathbb{Z}}. \end{aligned} \tag{31}$$

Proof is by direct calculation using (30).

The *Aubry graph* of a configuration $\Theta = \{\tilde{\theta}_n\}_{n\in\mathbb{Z}}$ is a graph of piecewise linear function $\mathcal{A}(\Theta)$ equal to $\tilde{\theta}_n$ at each integer n and linearly interpolated in between. A configuration $\Theta \in \mathbb{R}^{\mathbb{Z}}$ has *rotation number* $\rho(\Theta)$ if the following limit exists $\rho(\Theta) = \lim_{n\to\pm\infty} \tilde{\theta}_n/n$. Similarly, a one-sided configuration $\Theta \in \mathbb{R}^{\mathbb{Z}_+}$ has rotation number if $\lim_{n\to+\infty} \tilde{\theta}_n/n$ exists. Action-minimizing configurations have the following properties.

Aubry-Mather Theorem. *Every minimal configuration $\Theta \in \tilde{\Sigma}$ has rotation number, i.e. $\rho(\Theta)$, exists and for every rotation number $\omega \in \mathbb{R}$ there is a minimal configuration $\Theta \in \tilde{\Sigma}$ with $\rho(\Theta) = \omega$. Moreover, there exists a circle homeomorphism ψ with rotation number ω such that Θ is the orbit of the lift $\tilde{\psi}$ of ψ.*

6.2 Proof of Exactness

We will use the Exactness Theorem to show that the return map F is exact. Fix E and write the Hamiltonian for PCR3BP as $H = H(\theta, J, \theta_r)$. Then H is periodic in θ and θ_r, and we consider H as a time-periodic Hamiltonian by treating θ_r as time. From the twist condition (14), we (locally) have that

$$\frac{\partial^2 H}{\partial J^2} = \frac{-3}{J^4} + \frac{\partial^2 \Delta H}{\partial J^2} < 0, \tag{32}$$

where treated the above inequality as roughly being a weaker inequality than (14). Indeed, if one takes $\lambda = -3$ in (14), then due to the excess of positive terms and suprema in (14), then in this approximation, we have (14) implies (32). More precisely, due to the smallness of derivatives of ΔH, we indeed have $\lambda \approx -3$ (see the Appendix), and so the intervals on which (14) holds and (32) holds have a large overlap. Indeed, applying the lemmas and corollaries of this section, one sees that we can always truncate the twist interval, if necessary, so that the twist condition and (32) holds simultaneously; thus, there is no harm in assuming that (32) holds throughout the twist interval. Numerically, however, we find that (32) holds on all the intervals computed in Table 1, and no readjustment of the twist intervals is necessary.

Thus, having established (32), we see that our Hamiltonian H is negative definite in the twist region. We can thus apply the Legendre transform to H and get a Lagrangian L. When we do this, it turns out that since our Poincaré map F is a time-2π map (in θ_r "time") for H, then the generating function h for F is given by the following least action-principle (see e.g. [A]):

$$h(\theta, \theta') = \inf_{\substack{\gamma(0)=\theta \\ \gamma(2\pi)=\theta'}} \int_\theta^{\theta'} L(\gamma(t), \gamma'(t), t)dt,$$

where the infimum is taken over all absolutely continuous curves γ that start at $\gamma(0) = \theta$ and arrive at $\gamma(2\pi) = \theta'$ in time 2π. Since H is periodic in θ, so is L, and therefore h. This establishes the periodicity condition for the Exactness Theorem.

Establishing $h_{12} > 0$ involves just a bit of equation manipulation. First, since (29) gives us that $J = -h_1$, then from looking at the angular component F_θ of our twist map (13), we have

$$0 \equiv \theta' - \theta - 2\pi(J^{-3} + 1 - f) = \theta' - \theta - 2\pi\Big((-h_1)^{-3} + 1 + f(\theta', -h_1)\Big).$$

Differentiating the above with respect to θ' and rearranging, we find that

$$h_{12} = -\frac{1}{2\pi}\left(-\frac{3}{J^4} + \frac{\partial f}{\partial J}\right)^{-1} = -\left(\frac{\partial F_\theta}{\partial J}\right)^{-1} > 0, \tag{33}$$

and this is indeed true, since the twist condition requires that $\frac{\partial F_\theta}{\partial J} < 0$.

This completes the verification that our return map F is locally an exact area-preserving twist map, for sufficiently large E or sufficiently small μ. We are now in the position to apply Aubry-Mather theory to find invariant Aubry-Mather sets.

6.3 Existence of Aubry-Mather Sets in PCR3BP

It is a basic fact (see e.g. [Ba]) that if a circle map ψ has rotation number ω, then every orbit of its lift $\{\theta_i\}_{i\in\mathbb{Z}}$ must satisfy the so-called "ordering condition":

$$|\theta_i - \theta_0 - i\omega| < 2\pi, \forall i. \tag{34}$$

Intuitively, this ordering condition says that in position–time space, the θ_i's cannot deviate too far from having slope ω (θ_i is position at time i). Thus, if a minimal configuration has rotation number ω, by the Aubry-Mather Theorem, the ordering condition allows us to place bounds on the values of J in its J–configuration, since the θ–configuration (which depends on J due to the equation of motion for θ) needs to be within a certain range so that (34) is satisfied.

Indeed, this will be our method of finding Aubry-Mather sets for PCR3BP. Namely, we have our Poincaré map F that is a twist inside an annulus of S_E. It has a generating function $h : \mathbb{R}^2 \to \mathbb{R}$. By the Aubry-Mather theorem, there exists minimal configurations of h of all rotation numbers. The lemma preceding this thereom tells us that such minimal configurations correspond to orbits of $\tilde{F}$. What must be checked is that the J-configuration of this orbit, as determined by (29), lies inside the annulus where F was originally defined, i.e., $\mathbb{S}^1 \times \mathcal{J}_{\text{twist}}$. (This is the step where we place bounds on the J-configuration). It will then follow that this $\tilde{F}$-orbit is also an F-orbit, and hence, an Aubry-Mather set of PCR3BP.

So to place the bounds we need, we need some estimates on the error terms f and g occurring in our twist map F in (13). Comparing the equations of motion (3) with the twist map F in (13), the error term f occurring in the twist map arises due to

(e1) $J \in [J - b\tau, J + b\tau]$ after a period of motion of θ_r (see Prop 3.1);

(e2) Approximating the time of return of θ_r as 2π;

(e3) $\dfrac{\partial \Delta H}{\partial J}$.

Similarly, the main terror term for g occurs through

(e4) $\dfrac{\partial \Delta H}{\partial \theta}$.

Since $\frac{\partial \Delta H}{\partial J}$ and $\frac{\partial \Delta H}{\partial \theta}$ are of order J^{-7} and J^{-8}, respectively, and since (e1) and (e2), being lower order errors, do not change this order of magnitude, we can thus place the bounds

$$\begin{aligned} |f| \leq & \quad \frac{2\pi\alpha(\mu, E, J)}{J^7} = \frac{2\pi\alpha(\mu, E, \rho)}{J^7} \\ |g| \leq & \quad \frac{2\pi\beta(\mu, E, J)}{J^6} = \frac{2\pi\beta(\mu, E, \rho)}{J^6}, \end{aligned} \tag{35}$$

where the functions α and β depend only on inverse powers of J due to the fact that we have already factored out the appropriate power of J from f and g (see Lemma 5.3). We did not denote α and β as functions of e in the above, since e can be bounded in terms of μ, E, and J. Here, the (ρ, E) variables are defined as in §5. From our description (e1)–(e4) of the error terms f and g, it follows that

$$\begin{aligned} \alpha \lessapprox & \quad J^7 \left\| \frac{\partial \Delta H}{\partial J} \right\|^* \\ \beta \lessapprox & \quad J^6 \left\| \frac{\partial \Delta H}{\partial \theta} \right\|^*. \end{aligned} \tag{36}$$

As we can see from the above formulas, the degeneracies of the twist map at $e = 0$ and $e = 1$ enter into α and β. But by Lemma 5.3, so long as we are bounded away from $e = 0$ or $e = 1$, i.e., α and β will be bounded and under control. To see more explicit bounds on α and β, see Appendix A.

The next lemma below gives us the desired bounds on the J-configuration using the ordering condition (34). First, we need a few definitions:

To bound e from above, for any $0 < \nu < 1$, define the interval

$$\mathcal{J}_\nu := [J^-_{\text{twist}}, J_\nu] \subset [J^-_{\text{twist}}, J^+_{\text{twist}}]$$

to be such that $e \leq \nu$ on $\mathcal{J}_\nu$. Using the approximate formula $e \approx \sqrt{1 - E^2/J^2}$, this occurs roughly when $J_\nu = E/\sqrt{1-\nu^2} \approx J^-_{\text{twist}}/\sqrt{1-\nu^2}$. Indeed, by bounding $e \leq \nu$, we can ensure that the error terms α and β are bounded.

Given any point $(\theta, J) \in \mathbb{S}^1 \times \mathcal{J}_{\text{twist}}$ in the region of the twist, define the *instantaneous rotation at* (θ, J) to be the change in the angular component under an iteration of F, i.e., $\tilde{F}_\theta(\tilde{\theta}, J) - \tilde{\theta}$. Since F is a twist on $\mathbb{S}^1 \times J_{\text{twist}}$, every vertical line $\{\theta = \theta^*\}$ in the twist region is monotonically twisted. Thus, for suitable ω, there is a unique point (θ^*, J^*) on the line $\{\theta = \theta^*\}$ which has instantaneous rotation ω. So for any ω, define $J_{\max}(\omega)$ and $J_{\min}(\omega)$ to be the supremum and infimum of all the J^* obtained in this way as θ^* varies over $\mathbb{S}^1$. Let $\mathcal{J}(\omega) = [J_{\min}(\omega), J_{\max}(\omega)]$. It follows that the annulus $\mathbb{S}^1 \times \mathcal{J}(\omega)$ contains all the possible points of the twist region that can have instantaneous rotation ω. It follows that an orbit with rotation number ω cannot stay bounded above or below $\mathbb{S}^1 \times \mathcal{J}(\omega)$ for all future time, since if the orbit always stayed below $\mathbb{S}^1 \times \mathcal{J}(\omega)$, its rotation number would have to be greater than ω, and conversely, if it always stayed above $\mathbb{S}^1 \times \mathcal{J}(\omega)$, its rotation number would have to be less than ω.

Finally, so that we can avoid writing -2π everywhere, given a rotation number ω, define the *inertial rotation number*

$$\bar{\omega} := \omega + 2\pi. \tag{37}$$

This is the rotation number in the non-rotating inertial frame.

Lemma 6.1 *With the notations above, fix μ, E, and $0 < \nu < 1$, with $J_\nu \leq J^+_{\text{twist}}$. Let*

$$\begin{aligned} \tilde{\alpha} &= \sup_{J \in \mathcal{J}_\nu} \alpha(\mu, E, J) \\ \tilde{\beta} &= \sup_{J \in \mathcal{J}_\nu} \beta(\mu, E, J), \end{aligned}$$

and assume $\tilde{\alpha} < \frac{3}{7}(J^-_{\text{twist}})^4$*. Then the set of rotation numbers* ω *that satisfy* $J^-_{\text{twist}} < J_{\min}(\omega) < J_{\max}(\omega) < J_\nu$ *and the conditions*

$$\left(\frac{1}{(J^-_{\text{twist}})^2} - \frac{1}{J_{\min}(\omega)^2}\right) + \frac{1}{3}\left(\frac{\tilde{\alpha}}{J_{\min}(\omega)^6} - \frac{\tilde{\alpha} + 12\pi\tilde{\beta}}{(J^-_{\text{twist}})^6}\right) \geq 2\Big(J_{\min}(\omega) - J^-_{\text{twist}}\Big)(\bar{\omega}/2\pi) \quad (38)$$

$$\left(\frac{1}{J_{\max}(\omega)^2} - \frac{1}{J_\nu^2}\right) + \frac{1}{3}\left(\frac{\tilde{\alpha} + 12\pi\tilde{\beta}}{J_{\max}(\omega)^6} - \frac{\tilde{\alpha}}{J_\nu^6}\right) \leq 2\Big(J_\nu - J_{\max}(\omega)\Big)(\bar{\omega}/2\pi) \quad (39)$$

contains an interval $[\omega^-(\mu, E, \nu), \omega^+(\mu, E, \nu)]$*. Furthermore, for each rotation number* $\omega^-(\mu, E, \nu) < \omega < \omega^+(\mu, E, \nu)$*, there exists a solution of PCR3BP with rotation number* ω *whose* J*-configuration stays inside* $\mathcal{J}_{\text{twist}}(\mu, E)$ *and whose eccentricity is bounded by* ν*.*

Proof Consider a minimal configuration $\Theta = \{\theta_i\}$ with rotation number ω. By hypothesis, $\mathcal{J}(\omega) \subset \mathcal{J}_\nu$, and so if the J-configuration of Θ reaches J^-_{twist} or J_ν, eventually one of the points of Θ will have to have instantaneous rotation greater than or less than ω, respectively. We will call the former situation "overspeeding" and the latter "underspeeding". The idea is that the more $\mathcal{J}(\omega)$ is safely contained inside $\mathcal{J}_\nu$, the more overspeeding or underspeeding that must take place in order for the J-configuration of Θ to reach the boundary of $\mathcal{J}_\nu$. This is because then the orbit corresponding to Θ must spend more time outside $\mathcal{J}(\omega)$ before it gets to the boundary of $\mathcal{J}_\nu$, i.e, it takes more iteratations of $\tilde{F}$ to get there. Thus, for J^-_{twist} sufficiently smaller than $J_{\min}(\omega)$, the configuration will have had to overspeed so much that by the time it reaches J^-_{twist}, it will have failed (34), i.e., for some M,

$$\theta_M - \theta_0 \geq M\omega + 2\pi. \quad (40)$$

This is because the successive angular displacements $\theta_{i+1} - \theta_i$ will exceed ω too often. The same remarks apply to the underspeeding case. The inequalities of this lemma are then essentially statements for when these sufficient conditions are met.

Let us consider the overspeeding case first. So consider $J_0 < J_{\min}(\omega)$. In the worst possible scenario, J_i is monotonically increasing to $J_{\min}(\omega)$, so that it can try to stop overspeeding as soon as possible. What we want to show is that the range of J_0 for which (40) occurs, even in this worst case scenario, includes J^-_{twist}. This will imply that a minimizer with rotation number ω cannot escape $\mathcal{J}_{\text{twist}}$ by crossing J^-_{twist}.

From the above bounds placed on the twist map earlier, we have

$$\theta_{i+1} - \theta_i \geq 2\pi(J_i^{-3} - 1 - \tilde{\alpha}J_i^{-7}),$$

and

$$J_{i+1} \leq J_i + 2\pi\tilde{\beta}J_i^{-6},$$

where we can use $\tilde{\alpha}$ and $\tilde{\beta}$ since $J_0 < J_\nu$. It follows that

$$J_i \leq J_0 + 2\pi\tilde{\beta}iJ_0^{-6} \quad (41)$$

and so

$$\theta_{i+1} - \theta_i \geq 2\pi\Big((J_0 + \tilde{\beta}iJ_0^{-6})^{-3} - 1 - \tilde{\alpha}(J_0 + \tilde{\beta}iJ_0^{-6})^{-7}\Big).$$

The last line follows because $J^{-3} - \tilde{\alpha}J^{-7}$ is a decreasing function of J when $\tilde{\alpha} < 3J^4/7$, which is true by hypothesis. Next,

$$\begin{aligned} \theta_M - \theta_0 - M\omega &= \sum_{i=0}^{M-1}\Big(\theta_{i+1} - \theta_i - \omega\Big) \\ &\geq \sum_{i=0}^{M-1}\Big(2\pi(J_0 + 2\pi\tilde{\beta}iJ_0^{-6})^{-3} - 2\pi\tilde{\alpha}(J_0 + 2\pi\tilde{\beta}iJ_0^{-6})^{-7} - \bar{\omega}\Big). \quad (42)\end{aligned}$$

We can approximate the above sum from below as an integral and then require that it exceed 2π. Namely,

$$\begin{aligned}
&\sum_{i=0}^{M-1}\left[2\pi\Big((J_0+2\pi\tilde{\beta}iJ_0^{-6})^{-3}-\tilde{\alpha}(J_0+2\pi\tilde{\beta}iJ_0^{-6})^{-7}\Big)-\bar{\omega}\right]\\
&\qquad\geq \frac{J_0^6}{2\pi\tilde{\beta}}\int_{J_0}^{J_0+2\pi\tilde{\beta}MJ_0^{-6}}2\pi(x^{-3}-\tilde{\alpha}x^{-7})dx-M\bar{\omega}\\
&\qquad= \frac{J_0^6}{2\tilde{\beta}}\left(\frac{1}{J_0^2}-\frac{1}{(J_0+2\pi\tilde{\beta}MJ_0^{-6})^2}-\frac{\tilde{\alpha}}{3J_0^6}+\frac{\tilde{\alpha}}{3(J_0+2\pi\tilde{\beta}MJ_0^{-6})^6}\right)-M\bar{\omega}\\
&\qquad\geq 2\pi. \qquad (43)
\end{aligned}$$

The sum in the first line is slightly greater than the integral since the sum is a “left-endpoint” Riemann sum of a decreasing function.[6]

The configuration will continue to overspeed until it reaches $J_{\min}(\omega)$. This gives an upper bound on M, namely, $J_M \leq J_0+2\pi\tilde{\beta}MJ_0^{-6}\leq J_{\min}(\omega)$. So take

$$M=\frac{(J_{\min}(\omega)-J_0)J_0^6}{2\pi\tilde{\beta}}$$

in the above inequality. Rearranging terms and setting $J_0=J_{\text{twist}}^-$, we find that failure of the ordering condition due to overspeeding at J_{twist}^- amounts to choosing ω such that

$$\left(\frac{1}{(J_{\text{twist}}^-)^2}-\frac{1}{J_{\min}(\omega)^2}\right)+\frac{1}{3}\left(\frac{\tilde{\alpha}}{J_{\min}(\omega)^6}-\frac{\tilde{\alpha}+12\pi\tilde{\beta}}{(J_{\text{twist}}^-)^6}\right)\geq 2\Big(J_{\min}(\omega)-J_{\text{twist}}^-\Big)(\bar{\omega}/2\pi).$$

This is what we want, since overspeeding at J_{twist}^- means that the configuration cannot fall below J_{twist}^-.

If we now redo everything “from above” and assume that we are underspeeding, i.e. $J_0>J_{\max}(\omega)$, we get virtually the same results. We have

$$\theta_{i+1}-\theta_i\leq 2\pi(J_i^{-3}-1+\tilde{\alpha}J_i^{-7}),$$

and

$$J_{i+1}\geq J_i-2\pi\tilde{\beta}J_i^{-6}.$$

However,

$$J_i\not\geq J_0-2\pi i\tilde{\beta}J_0^{-6},$$

since J_0 is now too large. So fix some test value $J_{\text{ovsp}}\geq J_{\max}(\omega)$ for which we want the minimizer to overspeed before it reaches J_{ovsp}. We then have

$$J_i\geq J_0-2\pi i\tilde{\beta}J_{\text{ovsp}}^{-6}, \qquad (44)$$

since we are trying to decrease to J_{ovsp}. Notice that this is a crude estimate that actually gets worse as we decrease J_{ovsp}. Once again, we have an estimate on M, via $J_M\geq J_0-2\pi\tilde{\beta}MJ_{\text{ovsp}}^{-6}\geq J_{\text{ovsp}}$, and so we can take

$$M=\frac{(J_0-J_{\text{ovsp}})J_{\text{ovsp}}^6}{2\pi\tilde{\beta}}$$

[6] One could refine the inequalities of this lemma, say, by separating our calculation into two halves, one on the first half of the interval of integration $[J_0, J_0+2\pi\tilde{\beta}MJ_0^{-6}]$ and the other on the second half. The crude overestimation (41) can then be redone on the second half, with J_0 replaced with the midpoint $J_0+\pi\tilde{\beta}MJ_0^{-6}$.

in the inequality (the derivation is just as before)

$$\begin{aligned}\theta_M - \theta_0 - M\omega &\leq \frac{J_{\rm ovsp}^6}{2\pi\tilde\beta}\int_{J_0-2\pi\tilde\beta M J_{\rm ovsp}^{-6}}^{J_0} 2\pi(x^{-3}+\tilde\alpha x^{-7})dx - M\bar\omega \\ &= -M\bar\omega + \frac{J_{\rm ovsp}^6}{2\tilde\beta}\left(\frac{1}{(J_0-2\pi\tilde\beta M J_{\rm ovsp}^{-6})^2} - \frac{1}{J_0^2} + \frac{\tilde\alpha}{2(J_0-2\pi\tilde\beta M J_{\rm ovsp}^{-6})^6} - \frac{\tilde\alpha}{2J_0^6}\right) \\ &\leq -2\pi.\end{aligned}$$

Letting $J_0 = J_\nu$ and rearranging gives us

$$\left(\frac{1}{J_{\rm ovsp}^2} - \frac{1}{J_\nu^2}\right) + \frac{1}{3}\left(\frac{\tilde\alpha + 12\pi\tilde\beta}{J_{\rm ovsp}^6} - \frac{\tilde\alpha}{J_\nu^6}\right) \leq 2\Big(J_\nu - J_{\rm ovsp}\Big)(\bar\omega/2\pi). \tag{45}$$

We obtain (39) by letting $J_{\rm ovsp} = J_{\max}(\omega)$. Thus, (39) is a stronger inequality than the one above in light of the fact that we use the crudest possible estimate in (44) by using $J_{\rm ovsp} = J_{\max}(\omega)$.

Altogether, we obtain the desired inequalities that define Ω. Let us analyze them. With ω fixed, define

$$\rho_{\min} = \frac{J_{\min}(\omega)}{J_{\rm twist}^-}, \qquad \rho_{\rm ovsp} = \frac{J_\nu}{J_{\rm ovsp}}.$$

In (38), multiply both sides by $J_{\min}(\bar\omega)^2$. Then observing that $\bar\omega \leq 2\pi(J_{\min}(\omega)^{-3} + \tilde\alpha J_{\min}(\omega)^{-7})$ (this follows from the definition of $\mathcal{J}_{\min}(\omega)$) we obtain the stricter inequality

$$\left(\rho_{\min}^2 - 1\right) + \frac{1}{3}\left(\frac{\tilde\alpha\rho_{\min}^{-4}}{(J_{\rm twist}^-)^4} - \frac{(\tilde\alpha + 12\pi\tilde\beta)\rho_{\min}^2}{(J_{\rm twist}^-)^4}\right) \geq 2\left(1 - \frac{1}{\rho_{\min}}\right)\left(1 + \frac{\tilde\alpha\rho_{\min}^{-4}}{(J_{\rm twist}^-)^4}\right) \tag{46}$$

The second term on the left-hand side of (46) is a negative term which can be made arbitrarily small for large E or small μ when $\rho_{\min}$ is held fixed, since $E < J_{\rm twist}^-(\mu, E) < J_{\min}(\bar\omega)$, and $\tilde\alpha$ and $\tilde\beta$ are proportional to μ and remain bounded as we increase E. Similarly, the second factor on the right-hand side can be made arbitrarily close to one. Since $x^2 - 1 \geq 2(1 - 1/x)$ for all $x \geq 1$, for large E or small enough μ, $\rho_{\min}$ needs to be only slightly larger than 1 for the inequality to hold. Furthermore, the larger E is or the smaller μ is (and hence smaller negative term on left-hand side), the closer $\rho_{\min}$ can be to one. This means we need only take $\omega^+(\mu, E, \nu)$ slightly less than $2\pi[(J_{\rm twist}^-)^{-3} - 1]$. Indeed, in determining (38), we were trying to force orbits that cross $J_{\rm twist}^-$ to overspeed. For E larger or μ smaller, we are closer to integrability, and so we can take ω^+ closer to $2\pi[(J_{\rm twist}^-)^{-3} - 1]$, which is the rotation number of an orbit at $J_{\rm twist}^-$ in the integrable case.

The situation is similar for the other inequality. Namely, multiplying (39) by $J_{\rm ovsp}^2$, we have

$$\left(1 - \frac{1}{\rho_{\rm ovsp}^2}\right) + \frac{1}{3}\left(\frac{(\tilde\alpha + 12\pi\tilde\beta)\rho_{\rm ovsp}^4}{J_\nu^4} - \frac{\tilde\alpha\rho_{\rm ovsp}^{-2}}{J_\nu^4}\right) \leq 2\Big(\rho_{\rm ovsp} - 1\Big)\Big[J_{\rm ovsp}^3 \cdot (\bar\omega/2\pi)\Big] \tag{47}$$

The second term on the left-hand side of (47) is a small positive term when E is sufficiently large or μ is sufficiently small. Since $1-x^{-2} \leq 2(x-1)$ for all $x \geq 1$, then ignoring the second factor on the right-hand side, we see that for sufficiently large $\rho_{\rm ovsp}$ the equality will hold. This corresponds to decreasing $J_{\rm ovsp}$. Taking into account of the second factor, we see that if we lower the threshold value $J_{\rm ovsp}$ at which orbits must overspeed, we need to increase ω. Since $\bar\omega/2\pi \geq J_{\max}(\omega)^{-3} + \tilde\alpha J_{\max}(\omega)^{-7}) \approx J_{\max}(\omega)^{-3}$, by taking ω large enough so that $J_{\max}(\omega)$ is sufficiently below $J_{\rm ovsp}$, we see that term in the brackets will eventually be greater than one, which serves to establish the inequality. Even if we take $J_{\rm ovsp} = J_{\max}(\omega)^{-3}$, the error term $\tilde\alpha J_{\max}(\omega)^{-7}$ can be made small by making μ small or E large as before, whence the term in brackets is still nearly one.

Thus, to summarize, (46) is satisfied when $\rho_{\min}$ is sufficiently large, i.e., when ω is sufficiently small. On the other hand, (47) is satisfied when ω is sufficiently large. We thus get an interval of rotation numbers ω which satisfy both inequalities, and this interval will be nonempty for large enough E or small enough μ. The interval in the statement of the Lemma, defined by (38) and (39), will be smaller, since (39) uses the worst possible estimate $J_{\text{ovsp}} = J_{\max}(\omega)$ in (44) and (45). ∎

While we had to do a lot of work to make the estimates necessary for this lemma, they are still visibly not optimal given the many crude estimates we have made. The biggest problem is in (46), where in evaluating $\rho_{\min}$, which involves rotation numbers with low J, we grossly overestimated by replacing α and β with $\tilde{\alpha}$ and $\tilde{\beta}$, which were bounded over the larger interval J_ν. However, for low J, e will be small, and these estimates will be bad. That is, whatever value of $\rho_{\min}$ that satisfies (46), we need only estimate α and β over $[J^-_{\text{twist}}, \rho_{\min} J^-_{\text{twist}}]$, since the potential minimizer we are considering has its J-configuration confined to that region. Thus, we should replace $\tilde{\alpha}$ and $\tilde{\beta}$ in (46) with

$$\begin{aligned}\tilde{\alpha}(\rho_{\min}) &= \sup_{J\in[J^-_{\text{twist}}, \rho_{\min} J^-_{\text{twist}}]} \alpha(\mu, E, J)\\ \tilde{\beta}(\rho_{\min}) &= \sup_{J\in[J^-_{\text{twist}}, \rho_{\min} J^-_{\text{twist}}]} \beta(\mu, E, J),\end{aligned}$$

We then seek to find the first value of $\rho_{\min}$ for which this modified inequality is satisfied. Call it $\rho^-_{\min}$, and by the definition of $\rho_{\min} = J_{\min}(\omega)/J^-_{\text{twist}}$, the value of $\rho^-_{\min}$ corresponds to some value of $\omega = \omega^+$. It follows from everything that we have done that

$$J^-_{\text{AM}} := \rho^-_{\min} J^-_{\text{twist}} \tag{48}$$

and ω^+ satisfy the following property: if $\omega \le \omega^+$, then $J_{\min}(\omega) \ge J^-_{\text{AM}}$, and a minimal configuration with rotation number ω stays above J^-_{twist}.

This gives us a lower bound on where Aubry-Mather sets for PCR3BP exist, namely, minimal configurations which stay inside $\mathcal{J}_{\text{twist}}$ must lie above J^-_{AM}. At the same time, we also obtained an upper bound ω^+ on the rotation numbers of these minimizers. For an upper bound on where the Aubry-Mather sets for PCR3BP exist, we take ν to be the value of e at J^+_{twist} in Lemma (6.1), i.e., we take $J_\nu = J^+_{\text{twist}}$. We must use this high value of ν that define the bounds $\tilde{\alpha}$ and $\tilde{\beta}$ since we are now considering the "underspeeding case" in which configurations start too high, i.e., they are near J^+_{twist} and have high eccentricity. Considering the worst possible case of (47), which is the inequality (39), in which we set $J_{\text{ovsp}} = J_{\max}(\omega)$, we can consider the first $\rho_{\text{ovsp}} = J^+_{\text{twist}}/J_{\max}(\omega)$ at which (47) holds. Call it ρ^+_{ovsp}, and by definition of ρ_{ovsp}, it corresponds to some $\omega = \omega^-$. It follows, in an analogous fashion to the previous calculation, that

$$J^+_{\text{AM}} := (\rho^+_{\text{ovsp}})^{-1} J^+_{\text{twist}}, \tag{49}$$

and ω^- satisfy the following property: if $\omega \ge \omega^-$, then $J_{\max}(\omega) \le J^+_{\text{AM}}$, and a minimal configuration with rotation number ω stays below J^-_{twist}.

We have thus proved the following:

Theorem AM. *Fix μ, E, and the notations above, and define the intervals*

$$\begin{aligned}\mathcal{J}_{\text{AM}}(\mu, E) &:= [J^-_{\text{AM}}, J^+_{\text{AM}}]\\ \Omega(\mu, E) &:= [\omega^-, \omega^+].\end{aligned}$$

These intervals are nonempty on an open set of parameters (μ, E), and they satisfy the following property: For every $\omega \in \Omega(\mu, E)$, there exists a solution of PCR3BP with rotation number ω which stays inside $S^1 \times S^1 \times \mathcal{J}_{\text{AM}}(\mu, E) \subset \Pi_E$. Such a solution arises as an Aubry-Mather set of the local twist map (13). In particular, for every rational $\omega \in \Omega(\mu, E)$ there is a periodic solution of rotation number ω.

We now discuss our numerics for this Theorem. First, we define $\bar{\Omega}(\mu, E) := \Omega(\mu, E)$ to be the corresponding interval of inertial rotation numbers for Theorem AM. To compute $\bar{\Omega}(\mu, E)$ from $\mathcal{J}_{\mathrm{AM}}(\mu, E)$, we approximate by

$$\left[\frac{1}{(J_{\mathrm{AM}}^{+})^3} + \frac{\alpha(\mu, E, J_{\mathrm{AM}}^{+})}{(J_{\mathrm{AM}}^{+})^7}, \frac{1}{(J_{\mathrm{AM}}^{-})^3} - \frac{\alpha(\mu, E, J_{\mathrm{AM}}^{-})}{(J_{\mathrm{AM}}^{-})^7}\right],$$

that is we consider the maximum and minimum instantaneous rotations at J_{AM}^{+} and J_{AM}^{-} (as given by our bounds on the twist map F) as bounds for the endpoints of $\bar{\Omega}$ (and correcting by 2π). By considering the two highest order terms for the coefficients of ΔH, as we did with the twist map, we obtain (highly) good approximations for the α and β functions. Using inequalities (46) and (47), we find $\rho_{\min}^{-}$ and ρ_{ovsp}^{+} and then compute J_{AM}^{-} and J_{AM}^{+} accordingly. This gives us the below table, which we stated in terms of (μ, C) in the introduction.

Table 2. Existence of Aubry-Mather Sets and Their Rotation Numbers.

μ	E	$\bar{\Omega}(\mu, C)$	$\mathcal{J}_{\mathrm{AM}}(\mu, C)$
1/1000	$E_0(2.4, 0)$	$[0.32, 0.37]$	$[2.57, 2.71]$
1/1000	$E_0(2.65, 0)$	$[0.18, 0.30]$	$[2.76, 3.29]$
1/1000	$E_0(3.5, 0)$	$[0.04, 0.14]$	$[3.55, 5.61]$
1/1000	$E_0(5, 0)$	$[0.01, 0.05]$	$[5.03, 10.3]$

As explained in the introduction, $E = E_0(2.65, 0)$, which means $C \approx 5.4$ corresponds to a fictitious Sun-Jupiter-planar Pluto regime. However, we check that at $J_{\mathrm{AM}}^{-} = 2.76$, the eccentricity is roughly 0.27, while Pluto's eccentricity is only 0.2. Thus, the eccentricity of Pluto is a bit too low ($e = 0$ is a degeneracy recall) with respect to our numerics. On the other hand, observe that our entire numerics scheme is based upon placing crude upper bounds everywhere; thus, it is possible that a more refined numerics scheme could attain better results.

As a whole, however, we see that $\mathcal{J}_{\mathrm{AM}}$ is a significant fraction of $\mathcal{J}_{\mathrm{twist}}$, even for E as low as $E_0(2.4, 0)$. From $\mathcal{J}_{\mathrm{AM}}^{+} = 2.71$ in this case, we see that we even get periodic and quasiperiodic motions in PCR3BP (with $\mu = 1/1000$) when the massless body is only at a distance $2.71^2(1 - e(1/1000, E_0(2.4, 0), 2.71)) \approx 4$. Such motions have inertial rotation number lying in $[0.32, 0.37]$.

Finally, observe that for larger E, J_{AM}^{-} can get closer to J_{twist}^{-}, as we see from the twist intervals in Table 1, since estimates become better. Thus, we can get closer to $e = 0$.

That we can obtain the above results for a massless particle considerably close to the primaries is due to the fact Aubry-Mather theory works for Hamiltonian systems that need not be near-integrable. In contrast, KAM theory, does require near-integrability. To conclude, we will apply a KAM result to prove the existence of quasiperiodic motions and invariant curves in PCR3BP. However, our estimates suggest that the applicability of our KAM result, and others similar to it, require a much less practical range of μ and C.

7 A KAM Result

In this section, we describe how to prove Theorem KAM from the introduction or, in other words, how to apply KAM theory and prove existence of KAM quasiperiodic solutions for PCR3CP in the outer region. Direct application of the standard KAM theorem for twist maps (see e.g. [SM] §32-33, [LM] §1-7) fails in a view of degeneracies of PCR3BP. Indeed, for small J, the effect of interaction of massless body with light primary destroys integrability and, therefore, invariant curves. For large J, as (3) shows, the twist is of order $\approx J^{-3}$ and the rotation number of possible invariant curve has to be close to J^{-3}, which tends to zero as $J \to \infty$. This is the reason we need to apply KAM theorem with a small twist parameter (see e.g. [SM] §34). A similar problem arises in the proof of stability of non-degenerate elliptic points [SM,C]. The following theorem is essentially contained in [SM] §34.

Theorem 7.1 *Let $F : (\theta, J) \to (\theta', J')$ defined for $\theta \in \mathbb{T}$, $J \in [a, b]$, $0 \le \varepsilon \le \varepsilon_0$ as follows:*

$$F : \begin{pmatrix} \theta \\ J \end{pmatrix} \to \begin{pmatrix} \theta + \varepsilon\, l(J) + \varepsilon^2\, \mathcal{F}(\theta, J, \varepsilon) \\ J + \varepsilon^2\, \mathcal{G}(\theta, J, \varepsilon) \end{pmatrix},$$

where l is a real analytic function of J, and $\mathcal{F}, \mathcal{G}$ are real analytic functions of θ, J, ε, and 2π-periodic in θ. Suppose F is an EAPT map and $l'(J) \neq 0$ for each $J \in [a, b]$. Then for a sufficiently small ε_0 there is a closed invariant curve C_ε of F parameterized by

$$\theta = \xi + u(\xi), \quad J = v(\xi),$$

where u and v are real analytic functions of $\xi \in \mathbb{R}$. The functions u and v remain small for all ξ. Moreover, under the parametrization $\Phi(\xi) = (\xi + u(\xi), v(\xi))$ of the invariant curve we have

$$F(\Phi(\xi)) = \Phi(\xi + \omega) \quad \text{for all} \quad \xi \in \mathbb{R}.$$

Fix μ, E large, $0 < \nu < 1$, and J_0 large so that $[a, b] := [J_0 - 1, J_0 + 1] \subset \mathcal{J}_\nu \subset \mathcal{J}_{\text{twist}}(\mu, E)$. Let $\varepsilon = J_0^{-3}$, $l(J) = \left(\frac{J_0}{J}\right)^3$. Consider the Poincaré return map F defined in (13) restricted to $[a, b]$. Define $\mathcal{F}(\theta, J, \varepsilon) = \varepsilon^{-2} f(\theta, J)$, $\mathcal{G}(\theta, J, \varepsilon) = \varepsilon^{-2} g(\theta, J)$, where f and g are the real analytic functions which enter in the definition of the EAPT map F. Then since $e \le \nu$, we have $\mathcal{F}(\theta, J, \varepsilon)$ and $\mathcal{G}(\theta, J, \varepsilon)$ are uniformly bounded over all large E. This is because by (35)

$$\begin{aligned} |\mathcal{F}(\theta, J)| &\le \frac{J_0^6}{J^7} \cdot 2\pi\alpha(\mu, E, J) \\ |\mathcal{G}(\theta, J)| &\le \frac{J_0^6}{J^6} \cdot 2\pi\beta(\mu, E, J), \end{aligned}$$

and α and β are uniformly bounded in E when $e \le \nu$ is bounded away from 1. So for E large enough and $J_0 \in \mathcal{J}_\nu$ large enough, the theorem applies. ■

A More Upper Bounds & Error Analysis

In this section, we place upper bounds on the estimates (24) and (25). They give us the upper bounds on the derivatives of ΔH that we use to numerically verify the twist condition (see Section 5.1). Moreover, they are also to demonstrate that the approximations we made in our numerics scheme are negligible.

We begin by bounding the first derivatives of ΔH. This will give us estimates on the size of b, τ, and L. This amounts to finding bounds on the error made in approximating powers of d_1 and d_2 with finite Taylor expansions, as we did in Section 4.2. To do this, we use the Lagrange remainder formula:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \ldots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x), \qquad R_n(x) = \int_0^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}dt.$$

To allow for a more uniform notation, define

$$\mu_i = \begin{cases} \mu & i = 1 \\ 1 - \mu & i = 2, \end{cases}$$

where i is defined modulo two. Let $d_i := r(1+\delta_i)^{1/2}$, $i = 1, 2$, where

$$\delta_i \quad = \quad \frac{2(-1)^i \mu_{i+1}\cos\varphi}{r} + \frac{\mu_{i+1}^2}{r^2}$$

is bounded by

$$\delta_i^- := -\frac{2\mu_{i+1}}{r} + \frac{\mu_{i+1}^2}{r^2} < \delta_i < \frac{2\mu_{i+1}}{r} + \frac{\mu_{i+1}^2}{r^2} =: \delta_i^+.$$

Let $O_k(X)$ denote the terms of order r^m of X, $m \le -k$. We write a function $f = f(r, \varphi)$ as $f(r)$ or $f(\varphi)$ when we want to consider φ or r as being fixed, respectively.

It follows from (24) that the terms appearing in $\frac{\partial \Delta H}{\partial u}$ are given by

$$O\left(\frac{\mu}{r^4}\right)\frac{\partial r}{\partial u} \quad = \quad \sum_{i=1}^{2} O_4\left(\frac{\mu_i}{d_i^2}\right)\left(\frac{\partial r}{\partial u} + O_2\left(\frac{\partial d_i(r)}{\partial u}\right)\right), \tag{50}$$

$$O\left(\frac{\mu}{r^3}\right)\frac{\partial \varphi}{\partial u} \quad = \quad \sum_{i=1}^{2} O_3\left(\frac{\mu_i}{d_i^2}\right)\left(\frac{\partial \varphi}{\partial u} + O_1\left(\frac{\partial d_i(\varphi)}{\partial u}\right)\right), \tag{51}$$

since, as we observed Section 4.2, there is cancellation among lower order terms.

We have the below bounds (these are bounds on absolute value, although we neglect writing absolute value signs for brevity):

$$\begin{aligned}
O_4\left(\frac{1}{d_i^2}\right) &= O_4\left(\frac{1}{r^2(1+\delta_i)}\right) \\
&= O_4\left(\frac{1}{r^2}\left(1 - \delta_1 + \frac{\delta_i^2}{1+\delta_i}\right)\right) \\
&= O_4\left(\frac{1}{r^2}\left(\frac{\mu_{i+1}^2}{r^2} + \frac{\delta_i^2}{1+\delta_i}\right)\right) \\
&\le \frac{1}{r^4}\left(\mu_{i+1}^2 + \frac{(r\delta_i^+)^2}{1+\delta_i^-}\right).
\end{aligned} \tag{52}$$

$$\begin{aligned}
O_2\left(\frac{\partial d_i(r)}{\partial u}\right) &= O_2\left(\frac{\partial}{\partial u} r\left(1 + \frac{1}{2}\delta_i + R_1(\delta_i)\right)\right) \\
&= O_2\left(\frac{\partial r}{\partial u}\left(\frac{1}{2}\delta_i + R_1(\delta_i)\right) + r\left(\frac{1}{2}\frac{\partial \delta_i}{\partial u} + \frac{\partial R_1}{\partial u}(\delta_i)\right)\right) \\
&= O_2\left(\frac{\partial r}{\partial u}\left(\frac{1}{2}\frac{\mu_{i+1}^2}{r^2} - r\frac{1}{2}\cdot\frac{2\mu_{i+1}^2}{r^3}\right)\right) + O_2\left(\frac{\partial r}{\partial u}\int_0^{\delta_i}\left[\frac{d^2}{dt^2}(1+t)^{1/2}\right]\frac{(\delta_i - t)^2}{2!}dt\right) + \\
&\quad + O_2\left(r\int_0^{\delta_i}\left[\frac{d^2}{dt^2}(1+t)^{1/2}\right](\delta_i - t)\frac{\partial \delta_i}{\partial u}dt\right) \\
&\le \frac{1}{r^2}\left(\frac{\mu_{i+1}^2}{2} + \frac{(1+\delta_i^-)^{-3/2}(r\delta_i^+)^2}{8}\right. \\
&\quad \left. + \frac{(1+\delta_i^-)^{-3/2} r\delta_i^+}{4}\cdot\left(2\mu_{i+1} + \frac{\mu_{i+1}^2}{r}\right)\right)\left\|\frac{\partial r}{\partial u}\right\|^*.
\end{aligned} \tag{53}$$

$$\begin{aligned}
O_3\left(\frac{1}{d_i^2}\right) &= O_3\left(\frac{1}{r^2}\left(1 - \delta_i + \frac{\delta_i^2}{1+\delta_i}\right)\right) \\
&\le \frac{1}{r^3}\left(2\mu_{i+1} + \frac{\mu_{i+1}^2}{r} + \frac{r\delta_i^{+2}}{1+\delta_i^-}\right).
\end{aligned} \tag{54}$$

$$
\begin{aligned}
O_1\left(\frac{\partial d_i(\varphi)}{\partial u}\right) &= O_1\left(r\left(\frac{1}{2}\frac{\partial \delta_i}{\partial u} + \frac{\partial R_1}{\partial u}(\delta_i)\right)\right)\\
&= O_1\left(0 + r\int_0^{\delta_i}\left[\frac{d^2}{dt^2}(1+t)^{1/2}\right](\delta_i - t)\frac{\partial \delta_i}{\partial u}dt\right)\\
&\leq \frac{1}{r}\left(\frac{\mu_{i+1}(1+\delta_i^-)^{-3/2} r\delta_i^+}{2}\right)\left\|\frac{\partial\varphi}{\partial u}\right\|^*. \qquad (55)
\end{aligned}
$$

To obtain explicit upper bounds in terms of E and J, one substitutes the lower bound $r = J^2(1-e)$ into the leading power the coefficients and then substitutes the bounds on $\left\|\frac{\partial r}{\partial u}\right\|^*$, $\left\|\frac{\partial\varphi}{\partial u}\right\|^*$ from Section 4. For $u = J$, however, one can improve matters by observing that since $\frac{\partial r}{\partial J} = 2J(1-e\cos\xi) - J^2\left(\frac{\cos\xi - e}{1-e\cos\xi}\right)\frac{\partial e}{\partial J}$, one can cancel the factor of $(1-e\cos\xi)$ appearing in the first term, with one of the factors of $(1-e\cos\xi)$ appearing in $1/r$.

Altogether, (24), (50), (51), and the above upper bounds give us sharp upper bounds on the first derivatives of ΔH in terms of E and J (and μ), which we use to verify numerically the twist condition.

To bound the second derivatives of ΔH, we see that (25) was obtained by differentiating (24). This gives us the following formula for the terms appearing in (24):

$$
\begin{aligned}
O\left(\frac{\mu}{r^5}\right)\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} &= \frac{\partial}{\partial v}\left[\sum_{i=1}^2 O_4\left(\frac{\mu_i}{d_i^2}\right)\left(1 + \left(\frac{\partial r}{\partial u}\right)^{-1} O_2\left(\frac{\partial d_i(r)}{\partial u}\right)\right)\right](r)\frac{\partial r}{\partial u} \qquad (56)\\
O\left(\frac{\mu}{r^4}\right)\left(\frac{\partial r}{\partial u}\frac{\partial\varphi}{\partial v} + \frac{\partial r}{\partial v}\frac{\partial\varphi}{\partial u}\right) &= \frac{\partial}{\partial v}\left[\sum_{i=1}^2 O_4\left(\frac{\mu_i}{d_i^2}\right)\left(1 + \left(\frac{\partial r}{\partial u}\right)^{-1} O_2\left(\frac{\partial d_i(r)}{\partial u}\right)\right)\right](\varphi)\frac{\partial r}{\partial u}\\
&\quad + \frac{\partial}{\partial v}\left[\sum_{i=1}^2 O_3\left(\frac{\mu_i}{d_i^2}\right)\left(1 + \left(\frac{\partial \varphi}{\partial u}\right)^{-1} O_1\left(\frac{\partial d_i(\varphi)}{\partial v}\right)\right)\right](r)\frac{\partial \varphi}{\partial u} \qquad (57)\\
O\left(\frac{\mu}{r^3}\right)\frac{\partial\varphi}{\partial v}\frac{\partial\varphi}{\partial u} &= \frac{\partial}{\partial v}\left[\sum_{i=1}^2 O_3\left(\frac{\mu_i}{d_i^2}\right)\left(1 + \left(\frac{\partial \varphi}{\partial u}\right)^{-1} O_1\left(\frac{\partial d_i(\varphi)}{\partial u}\right)\right)\right](\varphi)\frac{\partial \varphi}{\partial u}. \qquad (58)
\end{aligned}
$$

The previous computations allow one to compute the above coefficients and bound them as before. The corresponding bounds on the second derivatives of ΔH are what we use in verifying the twist condition.

To get a feel for actual numbers, we have for $\mu = 1/1000$ and $r = 3 \approx E_0(2.4, 0)^2/2$, one finds that

$$
\begin{aligned}
\left\|\frac{\partial \Delta H}{\partial u}\right\|^* &\leq \frac{0.1}{J^6(1-e)^{9/2}}, \qquad u \in \{\theta, \theta_r\}\\
\left\|\frac{\partial \Delta H}{\partial J}\right\|^* &\leq \frac{0.01}{J^7(1-e)^3} + \frac{0.01E}{J^8(1-e)^4}\\
\left\|\frac{\partial \Delta H}{\partial J_r}\right\|^* &\leq \frac{0.01E}{J^8(1-e)^4}
\end{aligned}
$$

Here, the estimates from Section 4 gave us the dependence on J and e occurring through bounding the derivatives of the polar variables. From this, we automatically have that

Proposition A. *$J_r = J_r(E, \theta, \theta', J)$ is a well-defined function.*

Proof We have

$$\frac{\partial H}{\partial J_r} = 1 + \frac{\partial \Delta H}{\partial J_r}.$$

Since the minimum radial distance for a given energy E is bounded by $(E-6\mu)^2/2$ (see Section 2.1), we have from the above estimate that

$$\frac{\partial H}{\partial J_r} \geq 1 - \frac{0.01E}{J^8(1-e)^4} > 1 - \frac{1}{E^7} > 0,$$

since $E > 2$. Thus, if $H(\theta, \theta_r, J, J_r) = H(\theta, \theta_r, J, J_r')$, we have $J_r = J_r'$ by monotonicity of H in J_r. This shows that $J_r = J_r(E, \theta, \theta', J)$ is a well-defined function. ■

Moreover, from the above explicit bounds, we can verify that for $E = E_0(2.4, 0)$, we have $b \lessapprox 10^{-3}$ and $|\tau - 2\pi| \lessapprox 10^{-4}$ are indeed small. Other choices of E show that they remain small and negligible, as we had assumed in Section 5.1.

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