LONG TIME BEHAVIOUR OF PERIODIC STOCHASTIC FLOWS.

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Abstract. We consider the evolution of a set carried by a space periodic incompressible stochastic flow in a Euclidean space. We report on three main results obtained in [8, 9, 10] concerning long time behaviour for a typical realization of the stochastic flow. First, at time t most of the particles are at a distance of order $\sqrt{t}$ away from the origin. Moreover, we prove a Central Limit Theorem for the evolution of a measure carried by the flow, which holds for almost every realization of the flow. Second, we show the existence of a zero measure full Hausdorff dimension set of points, which escape to infinity at a linear rate. Third, in the 2-dimensional case, we study the set of points visited by the original set by time t. Such a set, when scaled down by the factor of t, has a limiting non random shape.

1. INTRODUCTION.

We study of the long-time behavior of a passive substance (or scalar) carried by a stochastic flow. Motivation comes from applied problems in statistical turbulence and oceanography, Monin & Yaglom [18], Yaglom [21], Davis [7], Isichenko [11], and Carmona & Cerou [3]. The questions we discuss here are also related to the physical basis of the Kolmogorov model for turbulence, Molchanov [17].

1.1. **Local properties of the dynamics.**

The key element of our analysis is presence of *nonzero Lyapunov exponents.*

The physical mechanism of turbulence is still not completely understood. It was suggested in [19] that the appearance of turbulence could be similar to the appearance of chaotic behavior in finite-dimensional deterministic systems. Compared to other cases, the mechanism responsible for stochasticity in deterministic dynamical systems with nonzero Lyapunov exponents is relatively well understood. It is caused by a sensitive dependence on initial conditions, that is, by exponential

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divergence of nearby trajectories. It is believed that a similar mechanism can be found in many other situations, but mathematical results are scarce. Here we describe a setting where analysis similar to the deterministic dynamical systems with nonzero Lyapunov exponents can be used. We consider a flow of diffeomorphisms on an n-dimensional torus $\mathbb{T}^n$ (or more generally on a compact manifold), generated by solutions of stochastic differential equations driven by a finite-dimensional Brownian motion.

$$dx_t = \sum_{k=1}^{d} X_k(x_t) \circ d\theta_k(t) + X_0(x_t)dt \tag{1}$$

where $X_0, X_1, \dots, X_d$ are C^∞ smooth divergence free space periodic vector fields on $\mathbb{R}^n$ with period one and $\vec{\theta}(t) = (\theta_1(t), \dots, \theta_d(t))$ is a standard $\mathbb{R}^d$-valued Brownian motion.

We show that *the presence of non-zero exponents combined with certain non-degeneracy conditions* (amounting roughly speaking to the assumption that the noise can move the orbit in any direction) *implies almost surely chaotic behavior* in the following sense:

- Exponential, in time, decay of correlations between the trajectories with different initial data.
- Equidistribution of images of submanifolds.
- Central Limit Theorem, with respect to the measure on a "rich enough" subset, which holds for almost every fixed realization of the underlying Brownian motion.

In order to illustrate the last point, let us consider a periodic flow on $\mathbb{R}^n$, and let ν be a Lebesgue probability measure concentrated on an open subset. As a motivating example one may think of an oil spot on the surface of the ocean. The ultimate goal could be to remove the oil or at least to prevent it from touching certain places on the surface of ocean. Thus, we wish to predict the properties and the probability laws governing the dynamics of the spot in time.

Let ν_t be the measure on $\mathbb{R}^n$ induced from ν by time t map of the flow. We shall show that almost surely ν_t is asymptotically equivalent to a Gaussian measure with variance of order t. In other words, for a sufficiently large positive R for large time 99 percent of the oil spot is contained in the ball of radius $R\sqrt{t}$. See Theorem 1 below for an exact statement.

1.2. **Global properties of the dynamics.** Global and local properties are intimately related.

• Ballistic points.

As has been demonstrated for a large class of stochastic flows with zero mean, under some mixing conditions on the flow, the displacement of a single particle is typically of order $\sqrt{t}$ for large t. On the other hand, it has been shown in the work of Cranston, Scheutzow, Steinsaltz, and Lisei [5], [6], [13], [20] that in any open set there are points which escape to infinity at a linear rate. We sharpen these results for the stochastic flow (1) and show that the linear escape points form a set of zero measure and full Hausdorff dimension (see Theorem 4 below).

$$\mathbf{L}_\theta = \left\{ x \in \mathbb{R}^n : \liminf_{t\to\infty} \frac{||x_t||}{t} > 0 \right\} \tag{2}$$

for a.e. realization $\{\theta(t)\}_{t>0}$ we have $HD(\mathbf{L}_\theta) = n$.

• Shape of poisoned area.

Consider the planar case $n = 2$. Denote the original set by $\Omega \subset \mathbb{R}^2$. The evolution of the set under the action of the flow will be denoted by Ω_t.

We study the set of "poisoned" points, that is those visited by the image of Ω before time t

$$\mathcal{W}_t(\Omega) = \bigcup_{s\le t} \Omega_s \ .$$

As shown in [5] and [6], the diameter of this set grows linearly in time almost surely. It turns out that $\mathcal{W}_t(\Omega)$, if scaled down by t, converges almost surely to a nonrandom convex set $\mathcal{B} \subset \mathbb{R}^2$ independent of Ω

$$\frac{\mathcal{W}_t(\Omega)}{t} \to \mathcal{B}. \tag{3}$$

Now we give the precise statements.

2. Statement of the results

Consider a stochastic flow of diffeomorphisms on $\mathbb{R}^n$ generated by a finite-dimensional Brownian motion (1). We impose several assumptions on the vector fields $X_0, X_1, \dots, X_d$, which are stated in the next section. All those, except the assumption of zero drift, are *nondegeneracy assumptions* and are satisfied for a generic set of vector fields $X_0, X_1, \dots, X_d$. The main results are the following:

• Central Limit Theorems:

Let ν be a probability measure on $\mathbb{R}^n$, which has compact support, such that for some positive p it has finite p-energy

$$I_p(\nu) = \iint \frac{d\nu(x)d\nu(y)}{d^p(x,y)} < \infty. \tag{4}$$

In particular, this means that the Hausdorff dimension of the support of ν is positive (see [14], sect. 8).

Theorem 1. [8] *Let ν be a probability measure with finite p-energy for some $p > 0$ and with compact support, and let assumptions (A)–(D) below be satisfied. Let x_t be the solution of (1) with the initial measure ν. Then for almost every realization of the Brownian motion the distribution of $\frac{x_t}{\sqrt{t}}$ induced by ν converges weakly as $t \to \infty$ to a Gaussian measure on $\mathbb{R}^n$ with zero mean and some variance D.*

We also prove Central Limit Theorems for a 2-point motion:

Theorem 2. [8] *Let conditions (A)–(C) be satisfied. Let x_t^1 and x_t^2 be the solutions of (1) with different initial data. Then for some value of the drift v the vector $\frac{1}{\sqrt{t}}(x_t^1 - vt, x_t^2 - vt)$ converges as $t \to \infty$ to a Gaussian random vector with zero mean.*

and an m-point motion for any positive integer m:

Theorem 3. [8] *Let the conditions $(A), (B_m)$, and (C) be satisfied. Let $x_t^1, \ldots, x_t^m$ be the solutions of (1) with pair-wise different initial data. Then for some value of the drift v the vector $\frac{1}{\sqrt{t}}(x_t^1 - vt, ..., x_t^m - vt)$ converges as $t \to \infty$ to a Gaussian random vector with zero mean.*

Actually all these results can be stated more generally in terms of additive functionals of one, two, and m-point motion on a compact manifold (see [8]). In the proof of the above results we essentially use the work of Baxendale–Stroock [1, 2]. Baxendale also proposed an application of results of Meyn–Tweedie [15], which helped us to simplify the proofs.

• Ballistic points

Cranston–Scheutzow–Steinsaltz [5, 20] proved that the set of ballistic points $\mathbf{L}_\theta$ is uncountable. Here is an extension of their result.

Theorem 4. [9] *Let assumptions (A)-(D) be satisfied. For almost every realization of the Brownian motion $\{\theta(t)\}_{t\geq 0}$ we have that points of the flow (1) with linear escape to infinity $\mathbf{L}_\theta$ form a dense set of full Hausdorff dimension $HD(\mathbf{L}_\theta) = n$.*

• Shape of poisoned area

Theorem 5. [10] *Let the original set $\Omega \subset \mathbb{R}^2$ be bounded and contain a continuous curve with positive diameter and let assumptions (A)-(E) below be satisfied. Then there is a compact convex non random set $\mathcal{B}$, independent of Ω, such that for any $\varepsilon > 0$ almost surely*

$$(1-\varepsilon)t\mathcal{B} \subset \mathcal{W}_t(\Omega) \subset (1+\varepsilon)t\mathcal{B} \tag{5}$$

for all sufficiently large t.

2.1. **Nondegeneracy assumptions.** In this section we formulate a set of assumptions on the vector fields $X_0, X_1, \ldots, X_d$. Recall that $X_0, X_1, \ldots, X_d$ are assumed to be periodic and divergence free. We shall assume that the period for all of the vector fields is equal to one.

(A) (*Strong Hörmander Condition for x_t*) For all $x \in \mathbb{R}^n$ we have

$$\mathrm{Lie}(X_1, \ldots, X_d)(x) = \mathbb{R}^2,$$

where $\mathrm{Lie}(X_1, \ldots, X_d)(x)$ is linear span of all possible Lie brackets of all orders formed out of $X_1, \ldots, X_d$ at x. Strong Hörmander Condition for x_t basically implies that probability that a point after time one gets into an open set is positive.

Denote the diagonal in $\mathbb{T}^n \times \mathbb{T}^n$ by

$$\Delta = \{(x^1, x^2) \in \mathbb{R}^n \times \mathbb{R}^n : \ x^1 = x^2 \pmod 1\}.$$

(B) (*Strong Hörmander Condition for the two–point motion*) The generator of the two–point motion $\{(x_t^1, x_t^2) : \ t > 0\}$ is nondegenerate away from the diagonal Δ, meaning that the Lie brackets made out of $(X_1(x^1), X_1(x^2)), \ldots, (X_d(x^1), X_d(x^2))$ generate $T_{x^1}\mathbb{T}^n \times T_{x^2}\mathbb{T}^n = \mathbb{R}^n \times \mathbb{R}^n$.

(B_m) (*Strong Hörmander Condition for the m–point motion*) The generator of the m–point motion $\{(x_t^1, \ldots, x_t^m) \ : \ \ t > 0\}$ is nondegenerate away from the generalized diagonal $\Delta^{(k)}(\mathbb{T}^n)$, meaning that Lie brackets made out of
$(X_1(x^1), \ \ldots, \ X_1(x^m)), \ \ldots, \ (X_d(x^1), \ \ldots, \ X_d(x^m))$ generate $T_{x^1}\mathbb{T}^n \times \cdots \times T_{x^m}\mathbb{T}^n = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (m times).

To formulate the next assumption we need additional notations. Let $Dx_t : T_{x_0}\mathbb{R}^n \to T_{x_t}\mathbb{R}^n$ be the linearization of x_t at t. We need the Strong Hörmander Condition for the process $\{(x_t, Dx_t) : \ t > 0\}$. Denote by TX_k the derivative of the vector field X_k thought as the map on $T\mathbb{R}^n$ and by $S\mathbb{R}^n = \{v \in T\mathbb{R}^n : \ |v| = 1\}$ the unit tangent bundle on $\mathbb{R}^n$. If we denote by $\tilde{X}_k(v)$ the projection of $TX_k(v)$ onto $T_vS\mathbb{R}^n$, then the stochastic flow (1) on $\mathbb{R}^n$ induces a stochastic flow on the unit tangent bundle $S\mathbb{R}^n$ defined by the following equation:

$$d\tilde{x}_t = \sum_{k=1}^{d} \tilde{X}_k(\tilde{x}_t) \circ d\theta_k(t) + \tilde{X}_0(\tilde{x}_t)dt.$$

With these notations we have condition

(C) (*Strong Hörmander Condition for* (x_t, Dx_t)) For all $v \in S\mathbb{R}^n$ we have

$$\mathrm{Lie}(\tilde{X}_1, \ldots, \tilde{X}_d)(v) = T_v S\mathbb{R}^n \ .$$

Let $L_{X_k} X_k(x)$ denote the derivative of X_k along X_k at the point x. Notice that $\frac{1}{2}\sum_{k=1}^{d} L_{X_k} X_k + X_0$ is the deterministic component of the stochastic flow (1) rewritten in Ito's form. Below we show that conditions (A)-(C) guarantee that the flow (1) has Lyapunov exponents and at least one of them is positive. We require that the flow has no deterministic drift, which is expressed by the following condition.

(D) (*zero drift*)

$$\int_{\mathbb{T}^n} \left(\frac{1}{2} \sum_{k=1}^{d} L_{X_k} X_k + X_0 \right)(x) dx = 0 \ , \tag{6}$$

We further require that

$$\int_{\mathbb{T}^n} X_k(x) dx = 0 \ , \ k = 1, ..., d \ . \tag{7}$$

The last assumption is concerned with the geometry of the stream lines for one of the vector fields $X_1, ..., X_d$ in the 2-dimensional case. Fix a coordinate system on the 2-torus $\mathbb{T}^2 = \{x = (x_1, x_2) \bmod 1\}$. As the vector fields have zero mean and are divergence free, there are periodic *stream* functions $H_1, ..., H_d$, such that $X_k(x) = (-H'_{x_2}, H'_{x_1})$. We require the following

(E) (*Morse condition on the critical points of* H_1) All of the critical points of H_1 are non-degenerate.

Functions with this property are called *Morse functions.* It is a standard result that a generic function is a Morse function (see e.g. [16]).

2.2. **Nondegeneracy assumptions (A)-(C) imply positive Lyapunov exponents.** For measure-preserving stochastic flows with conditions (C) Lyapunov exponents $\lambda_1, \ldots, \lambda_{\dim M}$ do exist by *multiplicative ergodic theorem for stochastic flows* of diffeomorphisms (see [4], thm. 2.1). Under condition (A) the sum of the Lyapunov exponents is equal to zero. On the other hand, Theorem 6.8 of [1] states that under condition (A) all of the Lyapunov exponents can be equal to zero only if for almost every realization of the flow (1) one of the following two conditions is satisfied

(a) there is a Riemannian metric ρ' on $\mathbb{T}^n$, invariant with respect to the flow (1) or

(b) there is a direction field $v(x)$ on $\mathbb{T}^n$ invariant with respect to the flow (1).

However (a) contradicts condition (B). Indeed, (a) implies that all the Lie brackets of $\{(X_k(x^1),\ X_k(x^2))\}_{k=1}^d$ are tangent to the leaves of the foliation

$$\{(x^1, x^2) \in \mathbb{T}^n \times \mathbb{T}^n :\ \rho'(x^1, x^2) = Const\}$$

and don't form the whole tangent space. On the other hand (b) contradicts condition (C), since (b) implies that all the Lie brackets are tangent to the graph of v.

This positivity of λ_1 is crucial for our approach.

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