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Generic Diffeomorphisms with Superexponential Growth of Number of Periodic Orbits

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Abstract: Let M be a smooth compact manifold of dimension at least 2 and $\mathrm{Diff}^r(M)$ be the space of C^r smooth diffeomorphisms of M. Associate to each diffeomorphism $f \in \mathrm{Diff}^r(M)$ the sequence $P_n(f)$ of the number of isolated periodic points for f of period n. In this paper we exhibit an open set N in the space of diffeomorphisms $\mathrm{Diff}^r(M)$ such for a Baire generic diffeomorphism $f \in N$ the number of periodic points $P_n f$ grows with a period n faster than any following sequence of numbers $\{a_n\}_{n\in Z_+}$ along a subsequence, i.e. $P_n(f) > a_{n_i}$ for some $n_i \to \infty$ with $i \to \infty$. In the cases of surface diffeomorphisms, i.e. $\dim M = 2$, an open set N with a supergrowth of the number of periodic points is a Newhouse domain. A proof of the man result is based on the Gontchenko–Shilnikov–Turaev Theorem [GST]. A complete proof of that theorem is also presented.

1. Introduction

This paper is a continuation of the study we began in [K1]. We announced the results of this paper there. In the introduction, for the convenience of the reader, we repeat the introduction to the second part of [K1].

Let $C^r(M, M)$ be the space of C^r mappings of a compact manifold M into itself with the uniform C^r-topology and $\mathrm{Diff}^r(M)$ be the space of C^r diffeomorphisms of M with the same topology. It is well-known that $\mathrm{Diff}^r(M)$ is an open subset of $C^r(M, M)$. For a map $f \in C^r(M, M)$, consider the number of *isolated* periodic points of period n (i.e. the number of isolated fixed points of f^n)

$$P_n(f) = \#\{ \text{ isolated } \ x \in M : x = f^n(x)\}. \tag{1}$$

In 1965 Artin & Mazur [AM] showed that: *there exists a dense set $\mathcal{D}$ in $C^r(M, M)$ such that for any map $f \in \mathcal{D}$ the number $P_n(f)$ grows at most exponentially with n, i.e. for some number $C > 0$,*

$$P_n(f) \le \exp(Cn) \ \text{ for all } \ n \in \mathbb{Z}_+. \tag{2}$$

Notice that the Artin–Mazur Theorem does not exclude the possibility that a mapping f in $\mathcal{D}$ has a curve of periodic points γ, i.e. $\forall x \in \gamma,\ f^n(x) = x$ for some $n \in \mathbb{Z}_+$, because in this case γ consists of nonisolated periodic points of period n (see the last part of Theorem 2 for this nonisolated case).

Definition 1. *We call a mapping (resp. diffeomorphism) $f \in C^r(M, M)$ (resp. $f \in \mathrm{Diff}^r(M)$) an Artin–Mazur mapping (resp. diffeomorphism) or simply A-M mapping (resp. diffeomorphism) if $P_n(f)$ grows at most exponentially fast, i.e. satisfies (2) for some $C > 0$.*

In what follows we consider *not the whole space $C^r(M, M)$ of mappings of M into itself, but only its open subset* $\mathrm{Diff}^r(M)$. Notice that in [K1] a simple proof of the fact that A-M diffeomorphisms with only hyperbolic periodic points are dense in the space $\mathrm{Diff}^r(M)$ is presented. This fact is an extension of Artin–Mazur theorem, because it gives hyperbolicity of periodic points. This paper is based on the preprint [K2].

In [AM] Artin–Mazur also introduced the *dynamical ζ_f-function* defined by

$$\zeta_f(z) = \exp\left(\sum_{n=1}^{\infty} P_n(f)\frac{z^n}{n}\right).$$

For A-M diffeomorphisms the dynamical ζ_f-function is analytic in some disk centered at zero. It is well-known that the dynamical ζ_f-function of a diffeomorphism f satisfying Axiom A has an analytic continuation to a rational function (e.g. [Ba]).

Recall that a subset of a topological space is called residual if it contains a countable intersection of open dense subsets. We call a residual set a *topologically (Baire) generic* set. See the classical book of Oxtoby [O] on discussion of Baire spaces.

In 1967 Smale [Sm] posed the following question (Problem 4.5, p. 765):

Is the dynamical ζ_f-function rational for a topologically generic set of diffeomorphisms in $\mathrm{Diff}^r(M)$)?

In [Si] it is shown that for the 3-dimensional torus the ζ_f-function is not rational. It turns out that for manifolds of dimension greater than or equal to 2 it is not even analytic in any neighborhood of zero (see Theorem 2 below).

Finally, in 1978 R. Bowen asked the following question in his book [Bo]:

Let $h(f)$ denote the topological entropy of f. *Does*

$$h(f) = \limsup_{n\to\infty} \log P_n(f)/n$$

for a topologically generic set of diffeomorphisms with respect to the C^r topology?

It turns out the two above questions can be answered simultaneously for C^r diffeomorphisms with $2 \le r < \infty$. The second result is the following:

Theorem 1. *Let $2 \le r < \infty$. Then the set of A-M diffeomorphisms is* **not** *topologically generic in the space of C^r diffeomorphisms* $\mathrm{Diff}^r(M)$ *with the uniform C^r topology.*

We have the following consequences:

Corollary 1. *The property of having a convergent $\zeta_f(z)$ function is not topologically C^r-generic, nor is the equation $h(f) = \limsup_{n\to\infty} \log P_n(f)/n$.*

The first part is easy. To prove the second, notice that the topological entropy for any C^r ($r \geq 1$) diffeomorphism f of a compact manifold is always finite (see e.g. [HK]). Define the rate of growth of the number of periodic orbits by $\limsup_{n\to\infty} \log P_n(f)/n$. Then for diffeomorphisms which are not A-M diffeomorphisms, the rate of growth is always equal to infinity.

Since, an Axiom A diffeomorphism is an A-M diffeomorphism, we need to analyze the complement to Axiom A diffeomorphisms in the space of $\mathrm{Diff}^r(M)$. An example of a diffeomorphism with an arbitrarily fast growing number of periodic orbits is given in [RG]. Now we describe a "bad" domain, where the A-M property fails to be topologically generic.

In 1970 Newhouse found a domain in the space of C^r-smooth 2-dimensional diffeomorphisms $\mathrm{Diff}^r(M)$, where diffeomorphisms exhibiting homoclinic tangencies are dense [N]. Moreover, in any neighborhood of a diffeomorphism exhibiting a homoclinic tangency there is a Newhouse domain. Such a domain is called *a Newhouse domain* $\mathcal{N} \subset \mathrm{Diff}^r(M)$. Our third and main result is as follows:

Theorem 2. *Let $2 \leq r < \infty$ and M be a compact 2-dimensional manifold. Let $\mathcal{N} \subset \mathrm{Diff}^r(M)$ be a Newhouse domain. Then for an arbitrary sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ there exists a residual set $\mathcal{R}_a \subset \mathcal{N}$, depending on the sequence $\{a_n\}_{n=1}^{\infty}$, with the property that $f \in \mathcal{R}_a$ implies that*

$$\limsup_{n\to\infty} P_n(f)/a_n = \infty.$$

Moreover, there is a dense set $\mathcal{D}$ in $\mathcal{N}$ such that any diffeomorphism $f \in \mathcal{D}$ has a curve of periodic points.

Let us note that Theorems 1 and 2 follow from a theorem of Gonchenko–Shilnikov–Turaev which will be discussed in Sect. 2.

In a Newhouse domain Newhouse exhibited a residual set of diffeomorphisms with infinitely many distinct sinks [N,Rb], and [PT]. Now it is known as *Newhouse's phenomenon*. In a way Theorem 2 is similar to Newhouse's phenomenon in the sense that for a residual set a "bad" property holds true.

Continuing, Theorem 1 is a corollary of the first part of Theorem 2. To see this in the case $\dim M = 2$ fix the sequence $a_n = n^n$ and denote by $\mathcal{R}_a$ a set from Theorem 2 corresponding to this sequence. Assume that A-M diffeomorphisms form a residual set, then this set must intersect with $\mathcal{R}_a$ which is a contradiction.

In the case $\dim M > 2$ sufficiently simple arguments, based on Sacker-Fenichel's theorem about stability of invariant manifolds see [Sa,Fe,HPS] reduce the problem to 2-dimensional case (see [K1, Sect. 4.2]).

For diffeomorphisms of manifolds $\dim M \geq 3$ existence of Newhouse domains in any neighborhood of a diffeomorphism exhibiting a homoclinic tangency is proven by Romero [Rm].

It seems that based on Newhouse's phenomenon in the space $\mathrm{Diff}^1(M)$ with the C^1-topology, where $\dim M \geq 3$, found by Bonnati & Diaz [BD] one can extend Theorems 1 and 2 to the case $r = 1$ and $\dim M \geq 3$. The problem with this straightforward generalization is that the proof of the Gonchenko–Shilnikov–Turaev (GST) theorem is essentially two-dimensional. To generalize the GST theorem to the three-dimensional case one needs either to find an invariant two-dimensional surface and use the two-dimensional proof or find another proof. In personal communications, Lorenzo Diaz has shown to the author that an invariant two-dimensional surface can be constructed using the method from [BD]. However, this extension is not straightforward and might appear separately from this paper.

Analogs of Theorems 1 and 2 can be formulated for the case of vector fields on a compact manifold of dimension at least 3. Reduction from the case of diffeomorphisms to the case of vector fields can be done using the standard suspension of a vector field over a diffeomorphism [PM].

From now on we fix r such that $2 \leq r < \infty$ and keep r unchanged in what follows in the paper.

1.1. Newhouse phenomenon and Palis conjecture. Newhouse showed that a Newhouse domain exists under the following hypothesis:

Let a diffeomorphism $f \in \mathrm{Diff}^r(M)$ have a saddle periodic orbit p. Suppose stable $W^s(p)$ and unstable $W^u(p)$ manifolds of p have a quadratic tangency. Such a diffeomorphism f is called *a diffeomorphism exhibiting a homoclinic tangency*. Then arbitrarily C^r-close to f in $\mathrm{Diff}^r(M)$ there exists *a Newhouse domain*. In particular, it means that by a small C^r-perturbation of a diffeomorphism f with a homoclinic tangency one can generate arbitrarily quick growth of the number of periodic orbits.

On this account we would also like to mention the following conjecture, which is due to Palis [PT], about the space of diffeomorphisms of 2-dimensional manifolds:

Conjecture. *If* $\dim M = 2$, *then every diffeomorphism* $f \in \mathrm{Diff}^r(M)$ *can be approximated by a diffeomorphism which is either hyperbolic or exhibits a homoclinic tangency.*

This conjecture is proven for approximations in the C^1 topology by Pujals and Sambarino [PS]. If this conjecture is true, then in the complement to the set of hyperbolic diffeomorphisms those diffeomorphisms with arbitrarily quick growth of the number of periodic orbits form a topologically generic set.

Unfolding of homoclinic tangencies is far from being understood. In [GST] the authors describe the following important result: *there does not exist a finite number of parameters to describe all bifurcations occurring next to a homoclinic tangency* (see Sect. 2, Corollary 2 for details). This implies that the complete description of bifurcations of diffeomorphisms with a homoclinic tangency is impossible.

This paper is organized as follows. In Sect. 2.1 we state the Gonchenko–Shilnikov–Turaev result and give a proof of it in Sects. 2.2–2.5. Section 2.6 is devoted to the proof of Theorem 2 in the case $\dim M = 2$. For reduction from Theorem 2 in the case $\dim M = 2$ to Theorem 2 to the general case $\dim M \geq 2$ see [K1] or [K2].

From now on we consider the space of C^r-smooth diffeomorphisms of a 2-dimensional compact manifold M.

2. Degenerate Periodic Orbits in a Newhouse Domain and the Gonchenko–Shilnikov–Turaev Theorem [GST]

Assume that a C^r diffeomorphism f exhibits a homoclinic tangency. By the Newhouse theorem [N], in each C^r neighborhood of a diffeomorphism f exhibiting a homoclinic tangency there exists a Newhouse domain.

Let us define a degenerate periodic point of order k or a k-degenerate periodic point. Sometimes, it is also called a *saddlenode periodic orbit of multiplicity* $k+1$.

Definition 2. *Let f be a C^s diffeomorphism of a 2-dimensional manifold having a periodic orbit p of period m. A periodic point p is called k-degenerate, where $k < s$, if the linear part of f^m at point p has a multiplier $\nu = 1$ while the other multiplier is different in absolute value from the unit and a restriction of f to the central manifold in some coordinate system can be written in the form*

$$x \mapsto x + l_{k+1}x^{k+1} + o(x^{k+1}). \tag{3}$$

Let $s > r$. Then C^s diffeomorphisms are dense in the space $\mathrm{Diff}^r(M)$ and, therefore, in any Newhouse domain $\mathcal{N} \subset \mathrm{Diff}^r(M)$ (see e.g. [PM]).

Theorem 3 (*Theorem 4***,** [GST]**).** *For any positive integers $s > k \geq r$ the set of C^s diffeomorphisms having a k-degenerate periodic orbit is dense in a Newhouse domain $\mathcal{N} \subset \mathrm{Diff}^r(M)$.*

This theorem and Newhouse's theorem imply the following important result:

Corollary 2 ([GST]**).** *Let $f \in \mathrm{Diff}^r(M)$ be a diffeomorphism exhibiting a homoclinic tangency. There is no finite number s such that a generic s-parameter family $\{f_\varepsilon\}$ unfolding a diffeomorphism $f_0 = f$ is a versal family of f_0 meaning that the family $\{f_\varepsilon\}$ describes all possible bifurcations occurring next to f. Indeed, to describe all possible bifurcations of a k-degenerate periodic orbit one needs at least $k + 1$ parameters and k can be arbitrary large.*

Once Theorem 3 is proved the proof of Theorem 2 can be completed by inductive application of the following idea. Let f be a C^s diffeomorphism from a Newhouse domain $\mathcal{N} \subset \mathrm{Diff}^r(M)$ with a k-degenerate periodic orbit p of period, say n, of f for $s > k \geq r$, then p is *flat periodic point* along the central manifold with respect to the C^r topology, namely, by a C^r-perturbation one can make the restriction to the central manifold be *the identical map*. It allows us either to create a *curve of periodic orbits* or *split p* into *any ahead given number of* hyperbolic periodic orbits of the same period (or double the period of p) by a small C^r-perturbation. Since, created periodic orbits are hyperbolic they persist under perturbations. Moreover, after a perturbation we are still in a Newhouse domain one can iterate this procedure of creating a k-degenerate periodic orbits and splitting them without destroying what was done in previous stages (see Sect. 2.7).

In what follows we need a few notions related to a saddle periodic point. These definitions will be needed in the proof of Theorem 3.

Definition 3. *Let f be a C^s diffeomorphism of a 2-dimensional manifold M and let p be a saddle periodic point of period m, namely, $f^m(p) = p$ with eigenvalues λ and μ, $\lambda < 1 < \mu$. The saddle exponent of p is the number $\rho(p, f) = \frac{-\log\lambda}{\log\mu}$. We call p a ρ-shrinking saddle, where $\rho = \rho(p, f)$. If ρ is greater than some r, then p is also called at least r-shrinking.*

A saddle p is called nonresonant if for any pair of positive integers n and m such that the number $\lambda^n\mu^m$ is different from 1.

2.1. A scheme of a proof of Theorem 3. Theorem 3 is stated in ([GST], Thm. 4). A proof of this theorem is outlined there. Proof of several technical statements[1] is omitted there.

[1] Lemmas 1 and 2 in [GST] which correspond to Lemma 1 and Proposition 5 of the present paper respectively

We present a rigorous proof which essentially uses ideas given in [GST]. After the paper was submitted the author get a preprint from Gonchenko–Shilnikov–Turaev [GST2], where they describe a detail proof. In what follows a C^r-perturbation means a small C^r-perturbation. The proof of Theorem 3 consists of four steps.

The first step. From the existence of a homoclinic tangency of a dissipative saddle, we deduce the existence (after a C^r-perturbation) of a homoclinic tangency of an at least k-shrinking saddle, $k > r$.

The second step. From the existence of a homoclinic tangency of an at least k-shrinking saddle, we create a k-floor tower (defined in Sect. 2.4) after a C^r-perturbation (see Fig. 3 for $k = 3$).

The third step. From the existence of a k-floor tower, we show that a C^r-perturbation can make a k^{th} order homoclinic tangency.

The fourth step. From the existence of a k^{th} order homoclinic tangency we construct by a C^r-perturbation a k^{th} order degenerate periodic orbit of an arbitrarily high period.

Notice that the way we construct a k-tower is slightly different from the one in [GST].

The proof of Theorem 3 is given in Sects. 2.2–2.5 according to the following plan. In Sect. 2.2 we present some basic properties of a return map in a neighborhood of a quadratic homoclinic tangency. In Sect. 2.3 we realize the first step (Corollary 4) and calculate limits for return maps in a neighborhood of a k^{th} order homoclinic tangency, where $k \geq 2$. The second and the third steps are done in Sects. 2.4 and 2.5 respectively. The last fourth step consists in application of Corollary 3 proven in Sect. 2.3.

2.2. Basic properties of a return map in a neighborhood of a homoclinic tangency. Fix a positive integer $r \geq 2$. Consider a C^∞ smooth diffeomorphism $f : M^2 \to M^2$ with a saddle fixed point p, namely, $f(p) = p$ with the eigenvalues λ and μ. Assume the saddle p is dissipative and nonresonant. We can obtain all conditions by applying a C^r-perturbation (for $f \in C^r$, or/and $\lambda\mu = 1$, or/and by inverting f) if necessary. Then by the standard fact from the theory of normal forms e.g. [IY] the map f is C^r linearizable in a neighborhood U of p

$$f : (x, y) \mapsto (\lambda x,\ \mu y), \tag{4}$$

where $\lambda < 1 < \mu$ and $\lambda\mu < 1$. The larger is r, the smaller is the neighborhood U, where a C^r-normal form applicable.

Assume that the stable $W^s(p)$ and unstable manifold $W^u(p)$ of p in normal coordinates have a point of quadratic tangency q with coordinates $(1, 0)$ and for some N we have that $f^{-N}(q) = \tilde{q}$ has coordinates $(0, 1)$ (see Fig. 1). Assume also that in a neighborhood of the homoclinic point q the unstable manifold $W^u(p)$ lies in the upper half plane $\{y \geq 0\}$ and the directions of $W^u(p)$ and $W^s(p)$ at the point of tangency q are the same (see Fig. 1). Diffeomorphisms with homoclinic tangency are dense in a Newhouse domain by definition. Let $f \in \text{Diff}^r(M^2)$ have a homoclinic tangency. We show in two steps that f can be approximated by a diffeomorphism with the homoclinic tangency described in Fig.1. It what follows it is easier. First, it is shown (see e.g. [PT, Thm. 1 Sect. 3.4]) that for any $\mu \in [0, 2]$ there is a sequence of boxes in M converging to a point of homoclinic tangency and an appropriate sequence of rescalings such that the corresponding return map of a box into itself converges to the map $(x, y) \to (y, y^2 + \mu)$. Second, in Sect. 6.3, Prop. 3 of [PT] it is shown that by a small perturbation of such

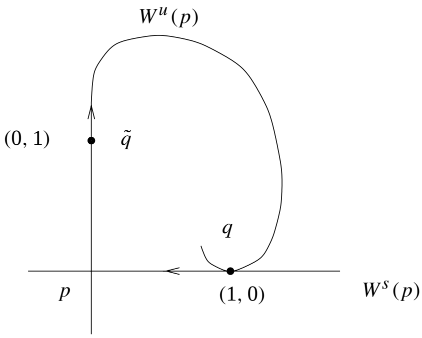

Fig. 1. Homoclinic tangency

a map into a 2-dimensional map one can create a homoclinic tangency of the required type (see Fig. 6.5 there).

Rescale sufficiently small neighborhoods U and $\tilde{U}$ of q and $\tilde{q}$ respectively. Denote by $W^u_{\rm loc}(p)$ the first connected component of the intersection $W^u(p) \cap U$ (see Fig. 2). Below we shall use the coordinate systems in U and $\tilde{U}$ induced by the normal coordinates of p and f. Write $W^u_{\rm loc}(p)$ in U as the graph of a function $y = cx^2 + g(x)$, where $g(x) = o(x^2),\ c > 0$. A rectangle in U (resp. $\tilde{U}$) is called *a right rectangle* if it has two sides that are parallel to the coordinate axis.

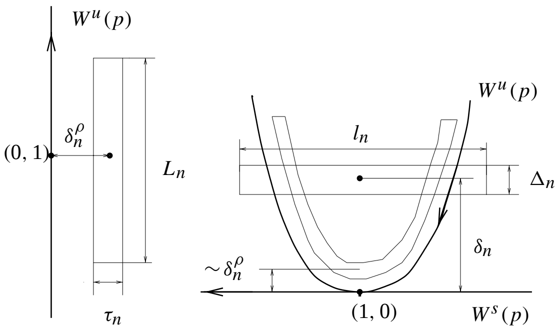

Fig. 2. Neighborhoods of points of homoclinic tangencies

Proposition 1. *Let $c > 0$ be the above constant and $n \in \mathbb{Z}_+$ be sufficiently large. Put $\delta_n = \mu^{-n}$, $\Delta_n = 2\delta_n^{3/2}$, and $l_n = 3c\delta_n^{1/2}$. Consider a right rectangle T_n centered at $(1, \delta_n)$ whose horizontal (resp. vertical) length is l_n (resp. Δ_n). Then the image $f^n(T_n)$ is the right rectangle centered at $(\delta_n^\rho, 1)$ whose horizontal (resp. vertical) length is $\tau_n = 3c\delta_n^{1/2}\lambda^n$ (resp. $L_n = 2\delta_n^{1/2}$). Moreover, T_n and $f^{n+N}(T_n)$ form a horseshoe which has a periodic saddle p' of period $n+N$ and the curvilinear rectangle $f^{n+N}(T_n)$ is $C\delta_n^\rho$ distant away from $W^s(p)$ for some $C > 0$ independent of n (see Fig. 2).*

Remark 1. The exponent $\rho = \frac{\log 1/\lambda}{\log \mu}$ gives a characteristic of distortion while a trajectory pass in a neighborhood of saddle p in the following sense. The rectangle T_n is δ_n-distance away from $W^s(p)$, but its image $f^{n+N}(T_n)$ is δ_n^ρ-distance away from $W^s(p)$, so the more exponent $\rho = \frac{\log 1/\lambda}{\log \mu}$ of saddle p, the deeper the horseshoe part $f^{n+N}(T_n)$ goes inside $W^u_{\text{loc}}(p)$ with respect to T_n and gets closer to $W^s(p)$.

Proof of Proposition 1. Use the fact that f is linear (4) in the unit square around p to prove the first part. Since N is fixed the ratio of {distance of the rectangle $f^n(T_n)$ to $W^u(x)$} and {distance of the curvilinear rectangle $f^{n+N}(T_n)$ to $W^u_{\text{loc}}(x)$} is bounded. This implies the second statement of the Proposition and completes the proof. □

2.3. The first step: Higher order homoclinic tangencies and appropriate limits. It is well-known from e.g. [MV], [PT], and [TY] that for a generic 1-parameter family $\{f_\varepsilon\}$ unfolding a quadratic (1^{st} order) homoclinic tangency q and for any number $c \in [-2, 1/4]$ there exist three sequences ($n \in \mathbb{Z}_+$): rectangles T_n next to q, rescalings $R_n : T_n \to [-2, 2] \times [-2, 2]$, and parameters ε_n such that a return map $f^n_{\varepsilon_n}$ from T_n into itself converges to the map $(x, y) \to (y, y^2 + c)$. The rectangle T_n and the rescaling R_n are independent of c, but $\varepsilon_n = \varepsilon_n(c)$ depends on c.

In this section we show that for a generic k-parameter family $\{f_\mu\}$ unfolding a $(k-1)^{\text{st}}$ order homoclinic tangency q and for any set of numbers $M = (M_0, \dots, M_{k-1}) \in \mathbb{R}^k$ there exist three sequences ($n \in \mathbb{Z}_+$): rectangles T_n next to q, rescalings $R_n : T_n \to [-2, 2] \times [-2, 2]$, and parameters $\mu(n) = (\mu_0(n), \dots, \mu_{k-1}(n))$ such that a return map $f^n_{\mu(n)}$ from T_n into itself under f_{ε_n} converges to $(x, y) \to (y, y^k + \sum_{i=0}^{k-1} M_i y^i)$. Moreover, convergence holds with respect to the C^r topology for any $r \in \mathbb{Z}_+$. The calculation presented below is in the spirit of [PT] and [TY].

Consider a C^∞ diffeomorphism f which has a dissipative saddle periodic point p exhibiting a homoclinic tangency of $(k-1)^{\text{st}}$ order. Figure 1 illustrates the topology for even k. We shall use notations of Fig. 1. Take coordinates $(\bar{x}, \bar{y}) = (x-1, y)$ around the homoclinic point q and coordinates $(\tilde{x}, \tilde{y}) = (x, y-1)$ around the homoclinic point $\tilde{q}$.

Because of a $(k-1)^{\text{st}}$ order tangency, the map $f^N : (\tilde{x}, \tilde{y}) \to (\bar{x}, \bar{y})$ from a neighborhood $\tilde{U}$ of $\tilde{q}$ with coordinates $(\tilde{x}, \tilde{y})$ to a neighborhood U of q with coordinates $(\bar{x}, \bar{y})$ can be written in the form:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \overset{f^N}{\longmapsto} \begin{pmatrix} \alpha\tilde{y} + \beta\tilde{x} + H_1(\tilde{x}, \tilde{y}) \\ \gamma\tilde{y}^k + \sigma\tilde{x} + H_2(\tilde{x}, \tilde{y}) \end{pmatrix}, \tag{5}$$

where α, β, and γ are constants such that for $\tilde{x} = \tilde{y} = 0$,

$$\begin{cases} H_1 = \partial_x H_1 = \partial_y H_1 = 0 \\ H_2 = \partial_x H_2 = \partial_y^j H_2 = 0, \quad j = 1, \dots, k. \end{cases} \tag{6}$$

To see that formula (5) holds consider images of lines $\{\tilde{x} = \text{constant}\}$.

Consider a generic k-parameter unfolding of a $(k-1)^{\text{st}}$ order homoclinic tangency:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \overset{f^N_\mu}{\longmapsto} \begin{pmatrix} \alpha\tilde{y} + \beta\tilde{x} + H_1(\tilde{x}, \tilde{y}) \\ \gamma\tilde{y}^k + \sum_{i=0}^{k-1} \mu_i \tilde{y}^i + \sigma\tilde{x} + H_2(\tilde{x}, \tilde{y}) \end{pmatrix}. \tag{7}$$

The main result of this section in the following:

Lemma 1. *With the above notations and $k \geq 2$ for an arbitrary set of real numbers $\{M_i\}_{i=0}^{k-1}$ there exists a sequence of parameters $\{\mu(n)\}_{n\in\mathbb{Z}_+}$ such that $\mu(n)$ tends to 0 as $n \to \infty$ and a sequence of change of variables $R_n : (\bar{x}, \bar{y}) \to (x_n, y_n)$ such that the sequence of maps: $\{R_n \circ f_{\mu(n)}^{n+N} \circ R_n^{-1} : [-2, 2] \times [-2, 2] \to [-2, 2] \times [-2, 2]\}$ converges to the 1-dimensional map*

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{\phi_M}{\longmapsto} \begin{pmatrix} y \\ y^k + \sum_{i=0}^{k-1} M_i y^i \end{pmatrix} \tag{8}$$

in the C^r topology for any r. The exact form of the sequence of $\{\mu(n)\}_{n\in\mathbb{Z}_+}$ is given in (11).

Corollary 3. *(The fourth step of the proof of Theorem 3) For $M_1 = 1$, and $M_j = 0$, $j = 0, 2, \dots, k-1$ by a C^r-perturbation of a C^∞ diffeomorphism f exhibiting a homoclinic tangency of order $(k-1)$ one can create a C^r diffeomorphism $\tilde{f}$ with a $(k-1)$-degenerate periodic orbit of an arbitrarily high period.*

Corollary 4. *For $k = 2$, $M_0 = -2$, and $M_1 = 0$ by a C^r-perturbation of a C^∞ diffeomorphism f exhibiting a quadratic homoclinic tangency one can create a C^∞ diffeomorphism f with a periodic saddle p exhibiting a homoclinic tangency and eigenvalues of p are close to 2 and to $+0$ respectively.*

To prove this corollary recall that for any r the map $(x, y) \to (y, y^2 - 2)$, $x, y \in [-2, 2]$ has a fixed point $(2, 2)$. One can show that by a C^r perturbation of this 2-dimensional map a fixed point $(2, 2)$ becomes a saddle near to $(2, 2)$ exhibiting a homoclinic tangency. In [PT] §6. 3, Prop. 3, Figs. 6.4 and 6.5 or [MV, p. 14], this is shown to be true. On 2-dimensional perturbations of the 1-dimensional map $y \mapsto y^2 - \mu$ see also [BC].

Proof of Lemma 1. We follow the standard method and split the return map f^{n+N} into the composition of two maps: the linear map $f^n : (x, y) \to (\lambda^n x, \mu^n y)$ and the map f_μ^N given by formula (7). The composition of f_μ^N and f^n has the form:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \overset{f_\mu^N \circ f^n}{\longmapsto} \begin{pmatrix} \alpha \bar{y}_n + \beta\lambda^n(1 + \bar{x}) + H_1(\cdot, \cdot) \\ \gamma \bar{y}_n^k + \sum_{i=0}^{k-1} \mu_i \bar{y}_n^i + \sigma\lambda^n(1 + \bar{x}) + H_2(\cdot, \cdot) \end{pmatrix}. \tag{9}$$

where $\bar{y}_n = \mu^n \bar{y} - 1$, $H_j(\cdot, \cdot) = H_j(\lambda^n(1 + \bar{x}), \bar{y}_n)$, $j = 1, 2$. Denote $\mu^{1/(k-1)}$ by τ. Introduce the change of variables $R_n : (\bar{x}, \bar{y}) \to (x_n, y_n)$, where

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \tau^n \bar{x} \\ \tau^n(\mu^n \bar{y} - 1) \end{pmatrix}. \tag{10}$$

In (x_n, y_n)-coordinates the map $f_\mu^N \circ f^n$ has the form:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} \overset{f_\mu^N \circ f^n}{\longmapsto} \begin{pmatrix} \alpha y_n + \beta\lambda^n(\tau^n + x_n) + \tau^n H_1(\cdot, \cdot) \\ \gamma y_n^k + \sum_{i=0}^{k-1} \mu_i \tau^{n(k-i)} y_n^i + \sigma\lambda^n\mu^n(\tau^n + x_n) + \tau^{kn} H_2(\cdot, \cdot) - \tau^n \end{pmatrix},$$

where $H_j(\cdot, \cdot) = H_j(\lambda^n(1 + \tau^{-n}x_n), \tau^{-n}y_n)$ for $j = 1, 2$.

Recall that p is dissipative, so $\lambda\mu < 1$ and $\lambda\tau < 1$ too. Thus, condition (6) and $0 < \lambda, \tau^{-1} < 1$ imply that terms $\lambda^n\mu^n x_n$, $\beta\lambda^n(\tau^n + x_n)$, $\tau^n H_1(\lambda^n(1+\tau^{-n}x_n), \tau^{-n}y_n)$, and $\tau^{kn} H_2(\lambda^n(1 + \tau^{-n}x_n), \tau^{-n}y_n)$ tends to 0 as $n \to \infty$ in the C^r topology for any positive integer r.

Put

$$\begin{aligned} \mu_0(n) &= \mu^{-kn/(k-1)} M_0 - \sigma\lambda^n + \mu^{-n} \\ \mu_i(n) &= \mu^{-(k-i)n/(k-1)} M_i \ \text{ for } \ i = 1, \dots, k-1. \end{aligned} \tag{11}$$

We see that all $\{\mu_i(n)\}$ tends to 0 as n tends to infinity. Therefore, in the limit as $n \to \infty$ we obtain

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \overset{\phi_M}{\longmapsto} \begin{pmatrix} \alpha y \\ \gamma y_n^k + \sum_{i=0}^{k-1} M_i y^i \end{pmatrix}. \tag{12}$$

Additional change of variables depending on α and γ completes the proof. □

2.4. The second step: Construction of a k-floor tower. Consider a C^∞ diffeomorphism f with a nonresonant saddle periodic point p exhibiting a homoclinic tangency at a point q. First, we give a definition of *a k-floor tower*. Recall that U denotes a neighborhood of the homoclinic tangency q. Let $\tilde{p}$ be a saddle periodic orbit of f, $\tilde{p} \in U$. Then denote by $W^s_{\text{loc}}(\tilde{p})$ (resp. $W^u_{\text{loc}}(\tilde{p})$) the first connected component of the intersection of stable (resp. unstable) manifold $W^s(\tilde{p})$ (resp. $W^u(\tilde{p})$) with U.

Definition 4. *A k-floor tower is a collection of k saddle periodic points $p_1, \dots, p_k$ (of different periods) such that $W^u_{\text{loc}}(p_i)$ is tangent to $W^s_{\text{loc}}(p_{i+1})$ for $i = 1, \dots, k-1$, and $W^u_{\text{loc}}(p_k)$ intersects $W^s_{\text{loc}}(p_1)$ transversally (see Fig. 3 for $k = 3$).*

For determines we shall consider towers with saddles posed in turn to the left and to the right, i.e. saddles $p_1, p_3, \dots$ are to the right of a point of homoclinic tangency and saddles $p_2, p_4, \dots$ are to the left of homoclinic tangency.

Construction of a k-floor tower is an intermediate step in the proof of Theorem 3. Notice that Fig. 3 might be misleading, because in order to get a natural shape of a 3-floor tower one should shrink the figure along the horizontal direction by a sufficiently large factor. If one draw a 3-floor tower with in a natural size such a figure it would become unreadable.

In this section we prove the following

Lemma 2. *For any positive integer k a C^r diffeomorphism f exhibiting a homoclinic tangency for an at least r-shrinking saddle periodic orbit p admits a C^r-perturbation $\tilde{f}$ such that $\tilde{f}$ has a k-floor tower. If q is a point of homoclinic tangency of f, then the aforementioned tower of $\tilde{f}$ is located in a neighborhood U of q.*

Proof. We prove this lemma using localized perturbation technique. As usual consider normal coordinates for a nonresonant saddle p. Induce coordinates in U by normal coordinates for the point p and the diffeomorphism f. Application of Proposition 1 gives existence of the contour described on Fig. 4 in the case $k = 3$. Indeed, consider an increasing sequence of numbers $n_1, \dots, n_k$ such that for each $i = 1, \dots, k$ the following two properties hold:

1) T_{n_i} intersects $f^{n_i+N}(T_{n_i})$ and they form a horseshoe;
2) n_{i+1} is the largest number such that $T_{n_{i+1}}$ and $f^{n_i+N}(T_{n_i})$ intersect in a horseshoe-like way, i.e., that they bound an open set.

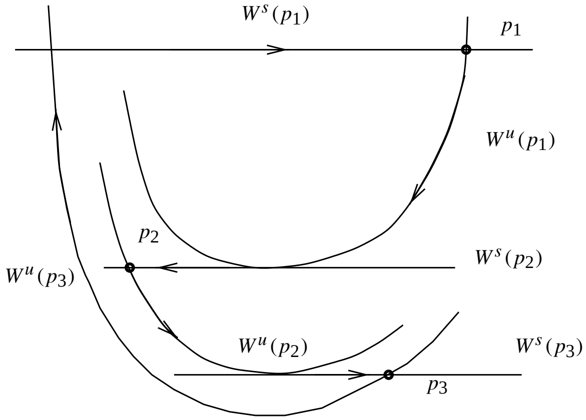

Fig. 3. A 3-floor tower

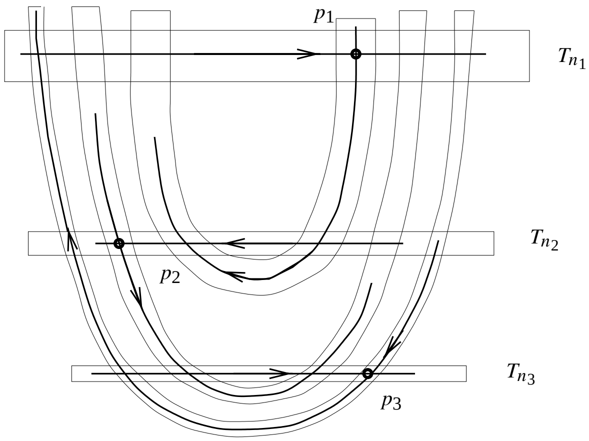

Fig. 4. An uncomplete 3-floor tower

For each $i = 1, \ldots, k$ condition 1) implies existence of a saddle periodic point $p_i \in T_{n_i} \cap f^{n_i+N}(T_{n_i})$ of period $n_i + N$ and condition 2) that $W^s_{\mathrm{loc}}(p_{i+1})$ and $W^u_{\mathrm{loc}}(p_i)$ intersect.

Let U be equipped with normal coordinates. Define the maximal distance in the vertical direction between $W^s_{\mathrm{loc}}(p_i)$ and $W^u_{\mathrm{loc}}(p_i)$, denoted by s_i, as the maximum of distance between any two points $x \in W^s_{\mathrm{loc}}(p_i)$ and $y \in W^u_{\mathrm{loc}}(p_i)$, which is below $W^s_{\mathrm{loc}}(p_i)$ such that x and y have the same $\bar{x}$-coordinate. Denote the vertical distance between centers of T_{n_i} and $T_{n_{i+1}}$ by t_i (see Fig. 5). So, $s_i - t_i$ is the distance by which one should lift $W^u_{\mathrm{loc}}(p_i)$ to create heteroclinic tangency with $W^s_{\mathrm{loc}}(p_{i+1})$. By calculation in Sect. 2.2 we get $t_i = \mu^{-n_i} - \mu^{-n_{i+1}}$.

Proposition 2. *If the saddle p having a homoclinic tangency is at least r-shrinking, then the ratio $\frac{s_i - t_i}{t_i^r}$ is arbitrarily small for each $i = 1, \ldots, k - 1$.*

Proof. Let us use notations and quantitative estimate obtained in Proposition 1. Let p be ρ-shrinking, $\rho > r$. Recall that the rectangle T_n is centered at $(1, \delta_n = \mu^{-n})$ and has length $3c\mu^{-n/2}$ and width $\mu^{-3n/2}$. Notice that the width is much less than μ^{-n}, the height of center μ^{-n}. Since p is ρ-shrinking and n_i and n_{i+1} satisfy the conditions $\mu^{-n_{i+1}} > const\ \mu^{\rho n_i} > \mu^{-n_{i+1}-1}$ it implies that $s_i - t_i < \delta_{n_{i+1}} + \Delta_{n_i} < C\delta_{n_i}^{\rho} < \varepsilon\delta_{n_i}^{r} = \varepsilon t_{n_i}$ for any $\varepsilon > 0$ and a sufficiently large n_i (see Fig. 2 right and Fig. 5). □

Proposition 3. *Let z be a point on $W^u_{\mathrm{loc}}(p_i)$ which is equidistant to both $W^s_{\mathrm{loc}}(p_i)$ and $W^s_{\mathrm{loc}}(p_{i+1})$. If the ratio $(s_i - t_i)/t_i^r$ is arbitrarily small, then there exists a small C^r-perturbation inside of the ball B centered at z of radius $t_i/3$ (see Fig. 5) such that $W^s_{\mathrm{loc}}(p_{n_{i+1}})$ and $W^u_{\mathrm{loc}}(p_{n_i})$ have a point of a heteroclinic tangency.*

Proof. Let ε be a sufficiently small positive number. Using the standard perturbation technic by a bump function inside of the ball B of radius $t_i/3$, one can find an ε-small C^r-perturbation lifting $W^u_{\mathrm{loc}}(p_{n_i})$ up by $\approx \varepsilon t_i^r$ and creating a heteroclinic tangency with $W^s_{\mathrm{loc}}(p_{n_{i+1}})$. □

Remark 2. Odd-even orientation of floors of towers we have choisen above requires i to be odd for the floor in Fig. 5. Since, saddles with odd indices have to be to the right and with even indices to the left.

In order to construct a k-floor tower one needs to create a heteroclinic tangency of $W^s_{\mathrm{loc}}(p_{n_{i+1}})$ and $W^u_{\mathrm{loc}}(p_{n_i})$ by a C^r-perturbation. We construct it by "bending" $W^u_{\mathrm{loc}}(p_{n_i})$.

Another way to construct a k-floor tower, used in [GST], is by fixing the eigenvalue $\mu > 1$ and varying the other eigenvalue $\lambda < 1$ of the saddle p exhibiting homoclinic tangency. See Proposition 1: the rectangle T_n is centered at $(1, \mu^{-1})$ and the curvilinear rectangle $f^{n+N}(T_n)$ is $C\delta_n^{\rho} = C\lambda^n$ distant away from $W^s(p)$, therefore, by changing λ one can vary the position of $f^{n+N}(T_n)$ without changing the position of T_n. But, in this case one needs some additional geometric argument to construct all heteroclinic tangencies of a k-tower simultaneously.

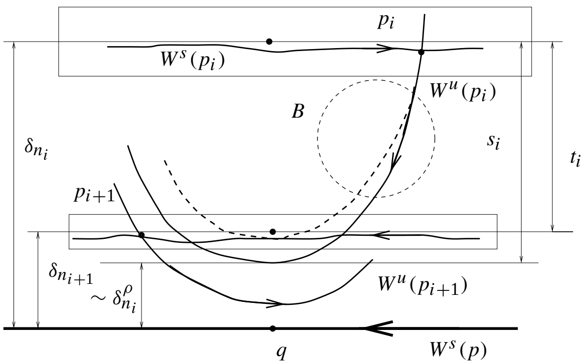

Fig. 5. A localized perturbation for a floor of a tower

2.5. The third step: Construction of a k^{th} order tangency. We shall prove that by a perturbation of a $(k+1)$-floor tower one can create a k^{th} order homoclinic tangency. Let us start with a 2^{nd} order tangency for a 3-floor tower and then use induction in k then.

Proposition 4. *[GST] A perturbation of a 3-floor tower can create a* 2^{nd} *order homoclinic tangency.*

Proof. Step 1. Let us create a 1^{st} order tangency of $W^u_{\text{loc}}(p_2)$ and $W^s_{\text{loc}}(p_1)$. Start with a 3-tower on Fig. 3. "Push" $W^u_{\text{loc}}(p_2)$ down $W^s_{\text{loc}}(p_3)$. Denote by γ the tongue (the part) of $W^u_{\text{loc}}(p_2)$ underneath $W^s_{\text{loc}}(p_3)$. The tongue γ is in the sector of the saddle hyperbolic point p_3, therefore, under iteration of f γ will be stretched along $W^u_{\text{loc}}(p_3)$ and for some s $f^s(\gamma) \cap W^s(p_1) \neq \emptyset$. Varying the size of the tongue γ we can create a heteroclinic tangency (see Fig. 6 a) with $i = 0$). Denote a point of tangency by q^*. Only two parts of $W^u(p_2)$ are depicted on Fig. 6 a): first part – starting part of $W^u(p_2)$ at p_2 and second – image of γ after a number of iterations under f (in above notations $f^s(\gamma)$).

Assume that saddle p_1 is nonresonant. Then there is normal coordinates around p_1 linearizing f. Induce by f normal coordinates in a neighborhood of U^* of q^*. In what follows we shall use these coordinate systems in U^*.

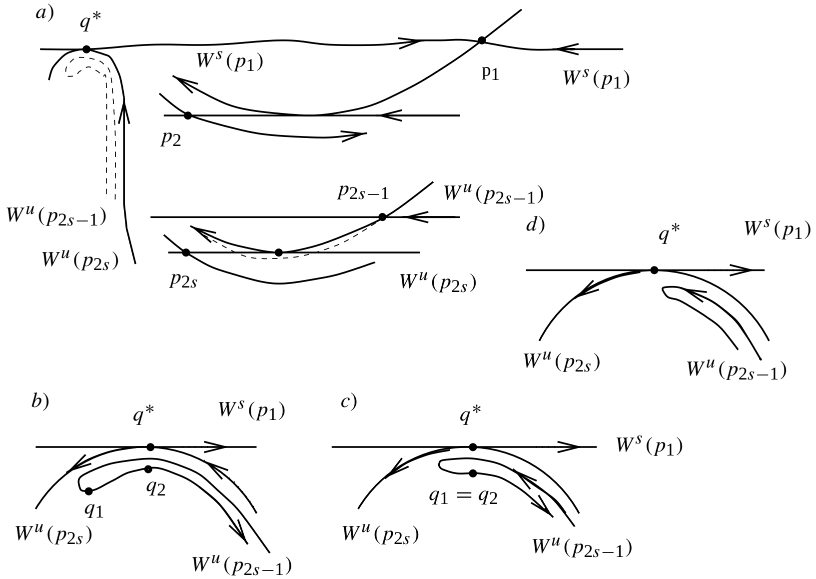

Fig. 6a–d. A 2^{nd} (even) order tangency

Step 2. Let us create a 2^{nd} order homoclinic tangency of $W^u_{\text{loc}}(p_1)$ and $W^s_{\text{loc}}(p_1)$. Start with a contour on Fig. 6 a). "Push" $W^u_{\text{loc}}(p_1)$ down $W^s_{\text{loc}}(p_2)$. Denote by γ^1 the tongue (the part) of $W^u(p_1)$ underneath $W^s(p_2)$. Some iterate of the tongue γ^1 of $W^u_{\text{loc}}(p_1)$ come into U^*. U^* has normal coordinates and naturally defined the horizontal and the vertical directions. Now our goal is varying the size of γ^1 construct a tangency of some iterate of γ^1 to the horizontal direction in U^*.

Fix some coordinates in a neighborhood of $\tilde{U}$ of a tangency $\tilde{q}$ of $W^u_{\text{loc}}(p_1)$ and $W^s_{\text{loc}}(p_2)$. Consider a 1-parameter family of diffeomorphisms $\{f_\varepsilon\}$, where ε is the maximal distance of $W^u_{\text{loc}}(p_1) \cap \tilde{U}$ and $W^s_{\text{loc}}(p_2) \cap \tilde{U}$ in the vertical direction.

Let $\gamma_\varepsilon = W^u_{\text{loc}}(p_1) \cap \{y \le 0\}$. Fix $\varepsilon > 0$ and $s = s(\varepsilon)$ such that $\gamma^s_\varepsilon = f^s(\gamma^1_\varepsilon) \cap U^* \neq \emptyset$ (see Fig. 6 b)). The curve γ^s_ε has two points q_1 and q_2 of tangency to the horizontal direction. As ε decreases q_1 and q_2 approach one to the other and for some critical value ε^* they collide and $q_1 = q_2$ (see Fig. 6 c)).

At the point $q_1 = q_2$ the local unstable manifolds $W^u_{\text{loc}}(p_1)$ has a 2^{nd} order tangency to the horizontal direction. Let this point have coordinates $(\varepsilon_1, \varepsilon_2)$ in U^*. Lifting $W^u_{\text{loc}}(p_2)$ by ε_2 we can create a 2^{nd} order tangency at point $(\varepsilon_1, 0)$. This completes the proof of the proposition. □

Now we construct a k^{th} order heteroclinic tangency using induction in k. In Proposition 4 we constructed a 2^{nd} order homoclinic tangency. Assume that for some $m < k$ having a $(k+1)$-floor tower consisting of saddle periodic points $p_1, \dots, p_{k+1}$ by a small C^r-perturbation we created m^{th} order heteroclinic tangency between $W^s_{\text{loc}}(p_1)$ and $W^u(p_{k-m+1})$ and preserved unbroken all 1^{st} order heteroclinic tangencies between $W^u_{\text{loc}}(p_i)$ and $W^s_{\text{loc}}(p_{i+1})$ for all $i = 1, \dots, k-m$. Suppose we proved that by a small C^r-perturbation we create an $(m+1)^{\text{st}}$ order tangency between $W^s_{\text{loc}}(p_1)$ and $W^u(p_{k-m})$ without destroying heteroclinic tangencies between $W^u_{\text{loc}}(p_i)$ and $W^s_{\text{loc}}(p_{i+1})$ for all $i = 1, \dots, k-m-1$. Iterative application of this procedure, i.e. by increasing m from 1 up to k, one can create a k^{th} order homoclinic tangency of $W^s(p_1)$ and $W^u(p_1)$ after $k-1$ steps.

Recall that we have a diffeomorphism f having a saddle periodic point p exhibiting a 1^{st} order tangency at a point q and in a neighborhood U of q there is a part of an undestroyed tower.

Proposition 5. *Let $p_1, \dots, p_{k-m+1}$ be saddle periodic points (of different periods) of a diffeomorphism f in U such that $W^u_{\text{loc}}(p_i)$ and $W^s_{\text{loc}}(p_{i+1})$ have heteroclinic tangencies for all $i = 1, \dots, k-m$ and $W^u(p_{k-m+1})$ has an m^{th} order tangency with $W^s(p_1)$*

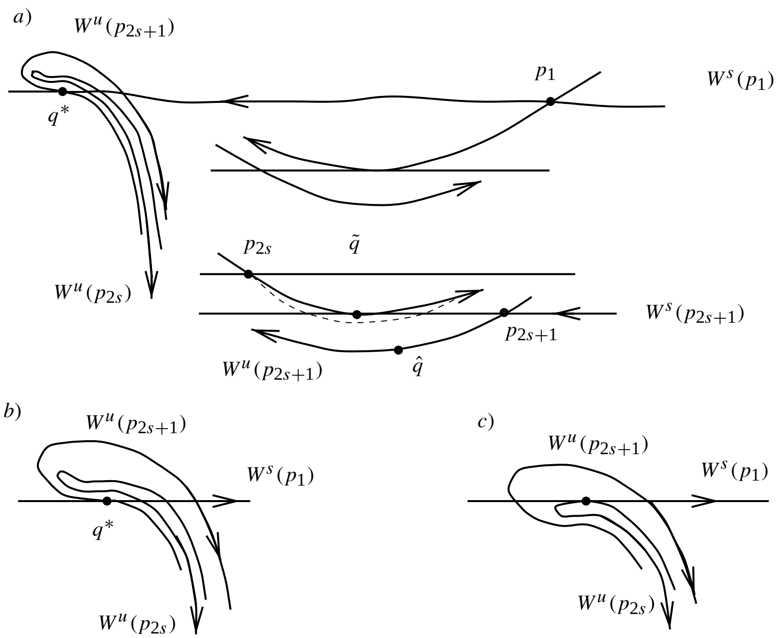

Fig. 7a–c. An odd order tangency

at point q^*. Then by a C^r-perturbation one can destroy only heteroclinic tangency between $W^u_{\text{loc}}(p_{k-m})$ and $W^s_{\text{loc}}(p_{k-m+1})$ at a point $\tilde{q}$ and create an $(m+1)^{\text{st}}$ order tangency between $W^u(p_{k-m})$ and $W^s_{\text{loc}}(p_1)$ (see Figs. 6 and 7 for different (odd/even) values of k and m).

Proof. Since, we need to prove approximation result without loss of generality one can assume that the initial diffeomorphism f is C^∞ smooth. Assume that p_1 and p_{k-m+1} are nonresonant saddles. Fix normal coordinate systems (x_1, y_1) (resp. (x_{k-m+1}, y_{k-m+1})) around p_1 (resp. p_2) so that f is linear there. Let $\lambda < 1 < \mu$ be eigenvalues of p_{k-m+1}.

Denote a preimage of the point q^* of an m^{th} order heteroclinic tangency of $W^u(p_{k-m+1})$ and $W^s_{\text{loc}}(p_1)$, which belongs to a neighborhood of p_{k-m+1}, by $\hat{q} = f^{-N}(q^*)$ for some N.

Fix the normal coordinate systems (x^*, y^*), $(\hat{x}, \hat{y})$, and $(\tilde{x}, \tilde{y})$ in neighborhoods U^* of q^*, $\hat{U}$ of $\hat{q}$, and $\tilde{U}$ of $\tilde{q}$ induced by the map f from the normal coordinate systems (x_1, y_1) and (x_{k-m+1}, y_{k-m+1}). So $q^* = (0, 0)$ in U^*, $\hat{q} = (0, 0)$ in $\hat{U}$, and $\tilde{q} = (0, 0)$ in $\tilde{U}$. Without loss of generality one can assume that points $\tilde{q}$ and $\hat{q}$ have coordinates $(1, 0)$ and $(0, -1)$ in the coordinate system around (x_{k-m+1}, y_{k-m+1})

Since by our assumption $W^u_{\text{loc}}(p_{k-m})$ has a 1^{st} order tangency to $W^s(p_{k-m+1})$ the tongue $W^u_{\text{loc}}(p_{k-m}) \cup \tilde{U}$ has the form $\hat{y} = a\bar{x}^2 + g(\bar{x})$, where $a > 0$ and $g(\bar{x}) = o(\bar{x}^2)$ at $\bar{x} = 0$.

Since, $W^s(p_1)$ and $W^u(p_{k-m+1})$ have an m^{th} order tangency the map $f^N : \tilde{U} \to U^*$ has the form

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \mapsto \begin{pmatrix} \alpha \hat{y}_n + \beta \hat{x} + H_1(\hat{x}, \hat{y}) \\ \gamma \hat{y}^k + \sigma \hat{x} + H_2(\hat{x}, \hat{y}) \end{pmatrix}, \tag{13}$$

where $H_1(\hat{x}, \hat{y})$ and $H_2(\hat{x}, \hat{y})$ satisfy condition (6) with $k = m + 1$.

The idea of the proof of the Proposition is to shift $W^u_{\text{loc}}(p_1)$ by ε down to $\bar{y} = a\bar{x}^2 - \varepsilon + g(\bar{x})$ and consider the versal $(m+1)$-parameter family $\{f_\mu\}$, $\mu = (\mu_0, \dots, \mu_m)$ of the form (7) unfolding the m^{th} order at heteroclinic tangency of $W^u(p_{k-m+1})$ and $W^s_{\text{loc}}(p_1)$ at q^*. The part of $W^u_{\text{loc}}(p_{k-m+1})$ given by $\{\tilde{y} \le 0\} \cap \{\tilde{y} = a\tilde{x}^2 - \varepsilon + g(\tilde{x})\}$ after a number of iterations under f come to a neighborhood U^* of q^*. It turns out that by varying $(m + 2)$ parameters $\varepsilon, \mu_0, \dots, \mu_m$ we can construct a $(m + 1)^{\text{st}}$ order homoclinic tangency in U^*. Let us prove the statement.

Calculate the composition map $f^N_\mu \circ f^n : \tilde{U}_n \to U^*$, which is defined in an open subset $\tilde{U}_n \subset \tilde{U}$, see Proposition 1 about choice of $\tilde{U}_n$,

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \overset{f^N_\mu \circ f^n}{\longmapsto} \begin{pmatrix} \alpha \tilde{y}_n + \beta \lambda^n (1 + \tilde{x}) + H_1(\cdot, \cdot) \\ \gamma \tilde{y}_n^k + \sum_{i=0}^{k-1} \mu_i \tilde{y}_n^i + \sigma \lambda^n (1 + \tilde{x}) + H_2(\cdot, \cdot) \end{pmatrix}, \tag{14}$$

where $\tilde{y}_n = \mu^n \tilde{y} + 1$, $H_j(\cdot, \cdot) = H_j(\lambda^n(1 + \tilde{x}), \tilde{y}_n)$ and $j = 1, 2$. Assume that after a change of coordinates in U^* preserving lines $\{y^* = const\}$ variable σ has an appropriate sign depending on whether k is odd or even and on high order derivatives of $y(x) = a\bar{x}^2 + g(\bar{x})$.

Consider the following parameterization of $W^u_{\text{loc}}(p_1) \cap \{\bar{y} \le 0\}$ by $\tilde{x}(t) = t$, $\tilde{y}(t) = at^2 - \varepsilon + g(t)$. Let $x^*_n(t)$ and $y^*_n(t)$ denote the first and the second coordinate function of the map (14). It is sufficient to find parameter values $\varepsilon(n)$, $\mu(n) = (\mu_0(n), \dots, \mu_{k-1}(n))$, and t_n such that

$$\tilde{y}_n(t) = y_n^*(t) = \frac{\partial y_n^*(t)}{\partial t}\bigg|_{t=t_n} = \cdots = \frac{\partial^k y_n^*(t_n)}{\partial t^k}\bigg|_{t=t_n} = 0, \qquad (15)$$

provided that $\frac{\partial x_n^*(t)}{\partial t}\big|_{t=t_n} \neq 0$. Indeed, in this case the image of t_n corresponds to the point of a $(m+1)^{\text{st}}$ order homoclinic tangency of $W^u(p_{k+m-1})$ and $W^s(p_1)$, because $f_\mu^N \circ f^n(W^u_{\text{loc}}(p_1)) \subset W^u(p_1)$.

In what follows we consider an appropriate rescaling limit of Eqs. (15). Using this limit we could say approximate values of all $m+3$ parameter values as $n \to \infty$.

To satisfy the first equation in (15) the following relation between t_n and ε_n is necessary:

$$\tilde{y}_n(t) = \mu^n(at^2 + g(t) - (\varepsilon - \mu^{-n})). \qquad (16)$$

Recall that $g(t)$ is $o(t^2)$ at $t = 0$. Thus, $\tilde{y}_n(t) = 0$ for $t_n \approx \sqrt{a^{-1}(\varepsilon - \mu^n)}$.

Now we look at Eqs. (15) after an appropriate rescaling. Consider the second coordinate function $y_n^*(t)$. Suppose $\varepsilon_n - \mu^{-n} = \rho_n^2 > 0$ and put $t = \rho_n\tau$ and $\tilde{g}_n(\tau) = g(\rho_n\tau)\rho_n^{-2}$. Rewrite $\tilde{y}_n(t)$ in the form

$$\tilde{Y}_n(\tau) = \tilde{y}_n(\rho_n\tau) = \mu^n\rho_n^2\left(a\tau^2 - 1 + \tilde{g}_n(\tau)\right).$$

One can see that $\tilde{g}_n(\tau)$ tends to 0 as $n \to \infty$, because $g(t)$ has a zero of third order.

Put $\pi_n = \sigma\lambda^n\rho_n$ and $\nu_n = \mu^n\rho_n$. Now we plug into the formula

$$y_n^*(t) = \gamma\tilde{y}_n^k(t) + \sum_{i=0}^{k-1}\mu_i\tilde{y}_n^i(t) + \sigma\lambda^n(1+t) + H_2(\lambda^n(1+t), \mu^n(at^2 + g(t) - \rho_n^2))$$

the function $\tilde{Y}_n(\tau) = \tilde{y}_n(\rho_n\tau)$, $\mu_0 = -\sigma\lambda^n$, and calculate $Y_n^*(\tau) = y_n^*(t)/\pi_n$. Then we get

$$Y_n^*(\tau) = \gamma\pi_n^{-1}\nu_n^{m+1}(a\tau^2 - 1 + \tilde{g}_n(\tau))^{m+1} + \sum_{p=0}^{m}\mu_p\pi_n^{-1}\nu_n^p(a\tau^2 - 1 + \tilde{g}_n(\tau))^p +$$
$$\tau + \pi_n^{-1}H_2(\lambda^n(1+\rho_n\tau), \nu_n(a\tau^2 - 1 + \tilde{g}_n(\tau))).$$

Assume that $\{\mu_p(n)\}_{p=0}^m$ and ρ_n are chosen in such a way that for some nonzero $\{a_p\}_{p=0}^{m+1} \subset \mathbb{R}\setminus\{0\}$,

$$\lim_{n\to\infty}\gamma\pi_n^{-1}\nu_n^{m+1} = a_{m+1}, \quad \lim_{n\to\infty}\mu_p(n)\pi_n^{-1}\nu_n^p = a_p \text{ for } p = 0,\ldots,m. \qquad (17)$$

Then in the limit as $n \to \infty$ we get

$$\lim_{n\to\infty} Y_n^*(\tau) = \sum_{p=1}^{m+1} a_p(a\tau^2 - 1)^p + \tau. \qquad (18)$$

We are going to show now that after such a rescaling there a set of nonzero $\{a_p\}_{p=0}^{m+1}$ satisfying (19). To justify that a solution $\{a_p\}_{p=0}^{m+1}$ to (19) would lead to a solution to (15) we need to estimate not only the remainder term $H_2(\lambda^n(1+\rho_n\tau), \nu_n(a\tau^2-1+\tilde{g}_n(\tau)))$, but also its derivatives with respect to τ of order $s \le m$,

$$\pi_n^{-1}\left|\frac{\partial^s}{\partial\tau^s}H_2(\lambda^n(1+\rho_n\tau), \nu_n(a\tau^2-1+\tilde{g}_n(\tau)))\Big|_{\tau=1/\sqrt{a}}\right|$$
$$\le \pi_n^{-1}\left|\sum_{j=0}^{s}\partial_\tau^{s-j}\ (\lambda^n\rho_n)^j(\partial_1^j H_2)(\lambda^n(1+\rho_n\tau), \nu_n(a\tau^2-1+\tilde{g}_n(\tau)))\Big|_{\tau=1/\sqrt{a}}\right|$$
$$\le C\nu_n^{[s/2]}.$$

Show that the last quantity is small. Using condition (17), which implies that $\nu_n^{m+1}\pi_n^{-1} \to a_{m+1}/\gamma$, and the fact $\pi_n = \sigma\lambda^n\rho_n \le \sigma\lambda^n$ we get that $\nu_n^{m+1} \le \sigma a_{m+1}/\gamma\lambda^n$. Therefore, the remainder term along with its derivatives of order up to m does not affect calculation of conditions (19). This show that if we find a collection of $\{a_p^*\}_{p=1}^{m+1}$ solving (19), then for each sufficiently large n the collection of $\{\mu_p(n)\}_{p=1}^n$ and ρ_n chosen using (17) solves Eqs. (15) with sufficient precision.

Lemma 3. *For any $m \in \mathbb{Z}_+$ there exists a collection $\{a_p^*\}_{p=1}^{m+1} \subset \mathbb{R}\setminus\{0\}$ such that*

$$Y(1/\sqrt{a}) = 0, \quad Y^{(s)}(1/\sqrt{a}) = 0, \quad s = 1, \dots, m+1. \tag{19}$$

Proof. The second derivative $Y^{(2)}(\tau)$ is an even function. The condition (19) implies that $Y^{(2)}(\tau)$ has two zeroes of order m each at $\tau = \pm 1/\sqrt{a}$. Therefore, $Y^{(2)}(\tau)$ has to be of degree at least $2m$. This shows that $a_{m+1} \ne 0$. Since, the sequence of $\{a_p^*\}_{p=0}^{m+1}$ can be constructed by induction in m neither of a_p's is zero. This completes the proof of the lemma. □

Because of the remarks before the lemma this completes the proof of Proposition 5. □

2.6. The fourth step of Theorem 3. To complete the proof of Theorem 3 we apply Corollary 3 which allows a diffeomorphism with a k^{th} order tangency by a C^r-perturbation turn into a diffeomorphism with an arbitrarily high period k-degenerate orbit.

2.7. A proof of Theorem 2. Fix a C^r metric ρ_r in $\text{Diff}^r(M)$ defined by the standard way (see e.g. [PM]). Let f be a C^r diffeomorphism which belongs to a Newhouse domain $\mathcal{N}$. Write $f \mapsto_{\varepsilon,r} g$ if g is a C^r-perturbation of size at most ε with respect to ρ_r. Consider an arbitrary sequence of positive integer numbers $\{a_n\}_{n=1}^{\infty}$.

Now for any ε we construct a 3ε perturbation f_3 of a diffeomorphism f such that for some n_1 the diffeomorphism f_3 *has $n_1 a_{n_1}$ hyperbolic periodic orbits of period n_1.* Hyperbolicity implies that the same is true for all diffeomorphisms sufficiently close to f_4.

Step 1. $f \mapsto_{\varepsilon,r} f_1$, where f_1 belongs to a Newhouse domain and is C^∞ smooth.

Step 2. By Theorem 3, there exists a C^r-perturbation $f_1 \mapsto_{\varepsilon,r} f_2$ such that f_2 has a k-degenerate periodic orbit q of an arbitrarily large period, where $k \ge r$.

Step 3. Let n_1 be a period of the k-degenerate periodic orbit q. It is easy to show that one can find $f_2 \mapsto_{\varepsilon,r} f_3$ such that in a small neighborhood of q f_3 has $n_1 a_{n_1}$ *hyperbolic periodic points* of period n_1.

Therefore, we show that an arbitrary C^r-close to f there exist a neighborhood $U \subset \mathrm{Diff}^r(M)$ with the following property for all $g \in U$:

$$\frac{\#\{x : g^n(x) = x\}}{a_n} \geq n. \tag{20}$$

If the diffeomorphism f_1 belongs to a Newhouse domain $\mathcal{N} \subset \mathrm{Diff}^r(M)$, then we can choose perturbation in Steps 1–3 so small that f_3 belongs to the same Newhouse domain $\mathcal{N}$. It is not difficult to see from Steps 1–3 that for an open dense set in $\mathcal{N}$ the condition (20) holds at least for one n. Iterating Steps 1–3 one constructs a residual set such that for each diffeomorphism f from that residual set the condition (20) holds for an infinitely many n's. This completes the proof of Theorem 2 in the case dim $M = 2$. □

Note that similar inductive argument leads to the well-known Newhouse's phenomenon on infinitely many coexisting sinks [N,PT,Rb], and [TY].

Remark 3. As we mentioned in the introduction, Theorem 2 implies Theorem 1.

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