BIFURCATION OF PLANAR AND SPATIAL POLYCYCLES: ARNOLD'S PROGRAM AND ITS DEVELOPMENT

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Working on the survey on Bifurcation Theory [AAIS] in 1985, Arnold insisted that the survey should attempt to predict the development of the theory during the next 25 years. The goal of this paper is to summarize some activity in the first half of this period related to the study of nonlocal planar bifurcations and to the extension of these results to spatial bifurcations.

1. The Hilbert-Arnold problem and bifurcation diagrams

1.1. **Arnold's conjecture and its consequences.** In Section 3.2.8 of [AAIS] Arnold proposed a general program for the investigation of nonlocal planar bifurcations. He stated several conjectures. One of them gave rise to the Hilbert–Arnold problem [I1], [I2], [I3].

Conjecture *(Arnold, 1985) In generic n-parameter families of vector fields on S^2 the bifurcation diagrams corresponding to the weak equivalence relation are locally homeomorphic to a finite number of samples; this number depends on n only.*

Weak equivalence of two families of vector fields means the existence of a family of homeomorphisms of the phase space *not necessarily continuous with respect to the parameter* that brings the phase portraits of the fields of one family to those of the other.

After the statement of the conjectures, Arnold wrote:

The proof or the disproof of these conjectures is definitely a necessary step in the study of nonlocal bifurcations in generic n-parameter families.

The Hilbert–Arnold problem was motivated in [I3] in the following way. "One can figure out why Hilbert has chosen the polynomial family for the study of limit cycles in his 16th problem. At the end of the last century polynomial families constituted probably the only natural example of finite–parameter families of vector fields. Now, when the fashion and viewpoints have reasonably changed, generic finite–parameter families became respectful. Therefore, a smooth version of the Hilbert 16th problem may be stated; it is written between the lines of some text due to Arnold [AAIS]."

The Hilbert–Arnold problem *Prove that the number of limit cycles in a generic finite-parameter family of vector fields on S^2 with a compact base is uniformly bounded with respect to the parameter by a constant depending on the family.*

1.2. **The Hilbert-Arnold problem and nonlocal bifurcations.** The Hilbert-Arnold problem is closely related to the notion of the cyclicity of a polycycle.

Consider a family of planar vector fields. Let a field of this family corresponding to a critical value of parameter have a polycycle.

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Definition 1. *The limit cycles is generated by a polycycle if it tends to the entire polycycle, as the parameter tends to the corresponding critical value.*

The cyclicity of polycycle in the family is the maximal number of limit cycles generated by this polycycle in the family.

The following conjecture implies a positive answer to the Hilbert-Arnold problem.

Finite cyclicity conjecture *The cyclicity of a polycycle that occurs in a generic n-parameter family of vector fields on the two-sphere is uniformly bounded by a constant depending only on n.*

The implication goes as follows.

For a given family of vector fields consider the integer valued function $L(\epsilon)$ equal to the number of limit cycles of the equation corresponding to the parameter value ϵ. This function is locally constant near the points corresponding to the structurally stable equations of the family. It may oscillate near the bifurcation values of the parameter. By the compactness of the two-sphere, this oscillation is no greater than the sum of cyclicities of a finite number of polycycles of the corresponding equation. Hence, L is bounded by a function with a finite oscillation at any point of the compact space of parameters.

On an intuitive level, the conjecture of 1.1 would imply the Finite cyclicity conjecture. But it turns out that the first conjecture is wrong, as is shown below in 1.3 and 1.4.

1.3. **The Kotova theorem.** Kotova found a polycycle in a generic three-parameter family whose bifurcation may produce an arbitrary number of limit cycles [KS]. Here we present a new modification of this example.

Consider a vector field with the following degeneracy of codimension three. The field has two saddlenode singular points, and separatrices of the two hyperbolic sectors coincide with each other. Some orbits emanating from the repelling parabolic sector of one of these points land at the attracting parabolic sector of the other one. See Figure 1. Thus a continuous family of polycycles named "lips" may occur in a generic three-parameter family of vector fields.

PSfrag replacements

Γ_1^-

Γ_1^+

Γ_2^-

Γ_2^+

λ

FIGURE 1. Lips

Theorem 2. *(Kotova,* [KS]*) For any positive integer k, a "lips" family of polycycles that occurs in a generic three-parameter family of vector fields may generate more than k limit cycles after bifurcation in this family.*

Consider now a hyperbolic saddle in the phase space of the lips polycycle. Without increasing the codimension of the degeneracy, the outgoing separatrix of the saddle may enter the attracting parabolic sector of one saddlenode, and the incoming separatrix may exit the repelling sector of the other saddlenode. The polycycle with these three singular points (it may be called the *"saddle lip"*) may generate a limit cycle close to the original polycycle.

But there may be an arbitrary number of saddles and broken saddle loops that contain the two saddlenodes. When the saddlenodes disappear, an arbitrary number of limit cycles may be generated by these saddle lips, see Figure 2.

PSfrag replacements

$a)$

$b)$

FIGURE 2. Multiple saddle lips (a) and their bifurcation (b)

The following problem occurs.

Problem *Describe the bifurcation diagram of the family with an arbitrary number of saddle lips.*

Bifurcation diagrams for lips and saddle lips were described by Stanzo [KS], [S]. Below we present his investigation of the lips polycycle.

1.4. **Diversity of germs of bifurcation diagrams (by Stanzo).** Consider once more the "lips" polycycle, see Figure 1. It may occur in a generic three-parameter family as was mentioned before. Let us construct a surface with singularities to be used below. Take an arbitrary monotonic increasing function $f \in C^2$ on a unit segment I. Let γ be the graph of f. Denote by $L(\gamma)$ the Legendre transform of γ, that is, the set of all $(a, b) \in \mathbb{R}^2$ such that the line $y = ax + b$ is tangent to γ. Let K_f be a cone over $L(\gamma)$, see Figure 3.

PSfrag replacements

$a)$

$b)$

$c)$

FIGURE 3. The cone K_f (a), the Legendre horn Σ_L (b), and the cylinder Z_L (c)

The coordinates in Figure 3 a–c are specified below in 1.5.

Theorem 3. *(Stanzo* [KS]*) For any function $f \in C^2(I)$ satisfying some genericity assumptions described below there exists a generic three-parameter family of planar vector fields such that a germ of a bifurcation diagram for this family at some point is a union of two smooth surfaces and a surface homeomorphic to K_f.*

Remark 1. *The germ in the theorem is called the Legendre horn.*

The genericity assumption for f is:

f has a finite number of nondegenerate inflection points. The nondegeneracy means that at the inflection point the graph of f has the cubic tangency with its tangent line.

1.5. **Smooth structure of germs of bifurcation diagrams.** The Legendre horn in Theorem 3 is exponentially narrow. Its smooth structure is described using special coordinates in the parameter space.

Consider a generic three–parameter family of planar vector fields with the "lips" polycycle. Suppose the value zero of the parameter corresponds to the field with this polycycle. The coordinates near the saddlenodes and the parameters of the family may be so chosen that the local families near the saddlenodes will take the form:

$$\dot{x} = (x^2 + \epsilon)(1 + a(\epsilon)x)^{-1},\ \dot{y} = -y$$

near O_1, see Figure 1,

$$\dot{u} = (u^2 + \delta)(1 + b(\delta)u)^{-1},\ \dot{v} = v$$

near O_2. Introduce the parameter λ such that λ measures the separatrix splitting. See Figure 1. When the saddlenode vanishes: $\epsilon > 0$ for O_1, $\delta > 0$ for O_2, then the map from the cross–section $\Gamma_j^+ \to \Gamma_j^-$ along the phase curves becomes well defined. It is given by

$$y \mapsto C(\epsilon)y,\ v \mapsto C(\delta)v,$$

where

$$C(\epsilon) = \varphi(\epsilon)\exp\left(-\frac{\pi}{\sqrt{\epsilon}}\right),\ \varphi(\epsilon) \in C^1(\mathbb{R}, 0).$$

Consider the rescaling

$$\Phi : (\epsilon, \delta, \lambda) \mapsto (p, \delta, q) = \left(\frac{C(\epsilon)}{C(\delta)}, \delta, -\frac{\lambda}{C(\delta)}\right).$$

Denote by Z_L the Legendre cylinder $L_\gamma \times [0, \delta_0]$ for some $\delta_0 > 0$.

Theorem 4. *(Stanzo* [KS]*) Let Σ_L be the Legendre horn from Remark 1. Then the image of Σ_L under the rescaling map Φ is diffeomorphic to a Legendre cylinder Z_L. The diffeomorphism is defined in the domain $p > 0,\ \delta > 0,\ q > 0$, maps it into itself, and may be C^1 extended across the planar domain $p > 0,\ \delta = 0,\ q > 0$.*

The theorem is illustrated by Figure 3 (b,c).

1.6. **The Generalized Legendre duality.** The diffeomorphism in Theorem 4 is closely related to the generalized Legendre transform. The latter is defined as follows.

Consider a local two-parameter family of planar curves. By definition [A], it is a divergent diagram of two germs of maps:

$$(\mathbb{R}^2, 0) \leftarrow (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^2, 0),$$

the left one p, the right one q, each one having rank 2. The map p is a *parameter map*, the map q is a *phase map.* The curve of the family is

$$\lambda_a = q(p^{-1}(a)),\ a \in (\mathbb{R}^2, 0).$$

The dual family occurs, when the phase and parameter maps in the same diagram are exchanged: q is now a parameter map, p is a phase one and the curve is

$$\mu_x = p(q^{-1}(x)), x \in (\mathbb{R}^2, 0).$$

The generalized Legendre transform of the germ of the curve $\gamma \subset (\mathbb{R}^2, 0)$ is a germ

$$L_\gamma = \{a \in (\mathbb{R}^2, 0) |\ \lambda_a \text{ is tangent to } \gamma\}.$$

Theorem 5. *(Stanzo,* [KS]*) Generalized Legendre transforms with respect to dual families are mutually inverse.*

2. The Kotova Zoo and cyclicity of corresponding elementary polycycles

2.1. **The Kotova Zoo.** Kotova [KS] gave a complete list of polycycles that may occur in generic two and three-parameter families. Below the list of elementary polycycles from the Kotova Zoo is given.

How does one obtain this list? Consider first the vertices of a polycycle in the given generic two (or three) parameter family. Assume that all vertices are elementary and the polycycle does not consist of one singular point only. The singular points on the polycycle may be nondegenerate (in this case they are hyperbolic saddles) or degenerate (in the latter case they are saddlenodes).

For the study of bifurcations one should distinguish saddles with zero and nonzero saddle value (this is the name for the sum of eigenvalues). The first ones are denoted by Z , the second ones by N; both may be marked by subscripts and superscripts.

The topology of the location of separatrices of a saddle vertex of the polycycle may be different. The polycycle divides the plane into several domains. One of them is unbounded. It is called the exterior component; the others are called interior. The hyperbolic saddle is of interior type provided that only one hyperbolic sector of this saddle belongs to the exterior domain of the polycycle. It is of exterior type if at least two sectors belong to the exterior domain of the polycycle. The corresponding notations are Z^e, Z^i, N^e, N^i. For example, Z^e denotes an exterior hyperbolic saddle with zero saddle value. These vertices are shown in Figure 4.

PSfrag replacements

$a)$
$b)$
$c)$

Figure 4. Interior (a) and exterior (b),(c) saddle vertices of the polycycle

A degenerate singular point (a saddlenode) on a polycycle met in generic two or three parameter family (that is, in the Kotova Zoo) may be of multiplicity no greater than 4. The saddlenodes of multiplicity 3 and 4 are denoted by D_3, D_4. If a polycycle from the Kotova Zoo contains a vertex of the type D_3 or D_4, it can contain no more singular points.

The location of the saddlenode of multiplicity two inside the polycycle may be different: the boundary between the parabolic and hyperbolic sectors of the saddlenode may or may not belong to the polycycle. In the first case the saddlenode is called the *bypassed* vertex, in the second case the *trespassed* vertex of the polycycle, see Figure 5. The bypassed vertices are denoted by B (with indices), the trespassed ones are denoted by T (with indices as well).

PSfrag replacements

$a)$

$b)$

FIGURE 5. Bypassed (a) and trespassed (b) saddlenode vertices of a polycycle

Any elementary polycycle from the Kotova Zoo generates a word in the alphabet

$$N^e,\ N^i,\ Z^e,\ Z^i,\ B,\ T\ (or\ N^e_j,\ N^i_j,\ Z^e_j,\ Z^i_j,\ B_j,\ T_j),\ D_3,\ D_4 \tag{1}$$

The list of these words together with corresponding pictures constitutes the Kotova Zoo of elementary polycycles. It is shown in section 2.2 starting on page 8, together with the cyclicities of the polycycles.

The completeness of this list is proved by Stanzo [KS]. The cyclicities of various polycycles from the list were found by many authors beginning with classical works of Andronow and Leontovich and continued by Cherkas, Roitenberg, Nozdracheva, Khibnik, Grozovskii, Shishkov, Dumortier, Roussarie, Russeau, Lukianov, Schechter, Li Weigu, Zhang Zhifen, Mourtada, Jebrane, and Stanzo. The final investigation of the cyclicity of all the 30 elementary polycycles from the Kotova Zoo was carried out by Trifonov [T]. He reproduced in a concentrated way the proof of the cyclicity theorems for 21 polycycles from the Zoo investigated by his predecessors. Moreover, he resolved the cyclicity for the new 9 cases.

The main result about the cyclicity of all elementary polycycles of the Kotova Zoo consisting of more that one point, are summarized in the table below.

2.2. **Elementary polycycles from the Kotova Zoo and their cyclicity.**

Definition 6. *Two words of the alphabet (1) are equivalent, if one may be transformed to another by one of the following operations:*
- cyclic permutation of symbols;
- order reversing;
- replacing all the superscripts i by e and vice versa.

Theorem 7. *Any elementary polycycle that may occur in a generic two or three parameter family of planar vector fields generates a word that is equivalent to one of the words of the Table below.*

This theorem in slightly different terms, is proved in [KS].

Remark 2. *The polycycles corresponding to the same word do not necessary bifurcate in the same way. The families that contain these polycycles may split to several domains in the functional space, each domain corresponding to a different bifurcation scenario. This corresponds to multiple entries in the last column. Some classes of families may contain polycycles that are encoded by the same word, but have different characters of connection between singular points. This is the case for two classes encoded as T_1T_2, and two others encoded as $T_1N^eT_2$.*

In the third column of the table some genericity assumptions are presented. The violation of these assumptions for polycycles with degeneracy of codimension two gives rise to polycycles with a higher degeneracy; the latter polycycles are included in the table as well.

The letter P stands for the Poincaré map of the polycycle. Zero corresponds to the vertex of the half interval which is the domain of this map.

The numbers r_j are the ratios of the eigenvalues of hyperbolic saddles. The numerator in this ratio corresponds to the separatrix that is met first according to the orientation of the polycycle. The restrictions for these ratios presented in the table are preserved under orientation reversal.

Theorem 8. [T] *The cyclicities of elementary polycycles in generic two and three parameter families may take values shown in the table below; the maximal values in the cell give the upper estimate.*

Theorem 9. *(Trifonov, unpublished) The estimates of cyclicity from the Table are sharp.*

Encoding	Figure	Comments	Upper bounds of cyclicity
N^e			**1**
$T\&D_4$			**1**
Z^e		$P'(0) \neq 1$	**2**
$N_1^e N_2^e$		$r_1 r_2 \neq 1$	**1,2**
....	....	....	....
$N_1^e N_2^i$			**1,2**
$N^e N^e$			**1**
$N^e T$			**1,2**
$D_3\&D_4$			**1**

Encoding	Figure	Comments	Upper bounds of cyclicity
T_1T_2			**1**
Z^i		$P'(0)=1$	**3**
N^eZ^e			**2**
$N_1^eN_2^e$		$r_1r_2=1$, $P'\neq 1$	**3****
N^eZ^i			**2**
Z^eZ^e		$P'(0)\neq 1$	**2**
$N_1^eN_2^eN_3^e$		$r_ir_j\neq 1$, $r_1r_2r_3 \neq= 1$	**1,2,3**
$N_1^eN_2^eN_3^i$		$r_1r_2\neq 1$	**1,2,3***

Encoding	Figure	Comments	Upper bounds of cyclicity
$N_1^e N_2^e N_2^e$			**1,2**
$N_1^e N_2^i N_2^i$		$r_1 r_2 \neq r_1 r_2^2 \neq 1$	**1,2,3***
$T_1 T_2 T_3$			**1**
$Z^e T$			**2**
$T_1 N^e T_2$			**1,2***
$N^e B$			**1,2**
$N^i B$			**1,2***
TB			**1**

Encoding	Figure	Comments	Upper bounds of cyclicity
$TN_1^eN_2^e$		$r_1r_2 \neq 1$	**1,2,3***
$TN_1^eN_2^i$			**1,2,3***
TN^eN^e			**1,3***
T_1T_2	Δ p	$\Delta''(0) \neq 0$	**2**
T_1T_2	Δ p	$\Delta''(0) = 0, \Delta'''(0) \neq 0$	**3**
$T_1N^eT_2$			**2**
TB			**2***

The map Δ in the third column is the map from a cross-section of a parabolic sector of one saddlenode to the likely cross-section for the other saddlenode, p is the point on the polycycle. The charts on the cross-sections are the restrictions of so-called normalizing coordinates near the saddlenode singular points. The latter coordinates are not unique but they define invariant affine structures on the cross-sections. Therefore, the restrictions to the derivatives of Δ mentioned in the table are invariant.

3. The Local Hilbert-Arnold problem and An Estimate of cyclicity of elementary polycycles

3.1. **The Bifurcation number and an estimate of cyclicity of polycycles.** As we described earlier the Hilbert-Arnold conjecture may be reduced to the problem of finite cyclicity of polycycles of finite codimension. It turns out that at least for a small number of parameter n, the cyclicity of polycycles occurring in a generic n-parameter family admits an upper estimate in terms of n. Sometimes, we shall use the term "*a polycycle γ of codimension n*" meaning that the polycycle γ occurs in a generic n-parameter family.

Examples

1) In a generic n-parameter family, the maximal multiplicity of a degenerate limit cycle does not exceed $n+1$, e.g. in codimension 1 a semistable limit cycle has cyclicity 2. Thus, the cyclicity of a trivial polycycle (a polycycle without singular points) in a generic n-parameter family does not exceed $n+1$.

2) (Andronow-Leontovich, 1930s; Hopf, 1940s). A nontrivial polycycle of codimension 1 has cyclicity at most 1.

3) (Takens, Bogdanov, Leontovich, Mourtada, Grozovskii, early 1970s-1993 (see [G], [KS] and references there)). A nontrivial polycycle of codimension 2 has cyclicity at most 2.

Definition 10. *A polycycle γ of a vector field on the sphere $\mathbb{S}^2$ is a cyclically ordered collection of equilibrium points $p_1, \dots, p_k$ (with possible repetitions) and different arcs $\gamma_1, \dots, \gamma_k$ (integral curves of the vector field) connecting them in the specific order: the j-th arc γ_j connects p_j with p_{j+1} for $j = 1, \dots, k$.*

Important feature of the definition is that arcs of a polycycle are different and for example the polycycle in Figure 2 (a) can not contain more than one saddle loop.

Definition 11. *Let $\{\dot{x} = v(x,\epsilon)\}_{\epsilon \in B^n},\ x \in \mathbb{S}^2$, be an n-parameter family of vector fields on $\mathbb{S}^2$ having a polycycle γ for the critical parameter value ϵ_*. The polycycle γ has cyclicity μ in the family $\{v(x,\epsilon)\}_{\epsilon \in B^n}$ if there exist neighborhoods U and V such that $\mathbb{S}^2 \supseteq U \supset \gamma,\ B \supseteq V \in \epsilon_*$ and for any $\epsilon \in V$ the field $v(\cdot,\epsilon)$ has no more than μ limit cycles inside U and μ is the minimal number with this property.*

Definition 12. *The bifurcation number $B(n)$ is the maximal cyclicity of a nontrivial polycycle occurring in a generic n-parameter family.*

The definition of $B(n)$ does not depend on a choice of the base of the family, it depends only on the number n of parameters.

The Local Hilbert-Arnold Problem *Prove that for any finite n, the bifurcation number $B(n)$ is finite and find an upper estimate for $B(n)$.*

This is the stronger version of the Finite Cyclicity conjecture stated in section 1.2. As it was pointed out a solution to this problem would imply a solution to the Hilbert-Arnold problem.

Definition 13. *A singular (equilibrium) point of a vector field on the two-sphere is called elementary if at least one eigenvalue of its linear part is nonzero. A polycycle is called an elementary polycycle if all its singularities are elementary.*

The Local Hilbert-Arnold problem was solved under the additional assumption that a polycycle have elementary singularities only.

Definition 14. *The elementary bifurcation number $E(n)$ is the maximal cyclicity of a nontrivial elementary polycycle occurring in a generic n-parameter family.*

From examples 2) and 3) above it follows that

$$E(1) = 1, \quad E(2) = 2.$$

Information about behavior of the function $n \mapsto E(n)$ has been obtained recently. The First crucial step was done by Ilyashenko and Yakovenko:

Finiteness Theorem *(Ilyashenko and Yakovenko* [IY]*) For any n the elementary bifurcation number $E(n)$ is finite.*

Corollary 1. *Under the assumption that families of vector fields have elementary singularities only the global Hilbert-Arnold conjecture is solved, i.e. any generic finite parameter family of vector fields on the sphere S^2 with a compact base and only elementary singularities has a uniform upper bound for the number of limit cycles.*

Application of some ideas from singularity theory and stratification theory along with the methods developed by Ilyashenko & Yakovenko allowed the second author to get the first known estimate for the elementary bifurcation number $E(n)$.

Theorem 15. *(Kaloshin* [K1]*) For any positive integer n*

$$E(n) \leq 2^{25n^2}. \tag{2}$$

Corollary 2. *Under the assumption that all the polycycles are elementary the last Theorem gives a solution to the Local Hilbert-Arnold problem.*

A vector-polynomial will be called for brevity simply a polynomial. Let us call a polynomial $P : \mathbb{R}^N \to \mathbb{R}^n$, $N \geq n$ *nontrivial* if there is a point $x \in \mathbb{R}^N$ such that $dP(x)$ has full rank. By *a chain map* we mean a composition map of the form

$$\mathcal{F} = P \circ F : \mathbb{B}^n \to \mathbb{R}^n, \ F : \mathbb{B}^n \to \mathbb{R}^N, \ P : \mathbb{R}^N \to \mathbb{R}^n, \tag{3}$$

where $\mathbb{B}^n \subset \mathbb{R}^n$ is a ball, P is a nontrivial polynomial of degree at most d, F is either a generic map or a jet of a generic map.

Using polynomial normal forms and the Khovanskii reduction method Ilyashenko and Yakovenko proved the following reduction

(4) $\boxed{\text{An Estimate for the maximal \# of small solutions to } P \circ F = const} \implies \boxed{\text{An Estimate for the maximal cyclicity } E(n)}$

The proof of the Finiteness Theorem [IY] also deals with chain maps, but treats them in a more complicated way. We shall describe an outline of the proof of Theorem 15. From now on we shall discuss the problem:

Given a polynomial P, estimate in terms of the degree of P the maximal number of small isolated preimages of a chain map $P \circ F(x) = const$, provided that the interior map F is generic.

Now we would like to raise another interesting question which appears while investigating chain maps of the form (3).

3.2. **Geometric multiplicity of germs of generic maps.** Let $F:\mathbb{R}^n \to \mathbb{R}^n$ be a generic C^k smooth map, $k \geq n+1$. Fix a point $a \in \mathbb{R}^n$ and denote $F(a)$ by b.

Definition 16. *A geometric multiplicity of a map germ $F:(\mathbb{R}^n, a) \to (\mathbb{R}^n, b)$ at a, denoted by $\mu_a^G = \mu_a^G(F)$, is the maximal number of isolated preimages $F^{-1}\left(\tilde{b}\right)$ close to a:*

$$\mu_a^G(F) = \limsup_{r\to 0} \sup_{\tilde{b}\in\mathbb{R}^n} \#\{x \in B_r(a) : F(x) = \tilde{b}\}. \tag{5}$$

For example, the geometric multiplicity of the function $f : x \to x^2$ at 0 is two, but the geometric multiplicity of $f : x \to x^3$ at 0 is one, even though 0 is a degenerate point of the second order.

In the complex case the geometric multiplicity equals the usual multiplicity (see [AGV]). In the real case the first is no greater than the second.

Definition 17. *Define geometric multiplicity of n-dimensional germs, $\mu^G(n)$, as follows. Let $F:\mathbb{R}^n \to \mathbb{R}^n$ be a generic map. The geometric multiplicity of F equals the least upper bound of geometric multiplicities of $\mu_a^G(F)$ taken for all points $a \in \mathbb{R}^n$. Then the geometric multiplicity of n-dimensional germs is the maximum of the geometric multiplicities of all generic maps F from $\mathbb{R}^n$ to $\mathbb{R}^n$*

$$\mu^G(n) = \sup_{F-generic,\ a\in\mathbb{R}^n} \mu_a^G(F). \tag{6}$$

It turns out that the geometric multiplicity of n-dimensional germs is finite for all positive integer n and depends only on the dimension n.

Remark 3. *For example, for $n=2$ the Whitney Theorem about maps of surfaces states that a generic map of two dimensional manifolds $F:M^2 \to N^2$ can have only three different types of germs: 1-to-1, a fold, and a pleat (see e.g.* [AGV]*). This implies that $\mu^G(2) = 3$.*

A natural problem is

Give estimates for the geometric multiplicity $\mu^G(n)$ of n-dimensional germs.

The problem about the upper estimate is closely related to the problem to estimate cyclicity of elementary polycycle, because a germ of a generic map is a particular case of a chain map (3) with the exterior polynomial $P = id$. The analytic case was considered by [GK]. The upper bound for the geometric multiplicity for n-dimensional smooth germs of generic maps is given by

Theorem 18. [K1] *The geometric multiplicity of germs of a generic C^k smooth map $F:\mathbb{R}^n \to \mathbb{R}^n$, $k > n$ admits the following upper estimate:*

$$\mu_a^G(F) \leq 2^{n(n-1)/2+1} n^n, \ \forall\ a \in \mathbb{R}^n. \tag{7}$$

Using the same method one can prove

Theorem 19. [K1] *Let $F:\mathbb{R}^n \to \mathbb{R}^N$ be a generic C^k smooth map with $k > n$, $N \geq n$ and $P:\mathbb{R}^N \to \mathbb{R}^n$ be a polynomial of degree d. Then the geometric multiplicity of germs of a chain map $P\circ F:\mathbb{R}^n \to \mathbb{R}^n$ admits the following upper estimate:*

$$\mu_a^G(P\circ F) \leq 2^{n(n-1)/2+1} (dn)^n, \ \forall\ a \in \mathbb{R}^n. \tag{8}$$

An interesting feature of this theorem is that the geometric multiplicity does not depend on the dimension N of the intermediate space.

Theorem 19 and reduction (4) give an estimate for the elementary bifurcation number $E(n)$ (see (2)). We shall sketch the proof of Theorem 18 in the next section.

3.3. **An estimate of geometric multiplicity of germs of generic maps.** In this section we split the proof of the estimate of multiplicity of germs of generic maps (Theorem 18) into two steps.

Denote by $J^k(\mathbb{R}^n, \mathbb{R}^n)$ the space of k-jets of maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. Fix coordinates in the source $(x_1, \dots, x_n)$ and the target $(y_1, \dots, y_n)$. Then the space $J^k(\mathbb{R}^n, \mathbb{R}^n)$ consists of

$$(x_1, \dots, x_n,\ (F_1(x), \dots, F_n(x)),$$

$$\frac{\partial^\alpha F_i}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n},\ \forall i = 1, \dots, n,\ \alpha_j \geq 0,\ \text{such that} \sum_{j=1}^{n} \alpha_j \leq k. \tag{9}$$

We shall call these coordinates on the k-jets space $J^k(\mathbb{R}^n, \mathbb{R}^n)$ *the natural coordinates.*

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a generic C^k-smooth map, $k > n$, and a be a point in $\mathbb{R}^n$. We shall describe how to estimate the maximal number of isolated preimages $\#\{x \in B_r(a) :\ F(x) = b\}$ of *regular values* b of the map F inside of a small enough ball $B_r(a)$ of radius r. The case of an arbitrary b can be treated in a similar fashion. By the definition of the geometric multiplicity if we can estimate the maximal number of preimages in the ball $B_r(a)$, then we can estimate the geometric multiplicity.

The first stage is an application of the Khovanskii method.

Theorem 20. [K1] *There exists a set of $(n+1)$ explicitly computable nontrivial polynomials $P^k = (P_1^k, \dots, P_n^k) : J^n(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^n$, $k = 0, 1, \dots, n$ defined on the space of n-jets $J^n(\mathbb{R}^n, \mathbb{R}^n)$ such that for a generic C^{n+1} smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$, for any regular value $b \in \mathbb{R}^n$ of the map F and any positive $r > 0$*

$$\begin{aligned} \#\{x \in B_r(a) : F(x) = b\} &\leq \#\{x \in B_r(a) : P_1^0 \circ j^n F(x) = \epsilon_1, \dots, P_n^0 \circ j^n F(x) = \epsilon_n\} \\ &+ \frac{1}{2} \sum_{k=1}^{n} \#\{x \in B_r(a) : P_1^k \circ j^n F(x) = \epsilon_1, \dots, P_n^k \circ j^n F(x) = \epsilon_n\}, \end{aligned} \tag{10}$$

where $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ decrease to zero sufficiently fast. The degrees of the polynomials satisfy inequalities $\deg P_i^k \leq 2^i n$ *for all k and i.*

Remark 4. *We can not find a direct reference in the book of Khovanskii* [Kh]*, but this theorem is in the spirit of the results about perturbations discussed in section* 5.2 *of this book. In fact this theorem is due to Khovanskii.*

Remark 5. *This theorem along with inequality (10) can also be proven for a chain map (3). If the degree of the exterior polynomial P of the chain map is at most d, then the degree of polynomials $\{P_i^k\}$ in inequality (10) satisfy inequalities* $\deg P_i^k \leq 2^i(dn)$ *for all k and i.*

A bit of terminology: "Replace an n-th jet $j^n F$ by its linear part at a point $a \in \mathbb{R}^n$" means "replace the map $j^n F : \mathbb{R}^n \to J^n(\mathbb{R}^n, \mathbb{R}^n)$ by its linear part $L_{F,a,n}$ at the point a".

By the phrase "a map $G: M \to N$ of manifolds satisfies *a transversality condition*" we mean that for some manifold (resp. a collection of manifolds) in the image N the map G is transversal to this manifold (resp. these manifolds).

The second stage consists in constructing a stratification of the n-jet space $J^n(\mathbb{R}^n, \mathbb{R}^n)$ (a decomposition into a disjoint union of manifolds described below) such that if the n-jet j^nF is transversal to all manifolds of this stratification, then the following theorem is true:

Theorem 21. *Let $P = (P_1, \dots, P_n)$ be a nontrivial polynomial defined on the space of n-jets $P: J^n(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^n$ and let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a C^k smooth map, $k > n$. Suppose the n-jet j^nF satisfies a transversality condition depending only on P. Then for a sufficiently small r one can replace in the statement of the previous theorem the n-jet j^nF at the point a by its linear part $L_{F,a,n}$. Namely,*

$$\begin{aligned} \#\{x \in B_r(a): \quad P_1 \circ j^nF(x) = \epsilon_1, \dots, P_n \circ j^nF(x) = \epsilon_n\} = \\ \#\{x \in B_r(a): \quad P_1 \circ L_{F,a,n}(x) = \epsilon_1, \dots, P_n \circ L_{F,a,n}(x) = \epsilon_n\}, \end{aligned} \tag{11}$$

where $\epsilon_1, \dots, \epsilon_n$ go to zero sufficiently fast. By Bezout's theorem the number of solutions to the equation in the right-hand side of (11) can be bounded by the product $\prod_{i=1}^n \deg P_i$.

The classical transversality theorem [AGV] says that for a generic map F its n-jet j^nF satisfies any ahead given transversality condition. It is easy to see that for a generic map $F: \mathbb{R}^n \to \mathbb{R}^n$ consecutive applications of Theorems 20 and 21 give the uniform estimate for the number of preimages of any regular value of the map F in a small ball $B_r(a)$.

In the next section we apply the Khovanskii method to prove Theorem 20 and exhibit the nature of the polynomials $\{P_i^k\}$ mentioned in Theorem 20. In section 4 we recall the notion of a_P-stratification and present the geometric idea which allows us to replace the jet map j^nF by its linear part (Theorem 21).

3.4. **The Khovanskii reduction method and Theorem 20.** The central idea of the Khovanskii reduction method is the Rolle lemma stated in the following way.

Lemma 1. *Consider C^2 Morse functions $f: S^1 \to \mathbb{R}^1$ on the circle and $g: [0,1] \to \mathbb{R}^1$ on the segment, i.e., functions with a finite number of critical points all of which are nondegenerate. Then for any $a \in \mathbb{R}$ and any small ϵ*

$$\begin{aligned} \#\{x: \ f(x) = a\} \le \#\{x: \ f'(x) = \epsilon\} \\ \#\{x: \ g(x) = a\} \le \#\{x: \ g'(x) = \epsilon\} + 1. \end{aligned} \tag{12}$$

Proof One proves first the formula for $\epsilon = 0$ and then uses nondegeneracy of critical points. Details are left to the reader.

Now using the Khovanskii reduction method we prove Theorem 20 in the case $n = 2$.

3.4.1. *A proof of Theorem 20 in the case $n = 2$.* Let us explain the inductive procedure that makes the Khovanski method work.

Denote by $B_r(a)$ the r-ball centered at a. Fix coordinate systems in the source and the target. Consider the polynomial function $\rho_r(x) = r - (x_1 - a_1)^2 - (x_2 - a_2)^2$. This function is positive inside $B_r(a)$ and vanishes on the boundary of $B_r(a)$. The function ρ_r is called *the covering function* of the ball $B_r(a)$.

Consider a C^k smooth function $G : \mathbb{R}^2 \to \mathbb{R}^2,\ k > 2$. Denote by G_1 and G_2 its coordinate functions. Denote the Jacobian of the map G by

$$J_G(x) = J_{(G_1,G_2)}(x) = \det \begin{pmatrix} G_{1x_1} & G_{1x_2} \\ G_{2x_1} & G_{2x_2} \end{pmatrix} = P_1 \circ j^1 G. \tag{13}$$

where $P_1 : J^1(\mathbb{R}^2, \mathbb{R}^2) \to \mathbb{R}$ is a polynomial, defined in the natural coordinates on the space of 1-jets; $\deg P = 2$[1].

Recall that a point $x \in \mathbb{R}^2$ is *nondegenerate or regular* if the rank of the derivative dG is maximal or equivalently $J_G(x) \neq 0$.

Lemma 2. *For any regular value $\tilde{b} = (\tilde{b}_1, \tilde{b}_2) \in \mathbb{R}^2$ of a map G and any positive r*

$$\#\{x \in B_r(a) :\ (G_1, G_2)(x) = (\tilde{b}_1, \tilde{b}_2)\} \leq$$
$$\#\{x \in B_r(a) :\ (G_1, J_G)(x) = (\tilde{b}_1, \epsilon)\} + \frac{1}{2}\#\{x \in B_r(a) :\ (G_1, \rho_r)(x) = (\tilde{b}_1, \epsilon)\}. \tag{14}$$

for any sufficiently small ϵ. The function $J_G(x)$ can be written in the form $J_G(x) = P_1 \circ j^1 G$, where P_1 is the polynomial, defined in the natural coordinates of the space of 1-jets. Moreover, $\deg P = 2$ (see (13)).

The function J_G is called *the contact function* for the system $(G_1, G_2)(x) = (\tilde{b}_1, \tilde{b}_2)$.

Proof Since $\tilde{b}$ is nondegenerate, the preimage $L_{\tilde{b}_1} = G_1^{-1}(\tilde{b}_1)$ is a 1-dimensional manifold consisting of compact connected parts— topological circles, denoted by $\{S_i\}_i$, and noncompact curves $\{\gamma_j\}_j,\ j = 1, \dots, k$ reaching the boundary ∂B_r. It is easy to see that

$$\#\{x \in B_r(a) :\ (G_1, G_2)(x) = (\tilde{b}_1, \tilde{b}_2)\} =$$
$$\sum_i \#\{x \in S_i :\ G_2(x) = \tilde{b}_2\} + \sum_j \#\{x \in \gamma_j :\ G_2(x) = \tilde{b}_2\}. \tag{15}$$

Let us estimate the first sum on the right-hand side. Fix a circle S_i. Restrict the second coordinate function G_2 to S_i and denote the result by $f_i = G_2|_{S_i} : S^1 \to \mathbb{R}$. We get a function f_i on the circle. Notice that *the condition $f_i'(x) = 0$ is equivalent to* $J_G(x) = 0$. So, we can apply Lemma 1 with $f = f_i$ and obtain

$$\sum_i \#\{x \in S_i :\ G_2(x) = \tilde{b}_2\} \leq \sum_i \#\{x \in S_i :\ J_G(x) = \epsilon\} \tag{16}$$

The second sum can be estimated in almost the same way. Instead of using Lemma 1 with $f = f_i$ we need to use Lemma 1 with $g = g_i = G_2|_{\gamma_j} : [0, 1] \to \mathbb{R}$. Thus

$$\sum_{j=1}^{k} \#\{x \in \gamma_j :\ G_2(x) = \tilde{b}_2\} \leq \sum_{j=1}^{k} \#\{x \in \gamma_j :\ J_G(x) = \epsilon\} + k \tag{17}$$

In order to find the number of component reaching the boundary notice that each such component intersects the sphere $\rho_r = \epsilon > 0$ in at least two points. This completes the proof of the Lemma. Q.E.D.

The following inductive application of Lemma 2 proves Theorem 20 in the case $n = 2$:

Take a regular value $b \in \mathbb{R}^2$ for a smooth map $F : \mathbb{R}^2 \to \mathbb{R}^2$.

[1] the meaning of the word “natural” see (9)

Step 1. Apply Lemma 2 with $(G_1, G_2) = (F_1, F_2)$, $(b_1, b_2) = (\tilde{b}_1, \tilde{b}_2)$, and $\epsilon_1 = \epsilon$. We get two additional functions J_F and ρ_r.

Step 2. Apply Lemma 2 with $(G_1, G_2) = (J_F, F_1), (\epsilon_1, b_2) = (\tilde{b}_1, \tilde{b}_2)$, and $\epsilon_1 = \epsilon$ and apply Lemma 2 with $(G_1, G_2) = (\rho_r, F_1)$, and $\epsilon_2 = \epsilon$. Notice that ϵ_2 could be much smaller than ϵ_1. The result is

$$\begin{aligned}\#\{x \in B_r(a) : (F_1, F_2)(x) = (b_1, b_2)\} &\leq \#\{x \in B_r(a) : (J_F, J_{(F_1, J_F)})(x) = (\epsilon_1, \epsilon_2)\} \\ &+ \frac{1}{2}\left(\#\{x : (J_F, \rho_r)(x) = (\epsilon_1, \epsilon_2)\} + \#\{x : (\rho_r, J_{(F_1, \rho_r)})(x) = (\epsilon_1, \epsilon_2)\}\right) \qquad (18)\end{aligned}$$

In order to represent the answer in the form of a chain map we use formula (13). Direct calculation shows that the functions J_F, $J_{(F_1, \rho_r)}$, and $J_{(F_1, J_F)}$ can be written in the form $J_F(x) = P_1 \circ j^1 F(x)$, $J_{(F_1, \rho_r)}(x) = P_2 \circ j^1 F(x)$, and $J_{(F_1, J_F)}(x) = P_3 \circ j^2 F(x)$ respectively, where the polynomials P_1, P_2 an P_3 can be computed explicitly in the natural coordinate system of the 2-jets space $J^2(\mathbb{R}^2, \mathbb{R}^2)$. This completes the proof of Theorem 20 in the case $n = 2$.

3.4.2. *The general case $n > 2$.* Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a generic C^k-smooth map, $k > n$, and let a be a point in $\mathbb{R}^n$. Fix coordinate systems in the image and the preimage. Let $F_1, \ldots, F_n$ denote the coordinate functions of F.

For a C^2 smooth function $G : \mathbb{R}^n \to \mathbb{R}^n$ we denote by $J_G(x)$ the determinant of the linearization of G at the point x. $\rho_r(x) = r^2 - \|x - a\|^2$ denotes the covering function for the r-ball $B_r(a)$ centered at the point a.

By analogy with Lemma 2 one can prove that for a regular value $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ of the map G, small ϵ, and any positive $r > 0$:

$$\begin{aligned}&\{x \in B_r(a) : \ (G_1, \ldots, G_s, \ldots, G_n)(x) = (b_1, \ldots, b_s, \ldots, b_n)\} \leq \\ &\{x \in B_r(a) : (G_1, \ldots, J_G, \ldots, G_n)(x) = (b_1, \ldots, \epsilon, \ldots, b_n)\} + \qquad (19) \\ &1/2\{x \in B_r(a) : (G_1, \ldots, \rho_r, \ldots, G_n)(x) = (b_1, \ldots, \epsilon, \ldots, b_n)\}.\end{aligned}$$

We shall say that we apply inequality (19) to the s-th coordinate function G_s.

The inductive application of this statement proceeds as follows: Start with a generic map $F : \mathbb{R}^n \to \mathbb{R}^n$. Take a regular value $\tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_n) \in \mathbb{R}^n$.

Step 1. Apply inequality (19) with $(\tilde{b}_1, \ldots, \tilde{b}_n) = (b_1, \ldots, b_n)$, $(F_1, \ldots, F_n) = (G_1, \ldots, G_n)$, and $\epsilon_1 = \epsilon$ to the first coordinate function. Let us estimate only the first term in the right-hand side. Denote the contact function J_F, which we get in this step, by J_1.

Step 2. Apply inequality (19) with $(\epsilon_1, \tilde{b}_2, \ldots, \tilde{b}_n) = (b_1, \ldots, b_n)$, $(J_1, F_2, \ldots, F_n) = (G_1, \ldots, \ldots, G_n)$, and $\epsilon_2 = \epsilon$ to the second coordinate function. Let us estimate only the first term on the right-hand side. Denote the contact function, which we get in this step, by J_2. Notice that the value of $\epsilon_2 = \epsilon$ for which inequality (19) applicable could be much smaller than ϵ_1.

Step s. Denote by $J_1, \ldots, J_{s-1}$ the contact functions from the previous $(s-1)$ steps. Apply inequality (19) with $(\epsilon_1, \ldots, \epsilon_{s-1}, \tilde{b}_s, \ldots, \tilde{b}_n) = (b_1, \ldots, \ldots, b_n)$, $(J_1, \ldots, J_{s-1}, F_s, \ldots, F_n) = (G_1, \ldots, \ldots, G_n)$, and $\epsilon_s = \epsilon$ to the coordinate function with index s. Notice that the value of $\epsilon_s = \epsilon$ for which inequality (19) applicable could be much smaller than all previous ϵ_i.

Using this inductive procedure one easily proves Theorem 20. All estimates for the degrees for the polynomial $\{P_j^k\}$ which we obtain by this procedure (compare with (18)), and which are given in the statement of Theorem 20, follow from

Proposition 1. *Let G be a C^{s+2} smooth map and let $\{Q_j\}_{j=1}^n$ be a set of polynomials defined on the space of s-jets (9) $Q_j : J^s(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$. Consider the map H defined by the coordinate functions $H = (Q_1 \circ j^s F, \dots, Q_n \circ j^s F)$. The Jacobian of this map $J_H(x)$ has the form $J_H(x) = P \circ j^{s+1} F$, where the polynomial P is defined on the space of $(s+1)$-jets. The degree of P is bounded by the sum of degrees of Q_j: $\deg P \le \sum_{j=1}^n \deg Q_j$.*

If a polynomial P is defined on the space of k-jets $J^k(\mathbb{R}^n, \mathbb{R}^n)$, then by the natural suspension of P it is also defined on the space of n-jets $J^n(\mathbb{R}^n, \mathbb{R}^n)$ for $n > k$. Therefore, in the statement of Theorem 20 on can consider all polynomials $\{P_j^k\}$ defined on the space of n-jets $J^n(\mathbb{R}^n, \mathbb{R}^n)$.

4. a_P-stratification and an estimate of the number of small inverse images of a chain map $P \circ F$ for a generic F.

In this section we shall sketch the proof of Theorem 21. The problem of estimating the maximal number of small isolated preimages is equally difficult for a chain map $P \circ F : \mathbb{R}^n \to \mathbb{R}^n$ with a generic map $F : \mathbb{R}^n \to \mathbb{R}^N$, $N \ge n$ and for a chain map $P \circ j^n F : \mathbb{R}^n \to \mathbb{R}^n$ with the n-jet of a generic map. We shall show that if the map F (resp. $j^n F$) satisfies a transversality condition in an appropriate space, then F (resp. $j^n F$) can be replaced by its linear part and we can apply the Bezout theorem to estimate the maximal number of small inverse images of the chain $P \circ F$ (resp. $P \circ j^n F$) uniformly over all sequences of numbers $\epsilon_1, \dots, \epsilon_n$ decreasing sufficiently fast to 0. So, to simplify notations we shall consider a chain map of the form $P \circ F : \mathbb{R}^n \to \mathbb{R}^n$.

4.1. **A Heuristic description.** Consider a chain map $P \circ F : \mathbb{R}^2 \to \mathbb{R}^2$, where $F : \mathbb{R}^2 \to \mathbb{R}^N$ is a generic C^k smooth map, $k > 2$ and $P = (P_1, P_2) : \mathbb{R}^N \to \mathbb{R}^2$ is a polynomial of degree d. Fix a small positive r. We would like to estimate the maximal number of small preimages

$$\#\{x \in B_r(0) : \ P_1 \circ F(x) = \epsilon, \ P_2 \circ F(x) = 0\} \tag{20}$$

for a small enough ϵ.

To show the idea put $N = 3$, $P_1(x, y, z) = x^2 + y^2$, and $P_2(x, y, z) = xy$. Assume also that $F(0) = 0$. Denote the level set by $V_\epsilon = \{P_1 = \epsilon, \ P_2 = 0\}$. The level set V_ϵ for $\epsilon > 0$ consists of 4 parallel lines (see Figure 6).

Notice that in our notation the number of intersections of $F(B_r(0))$ with V_ϵ equals the number of preimages of the point $(\epsilon, 0)$ (20).

It is easy to see from Figure 6 that if F is transversal to V_0 it is transversal to V_ϵ for any small $\epsilon > 0$. Moreover, the number of intersections $F(B_r(0))$ with V_ϵ equals 4 (see the points $P_1, \dots, P_4$ in Figure 6).

Another way to calculate the same number is as follows. Let us replace F by its linear part L_F at zero. Then $\#\{x \in B_r(0) : \ P_1 \circ F(x) = \epsilon, \ P_2 \circ F(x) = 0\} = \#\{x \in B_r(0) : \ P_1 \circ L_F(x) = \epsilon, \ P_2 \circ L_F(x) = 0\}$ and solving this polynomial system also yields 4.

The idea behind this picture is the following: Consider an arbitrary N and a polynomial $P = (P_1, P_2) : \mathbb{R}^N \to \mathbb{R}^2$ of degree at most d, $N > 2$. Define the semialgebraic variety $V_\epsilon = (P_1, P_2)^{-1}(\epsilon, 0)$ as the level set.

PSfrag replacements

V_ϵ
V_0
P_1
P_2
P_3
P_4
$F(B)$

FIGURE 6. The Idealistic Example

Assume for simplicity that for any small $\epsilon \neq 0$ the level set V_ϵ is a manifold of codimension 2. We shall get rid of this assumption later (see Theorem 30 b)). It turns out that there exists a stratification of V_0 by semialgebraic strata $(V_0, \mathcal{V}_0)$ (a decomposition of V_0 into a disjoint union of semialgebraic sets see definition 24), depending on P only, such that

$$\text{(21)} \quad \boxed{F \text{ is transversal to } (V_0, \mathcal{V}_0)} \quad \Longrightarrow \quad \boxed{F \text{ is transversal to } V_\epsilon}$$

Condition (21) is written for $n = 2$. Below we shall use its analogue for an arbitrary n.

Proposition 2. *Let $B_r(a)$ be the r-ball centered at the point $a \in \mathbb{R}^2$ and let $L_{F,a}$ denote the linearization of F at the point a. Under condition (21), the number of intersections of the image $F(B_r(a))$ with V_ϵ coincides with the number of intersections of the image $L_{F,a}(B_r(a))$ with V_ϵ, provided r is small enough. That is*

$$\begin{aligned} \#\{x \in B_r(0) :\ (P_1, P_2) \circ F(x) = (\epsilon, 0)\} = \\ \#\{x \in B_r(0) :\ (P_1, P_2) \circ L_{F,a}(x) = (\epsilon, 0)\}. \end{aligned} \tag{22}$$

The argument below is independent of the codimension of V_ϵ. We only need condition (21) and the fact that the codimension of V_ϵ coincides with the dimension of the preimage of a chain map $P \circ F$.

Proof

Consider the 1-parameter family of maps $F_t = tF + (1-t)L_F$ deforming the linear part of F into F. Clearly, $F_1 \equiv F$ and $F_0 \equiv L_F$. Fix a small $r > 0$. Since, F is transversal to V_0 at 0 all F_t are transversal to V_0 at 0. Condition (21) implies that for all small ϵ and all $t \in [0,1]$ F_t is transversal to V_ϵ.

Therefore, *the number of intersections of $F_t(B_r(0))$ with V_ϵ is independent of t.* Indeed, assume that $\#\{F_{t_1}(B_r(0)) \cap V_\epsilon\} \neq \#\{F_{t_2}(B_r(0)) \cap V_\epsilon\}$ for some $t_1 < t_2$. Then as t_1 increases to t_2 there is a point t^* where the number of intersections drops or jumps. At this point t^* the condition of transversality of F_{t^*} and V_ϵ must fail. This completes the proof of the proposition.

4.2. **Stratified manifolds.** Now we recall basic definitions from the theory of stratified sets.

Let M be a smooth manifold, which we call the *ambient manifold.* Consider a singular subset $V \subset M$. Roughly speaking a stratification of V is a decomposition of V into a disjoint union of manifolds (strata) $\{V_\alpha\}_\alpha$ such that *strata of bigger dimension are attached to strata of smaller dimension in a "regular" way.*

"Regular" will obtain a precise meaning in a moment, but the most important property is that *transversality to a smaller stratum implies transversality to an "attached" bigger stratum.* Now we are going to describe the standard language of stratified manifolds and maps of stratified manifolds. This goes back to Whitney and Thom [W], [Th].

Recall the Whitney Conditions (a) and (b). Condition (a) is similar to the notion of a_P-stratification due to Thom [Th] defined in the next subsection. We shall use a_P-stratification to prove condition (21).

Consider a triple (V_β, V_α, x), where $V_\beta,\ V_\alpha$ are C^1 manifolds, x is a point in V_β and $V_\beta \subseteq \bar{V}_\alpha \setminus V_\alpha$.

Definition 22. *A triple (V_β, V_α, x) satisfies the Whitney (a) condition if for any sequence of points $\{x_k\} \subset V_\alpha$ converging to a point $x \in V_\beta$ the sequence of tangent planes $T_k = T_{x_k}V_\alpha$ converges in the corresponding Grassmanian manifold of* $\dim V_\alpha$*-planes in* TM *and* $\lim T_k = \tau \supset T_x V_\beta$.

Definition 23. *A triple (V_β, V_α, x) satisfies the Whitney (b) condition if for any two sequences of points $\{x_k\} \subset V_\alpha$, $\{y_k\} \subset V_\alpha$ converging to a point $x \in V_\beta$ the sequence of "vectors" $\frac{y_k - x_k}{|y_k - x_k|}$ converges to a vector $v \in T_x M$ which belongs to a limiting position of* $\lim T_{x_k} V_\alpha = \tau$, *i.e.* $v \in \tau$.

Since condition (b) is local one can think of M as Euclidean. This explains how to interpret the vector $\frac{y_k - x_k}{|y_k - x_k|}$.

It is easy to show that condition (b) implies condition (a).

Definition 24. *A locally closed subset V in the ambient manifold M is called a stratified manifold (set, variety) in M, if it is represented as a locally finite disjoint union of smooth submanifolds V_α of M, called strata, of different dimensions in such a way that the closure of each stratum consists of itself and the union of some other strata of strictly smaller dimensions, and Condition (b) of Whitney is satisfied.*

Any union of submanifolds satisfying condition of this definition

$$V = \cup_\alpha V_\alpha \tag{23}$$

is called a *stratification* of V, and the submanifolds V_α are called *strata*. A set V is *stratifiable* if there is a "nice" partition into strata. By *a stratified manifold* we mean a pair $(V, \mathcal{V})$ consisting of a manifold V itself and a partition $\mathcal{V} = \{V_\alpha\}$.

4.3. **Stratified maps and a_P-stratification.** Now we define a smooth map of a stratified manifold $(V, \mathcal{V})$:

Definition 25. *Let $(V, \mathcal{V})$ be a stratified manifold in an ambient manifold M, $V \subseteq M$, then a map $f : V \to N$ is called C^2-smooth if it can be extended to a C^2 smooth map of the ambient manifold M $F : M \to N$ whose restriction to V coincides with f.*

A stratification $V = \cup_\alpha V_\alpha$ stratifies a smooth map $f : V \to \mathbb{R}^k$ if the restriction of f to any stratum V_α has constant rank, i.e., rank $df|_{V_\alpha}(x)$ is independent of $x \in V_\alpha$.

A map $G : L \to M$ is called transversal to a stratified set $(V, \mathcal{V})$ if G is transversal to each strata $V_\alpha \in \mathcal{V}$.

By the Rank Theorem, if a stratification $(V, \mathcal{V})$, $\mathcal{V} = \{V_\alpha\}_{\alpha \in I}$ stratifies a smooth map P, then for each strata V_α the number $d_\alpha(P) = \dim V_\alpha - \text{rank } dP|_{V_\alpha}$ is well defined.

Assume $d_\alpha(P) \geq d_\beta(P)$ for each $V_\beta \subseteq \bar{V}_\alpha \setminus V_\alpha$, i.e. nonempty level sets inside the bigger stratum V_α have dimension $d_\alpha(P)$ greater or equal to dimension of the level sets $d_\beta(P)$ in the smaller stratum V_β. We require that for any sequence of points $\{a_k\} \subset P(V_\alpha)$ converging to a point $a \in P(V_\beta)$, the nonempty level sets $\{P^{-1}(a_k) \cap V_\alpha\}$ approach the limiting level set $\{P^{-1}(a) \cap V_\beta\}$ "regularly". In other words, we *require that the level sets in the bigger stratum V_α approach the limit level set in the smaller stratum V_β nicely.*

Definition 26. *Let $P : M \to N$ be a C^2 smooth map of manifolds, and let V_β and V_α be submanifolds of M such that the restrictions $P|_{V_\beta}$ to V_β and $P|_{V_\alpha}$ to V_α have constant ranks $R_{V_\beta}(P)$ and $R_{V_\alpha}(P)$, respectively. Let x be a point in V_β.*

We call the manifold V_α a_P-regular over V_β with respect to the map P at the point x if for any sequence of points $\{x_n\} \subset V_\alpha$ converging to $x \in V_\beta$ the sequence of tangent planes to the level sets $T_k = ker\ dP|_{V_\alpha}(x_k)$ converges in the corresponding Grassmanian manifold of $(\dim V_\alpha - R_{V_\alpha}(P))$-dimensional planes to a plane τ and

$$\lim ker\ dP|_{V_\alpha}(x_k) = \tau \supseteq ker\ dP|_{V_\beta}(x) \tag{24}$$

Definition 27. *A C^2 smooth map $P : V \to N$ of a stratifiable manifold V to a manifold N is called a_P-stratifiable if there exist a stratification $(V, \mathcal{V})$ such that the following conditions hold:*

a) $(V, \mathcal{V})$ stratifies the map P (see definition 25);

b) for all pairs V_β and V_α from $\mathcal{V}$ such that $V_\beta \subseteq \bar{V}_\alpha \setminus V_\alpha$ the stratum V_α is a_P-regular over the stratum V_β with respect to P at point x for all $x \in V_\beta$.

The original definition of a_P-stratification requires an appropriate stratification of the image also [Ma], but we do not require stratification of the image for our purposes.

4.4. **Relation between existence of a_P-stratification and condition (21).** In section 4.1 we showed that the key to the proof of Theorem 21 is condition (21) (see Proposition 2). Now we are going to reduce the question whether condition (21) is satisfied to the question whether an a_P-stratification of the polynomial P exists.

Let $P = (P_1, P_2) : \mathbb{R}^N \to \mathbb{R}^2$ be a nontrivial polynomial, $V = P_2^{-1}(0)$ and $V_0 = (P_1, P_2)^{-1}(0)$ be level sets. Assume that there exists a stratification $(V, \mathcal{V})$ that stratifies the map $P|_V$ such that the zero level set V_0 can be represented as a union of strata from $\mathcal{V}$, i.e., $V_0 = \cup_{\alpha \in I_0} V_\alpha$. Denote this stratification of V_0 by $\mathcal{V}_0$. Recall that a map $F : \mathbb{R}^k \to \mathbb{R}^N$ is transversal to a stratification $(V_0, \mathcal{V}_0)$ if it is transversal to each strata $V_\alpha \in \mathcal{V}_0$. Associate to each level set V_ϵ, $\epsilon \neq 0$ a natural decomposition $\mathcal{V}_\epsilon = \{V_\epsilon \cap V_\alpha\}_{\alpha \in I}$.

Proposition 3. *With the above notation if a stratum $V_\alpha \in \mathcal{V} \setminus \mathcal{V}_0$ is a_P-regular over a stratum $V_\beta \in \mathcal{V}_0$ with respect to the polynomial P, then any C^2 smooth map $F : \mathbb{R}^n \to \mathbb{R}^2$ transversal to $(V_0, \mathcal{V}_0)$ is also transversal to $V_\epsilon \cap V_\alpha$ for any small ϵ. This is equivalent to condition (21).*

Proof

Pick a point x in $V_\beta \subset V_0$ and a point $y \in V_\alpha$. Notice that $ker\ dP|_{V_\beta}(x)$ is the tangent plane to the level set $\{P^{-1}(P(x)) \cap V_\beta\}$ at the point x and $ker\ dP|_{V_\alpha}(y)$ is the tangent plane to the level set $\{P^{-1}(P(y)) \cap V_\alpha\}$.

By condition (24) if a map $F : X \to \mathbb{R}^N$ is transversal to $ker\ dP|_{V_\beta}(x)$ at a point x, then F is transversal to $ker\ dP|_{V_\alpha}(y)$ for any $y \in V_\alpha$ near x.

Therefore, the condition "F is transversal to V_β at a point x" implies the condition "F is transversal to $V_\alpha \cap V_\epsilon$ for any small ϵ". This completes the proof.

4.5. **Existence of a_P-stratification for polynomial maps.** The existence of a_P-stratifications is not a trivial question. There are some obvious obstacles. For example, let $V \subset \mathbb{R}^n$ be an algebraic variety and let $P : \mathbb{R}^n \to \mathbb{R}^k$ be a polynomial map. Assume that $(V, \mathcal{V})$ stratifies P. If we have two strata V_α and V_β so that

V_α lies "over" $V_\beta \subseteq \bar{V}_\alpha \setminus V_\alpha$, then condition (24) can't be satisfied if dimension of the level sets $d_\alpha(P)$ in the upper stratum V_α is strictly less than that of $d_\beta(P)$ in the lower stratum V_β, i.e., $\dim ker\ dP|_{V_\alpha}(y) < \dim ker\ dP|_{V_\beta}$. In this case a plane $ker\ dP|_{V_\beta}(x)$ of the lower stratum V_β should belong to a plane τ of smaller dimension (see condition (24)), which is impossible. Thom constructed the first example when this happens [GWPL].

Thom's example

Consider the vector-polynomial P $P : (x, y) \to (x, xy)$. The line $\{x = 0\}$ is the line of critical points of P. Outside of the line $\{x = 0\}$ P is a diffeomorphism. Therefore, the preimage of any point $a \neq 0$ $P^{-1}(a)$ is 0-dimensional. On the other hand, the preimage of 0 is the line $\{x = 0\}$.

It turns out that even if we require that there exists a stratification $(V, \mathcal{V})$ which stratifies P and for any pair V_α and V_β from $\mathcal{V}$ such that $V_\beta \subseteq \bar{V}_\alpha \setminus V_\alpha$ dimensions of the levels $d_\beta(P)$ in the lower stratum V_β do not exceed dimensions of the level sets $d_\alpha(P)$ in the upper stratum V_α, a_P-stratifications still do not always exist. In [K1] we give a counterexample constructed by M.Grinberg. It seems that the existence of a counterexample was known before, but we did not find an appropriate reference.

Let us mention a positive result on existence of a_P-stratification.

Theorem 28. [Hir] *If $V \subset \mathbb{R}^n$ is a semialgebraic variety and $P : \mathbb{R}^n \to \mathbb{R}$ is a polynomial function, then there exists an a_P-stratification of $(V, \mathcal{V})$ with respect to P.*

Consider a nontrivial vector-polynomial $P = (P_1, \ldots, P_n) : \mathbb{R}^N \to \mathbb{R}^n$ defined by its coordinate functions. Recall that a polynomial P is nontrivial if the image $P(\mathbb{R}^N)$ has a nonempty interior in $\mathbb{R}^n$. However, it turns out that given level sets $\{P_1 = \epsilon_1, \ldots, P_n = \epsilon_n\}$, where $\epsilon_1, \ldots \epsilon_n$ decrease sufficiently fast we will still be able to construct an a_P-stratification.

The precise statement is as follows:

Definition 29. *Let $\mathbf{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{N}^{n-1}$, $\delta > 0$. We call the $(\mathbf{m}, \delta)$-cone $K_{\mathbf{m},\delta}$ the set of points of the form*

$$K_{\mathbf{m},\delta} = \{0 < a_{j+1} < (a_1 \ldots a_j)^{m_j},\ 0 < a_1 < \delta,\ j = 1, \ldots, n-1\}. \tag{25}$$

Define the following sets:

$$V_{\mathbf{m},\delta,P} = \text{closure}\left\{P^{-1}(K_{\mathbf{m},\delta})\right\},\ V_{0,\mathbf{m},P} = \cap_{\delta>0} V_{\mathbf{m},\delta,P}. \tag{26}$$

Then one has

Theorem 30. [K1] *For any nontrivial polynomial P there exists an integer vector $\mathbf{m} \in \mathbb{N}^n$ and positive δ such that the following conditions hold*

a) the set $V_0 = V_{0,\mathbf{m},P}$ (see (26)) is semialgebraic.

b) the set $V_{\mathbf{m},\delta,P}$ consists of regular points of P, i.e. if $b \in V_{\mathbf{m},\delta,P}$, then the level set $P^{-1}(b)$ is a manifold of codimension n.

c) there exists a stratification of V_0 by semialgebraic strata $(V_0, \mathcal{V}_0)$ satisfying the property: $V_{\mathbf{m},\delta,P}$ is a_P-regular over any strata $V_\alpha \in \mathcal{V}_0$ with respect to P.

Let F be a generic map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let us complete the proof of the fact that the maximal number of preimages of regular values of F in a small enough ball is bounded by $2^{n(n-1)/2+1}n^n$.

Let $\{P^k = (P_1^k, \dots, P_n^k)\}_{k=0}^n$ be nontrivial polynomials as in the statement of Theorem 20. All these polynomials are defined on the space of n-jets of the map F, $P^k : J^n(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^n$.

For each k from 0 to n by Theorem 30 (using the notation of that theorem) there exists an (m_k, δ_k)-cone $K_{m_k,\delta_k} \subset \mathbb{R}^n$ in the image of the k-th polynomial P^k and a stratified semialgebraic set $(V_0^k, \mathcal{V}_0^k) \subset J^n(\mathbb{R}^n, \mathbb{R}^n)$ such that the semialgebraic set V_{m_k,δ_k,P^k} is a_P-regular over $(V_0^k, \mathcal{V}_0^k)$ with respect to the polynomial P^k. Notice that $V_0^k \subseteq \left(P^k\right)^{-1}(0)$.

Assume that for each k from 0 to n the following is true: the n-jet j^nF is transversal to the semialgebraic set $(V_0^k, \mathcal{V}_0^k)$. If δ_k is small enough, then by Proposition 2 the n-jet j^nF is transversal to the level set $V_{\epsilon,k} = \left(P^k\right)^{-1}(\epsilon)$ for any $\epsilon \in V_{m_k,\delta_k,P^k}$. Consider the intersection of all (m_k, δ_k)-cones K_{m_k,δ_k} for $k = 0, 1, \dots, n$

$$K^* = \cap_{k=0}^n K_{m_k,\delta_k}.$$

Now take a regular value $\tilde{b} \in \mathbb{R}^n$. By Theorem 20 we can take a sequence of parameters $\epsilon_1, \dots, \epsilon_n$ decreasing sufficiently fast so that the point $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ belongs to the intersection cone K^* and inequality (10) holds.

By the transversality assumption made in the previous passage the n-jet j^nF is transversal to the level set $V_{\epsilon,k}$ for each $\epsilon \in K^*$ and for each $k = 0, 1, \dots, n$. Therefore, by Proposition 3 one can replace the map j^nF by its linear part $L_{F,n}$ in each chain map $P^k \circ j^nF$ $(k = 0, \dots, n)$ without changing the number of small preimages. Now we can apply Bezout's theorem to each map $P^k \circ L_{F,n}$ $k = 0, \dots, n$. To prove estimate (7) for all values of $\tilde{b}$ one has to use similar argument [K1].

4.6. **The Cartesian Transversality Theorem and Cartesian maps.** In order to apply the technic discussed above to the main problem of estimating the number of limit cycles (Theorem 15) we need a special transversality theorem.

Let $\mathbb{R}^m$ be Euclidean space with a fixed coordinate system $x = (x_1, \dots, x_k) \in \mathbb{R}^m$ (one variable x_i could denote more then one coordinate). Consider a map

$$F(x, \epsilon) = (f_1(x, \epsilon), \dots, f_k(x_k, \epsilon)) \tag{27}$$

depending on an n-dimensional parameter $\epsilon \in \mathbb{R}^n$. This map has the following property:

$$\frac{\partial f_i}{\partial x_j} \equiv 0, \quad i \neq j \tag{28}$$

Thus, a coordinate function f_i does not depend on the variable x_j for $j \neq i$. This is the form of a map F which appears inside of a chain map (3) after the Ilyashenko-Yakovenko reduction (4).

The space of maps with property (28) is called *the Cartesian space.* In the same way as we talk about genericity or transversality for usual maps we can talk about genericity or transversality for Cartesian maps. We can also define the space of jets of Cartesian maps as in the standard case; we call it *the space of Cartesian s-jets* denoted by $\mathbf{J}^s$.

The necessary ingredient of the proof of estimate for the elementary bifurcation number $E(n)$ (Theorem 15) is the following theorem.

The Cartesian Transversality Theorem *(Shelkovnikov* [IY]*) If S be a submanifold in the space of Cartesian s-jets $\mathbf{J}^s$, then the Cartesian maps whose Cartesian extension is transversal to S form a residual set in the corresponding space of C^{s+1} smooth Cartesian maps.*

Recall that a residual set is a set containing a countable intersection of open dense sets.

5. Bifurcation of spatial polycycles

When Ilyashenko and Yakovenko delivered the Finiteness Theorem for elementary polycycles (see section 3) in 1991, Arnold posed the question:

What can be said about bifurcations of spatial polycycles?

Another reason to look at this problem is the following compactness argument which is somewhat similar to the planar argument (the Poincaré-Bendixon theorem) given in section 1:

Consider a flow ϕ_t of finite codimension in $\mathbb{R}^3$ (3 will be replaced by an arbitrary dimension $N > 2$ below), i.e., a flow which occurs in a generic finite parameter family. Then ϕ_t has only isolated singular points. Fix a positive number L and assume that in a compact region of the phase space there are an infinite number of phase curves of length less L corresponding to periodic motions of ϕ_t. Then a subset of these periodic motions must accumulate to a separatrix polygon (polycycle).

The definition of a spatial polycycle of a flow ϕ_t in $\mathbb{R}^3$ is the same as definition 10.

Bifurcation properties of spatial polycycles are much richer then those of planar polycycles. The first important 3-dimensional feature is existence of limit cycles that winds several times around a polycycle. We call a periodic trajectory that "turns" around a polycycle m-times before closing up an *m-cycle.* On the plane only 1-cycles exist for topological reasons.

Consider an n-parameter family of flows $\{\phi_{t,\epsilon}\}_{\epsilon\in B^n}$ in $\mathbb{R}^3$.

Definition 31. *Suppose that for some ϵ^* the flow ϕ_{t,ϵ^*} has a polycycle γ. For any $m \in \mathbb{Z}_+$ we define m-cyclicity of the polycycle γ in the family $\{\phi_{t,\epsilon}\}_{\epsilon\in B^n}$, denoted by $C(m,\gamma)$, as the maximal number of m-cycles bifurcating in a tube neighborhood of γ in the family.*

Now we discuss a classical example of a polycycle which has infinite m-cyclicity for any $m \geq 1$.

5.1. **The Shilnikov polycycle.**

Consider a flow ϕ_t in $\mathbb{R}^3$ with a hyperbolic equilibrium point O that has one positive eigenvalue λ and two complex conjugates $\lambda+\mu$ with negative real part. Suppose that the sum of $\lambda + \mu$ is positive, and the unstable one-dimensional manifold returns to the stable one, which is two-dimensional. Thus, the equilibrium O has a homoclinic orbit that tends back to O along the unstable manifold as $t \to -\infty$, and spirals around O on the stable manifold as $t \to +\infty$. In 1965 Shilnikov discovered that the Poincaré map for such a polycycle restricted to a countable number of pairwise disjoint subdomains that form a Smale horseshoe on each subdomain. Any such horseshoe is structurally stable, therefore, the polycycle described above (the Shilnikov polycycle) has an infinite m-cyclicity for all $m \in \mathbb{Z}_+$ (see e.g.[GH]). The codimension of this polycycle is 1.

However, it seems reasonable to state the following

PSfrag replacements

Σ

O

FIGURE 7. The Shilnikov Polycycle

Conjecture *(Arnold-Ilyashenko-Yakovenko) If a spatial polycycle $\gamma \in \mathbb{R}^3$ has finite codimension k and all its equilibrium points are saddles with real eigenvalues or saddlenodes with at most one zero eigenvalue and the other eigenvalues are real, then the m-cyclicity of γ, $C(m, \gamma)$, is finite for all $m \in \mathbb{Z}_+$.*

Using the ideas and methods for the planar problem and some auxiliary results the Arnold-Ilyashenko-Yakovenko Conjecture has been solved in arbitrary dimension $N > 2$, but with additional assumptions on the equilibria.

5.2. **An estimate of the cyclicity of a quasielementary spatial polycycle.** In the planar case we considered polycycles with elementary equilibria only, now we define a class of points called *quasielementary equilibria*. The second author has shown that polycycles with quasielementary equilibria have finite m-cyclicity for any $m \geq 1$. Moreover, there exists an upper bound for m-cyclicity.

Recall some standard definitions from the normal form theory.

Definition 32. *The set of complex numbers $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ is called :*

a) nonresonant if there is no integral relation among the numbers λ_j of the form $\lambda_j = \sum_{i=1}^N k_i \lambda_i$, where $k_i \in \mathbb{Z}_+,\ i = 1, \dots, n$ and $\sum_{i=1}^N k_i \geq 2$.

b) strongly simply resonant if all the nontrivial resonance relations $\lambda_j = \sum_{i=1}^N k_i \lambda_i$ follows from the single one $\sum_{i=1}^N k_i^* \lambda_i = 0$, where $k_i \in \mathbb{Z}_+,\ i = 1, \dots, n$ and $\sum_{i=1}^N k_i \geq 2$.

Definition 33. *We shall call an equilibrium point of a differential equation quasielementary, if the linearization matrix of the equation at this point has only real eigenvalues, at most one of them is zero, and they satisfy one of the following conditions:*

1) they are nonresonant. We call such an equilibrium a nonresonant saddle;

2) they form a strongly simply resonant set of numbers— a strongly simply resonant saddle;

3) one eigenvalue is zero and the others form a nonresonant set— a saddlenode.

A polycycle is called quasielementary if all its vertices are quasielementary.

Note that the class of quasielementary points in the case of the plane ($N = 2$) coincides with the class of elementary points. Therefore, Theorem 35 below is a generalization of Theorem 15.

Definition 34. *The quasielementary bifurcation number $QE(N, n, m)$ is the maximal m-cyclicity of a quasielementary polycycle occurring in a generic n-parameter families of vector fields in $\mathbf{R}^N$.*

Theorem 35. [K2] *For any positive integer* $N, n, m,$ *and* $T = 6Nnm$

$$QE(N, n, w) \leq 2^{T^2}.$$

5.3. **An estimate of the cyclicity of a quasielementary polycycle.** In this section we briefly discuss the proof of Theorem 35. Let γ be a polycycle of codimension n in $\mathbb{R}^3$. We explain how to estimate the number of 2-cycles bifurcating from γ. The case of m-cycles $m > 2$ can be considered in a similar way.

Like in the planar case application of the normal form theory and the Khovanski method [K2] proves the following reduction.

First, we define a *multichain map* by

$$F : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}^{2n}, \quad (x, y) \mapsto P(f(x), f(y)), \tag{29}$$

where $\mathbb{B}^n \subset \mathbb{R}^n$ is the unit ball centered at the origin,

$$f : \mathbb{B}^n \to \mathbb{R}^N, \; P : \mathbb{R}^{2N} \to \mathbb{R}^{2n}, \tag{30}$$

P is a polynomial of degree at most d, f is either a generic map or a jet of a generic map.

The reduction is

$$\boxed{\text{An Estimate for the maximal \# of small solutions to } P \circ (f, f) = const} \implies \boxed{\text{An Estimate for the maximal 2-cyclicity of a quasielementary } \gamma} \tag{31}$$

Let us apply the strategy developed in section 3.3 to the multichain map (29). It turns out that Theorem 20 holds for a multichain map and after application we still have a set consisting of $(2n + 1)$ multichain maps. To apply Theorem 21 we have to make sure that for a generic map f a jet of 2-tuple $j(f, f)$ satisfies some transversality condition necessary for Theorem 21. This is necessary, since, the problem under consideration is to estimate the maximal number of small solutions to (29) for a *generic* f. However, the classical transversality theorem is not true for tuples (f, f). We shall avoid this obstacle below.

Let us call $J^0(\mathbb{B}^n, \mathbb{R}^N) \times J^0(\mathbb{B}^n, \mathbb{R}^N)$ *the multijet space* MJ_2. Suppose that f is sufficiently smooth.

Grigoriev and Yakovenko [GY] constructed the *space of divided differences or* DD_2*-space*, i.e., a smooth map $\nabla_2(f) : \mathbb{B}^n \times \mathbb{B}^n \to DD_2$, and an explicitly computable polynomial $\pi_2 : DD_2 \to MJ_2$ in such a way that the composition $\pi_2 \circ \nabla_2(f) : \mathbb{B}^n \times \mathbb{B}^n \to MJ_2$ coincides with the 2-tuple map $(f, f) : \mathbb{B}^n \times \mathbb{B}^n \to MJ_2$, i.e. the diagram in Figure 8 is commutative.

Moreover, by an arbitrary C^∞-small perturbation of f one can achieve any transversality condition for the map $\nabla_2(f) : \mathbb{B}^n \times \mathbb{B}^n \to DD_2$. Since transversality is an open property, it implies that for an open and dense set of f any ahead given transversality condition for the map $\nabla_2(f)$ holds.

Now we can represent the multichain map $P \circ (f, f)$ (29) in the form

$$P \circ (f, f) = (P \circ \pi_2) \circ \nabla_2(f), \tag{32}$$

where $P \circ \pi_2$ is a polynomial, since π_2 is a polynomial, and $\nabla_2(f)$ is a smooth map. Now we apply Theorem 21 to the chain map $(P \circ \pi_2) \circ \nabla_2(f)$. By the remark in the last passage for a generic f the map $\nabla_2(f)$ satisfies a transversality condition necessary to apply Theorem 21. In this way we can estimate the geometric multiplicity of the multichain map (29) and by reduction (31) prove Theorem 35 for $m = 2$.

PSfrag replacements $\mathbb{B}^n \times \mathbb{B}^n$ MJ_2 DD_2 π_2 $\nabla_2(f)$ (f, f)

FIGURE 8. Polynomial blow-up of the multijet space

In the case of m-cycles $m > 2$ the strategy is almost the same. Consider a multichain map (29) depending not on (f, f), but on the m-tuple $(f, \dots, f)$. Then one can prove the reduction (31) replacing 2-cycles by m-cycles and a multichain map $P \circ (f, f)$ depending on the 2-tuples (f, f) by a multichain map $P \circ (f, \dots, f)$ depending on the m-tuple $(f, \dots, f)$ [K2].

Let MJ_m be defined similarly to $m = 2$ (see above) and a Cartesian product $\mathbb{B}^n \times \cdots \times \mathbb{B}^n$ have m factors below. Grigoriev & Yakovenko constructed a space of divided differences DD_m such that if f is sufficiently smooth, then for a smooth map $\nabla_m(f) : \mathbb{B}^n \times \cdots \times \mathbb{B}^n \to DD_m$ and some polynomial $\pi_m : DD_m \to MJ_m$ the following formula holds:

$$P \circ (f, \dots, f) = (P \circ \pi_m) \circ \nabla_m(f). \tag{33}$$

Moreover, by an arbitrary C^∞-small perturbation of f one can achieve any transversality condition for $\nabla_m(f) : \mathbb{B}^n \times \cdots \times \mathbb{B}^n \to DD_m$ [GY]. Therefore, for a generic f one can apply theorems 20 and 21 consequently and obtain an estimate for the number of small inverse images of (29). This reduces the problem of the maximal number of m-cycles to the problem of multiplicity a multichain map. This completes the proof of Theorem 35 for $m > 2$.

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