SIMPLE RANDOM WALKS ALONG ORBITS OF ANOSOV DIFFEOMORPHISMS

V. YU. KALOSHIN , YA. G. SINAI *

1. Introduction

Consider an automorphism S of probability space $(M, \mathcal{M}, \mu)$. Any measurable function $p : M \to (0, 1)$ generates a Markov chain whose phase space is M and a particle at $x \in M$ jumps to Sx with probability $p(x)$ and to $S^{-1}x$ with probability $1 - p(x)$. We call such a Markov chain a *simple random walk* along orbits of S. Random walk is called *symmetric* if

$$\int \ln \frac{1 - p(x)}{p(x)} d\mu(x) = 0 \tag{1}$$

and non-symmetric otherwise. We implicitly assume that both integrals $\int \ln p(x)\ d\mu(x)$ and $\int \ln(1 - p(x))\ d\mu(x)$ are finite. It is natural to raise the following questions:

A) Does simple random walk have an invariant measure absolutely continuous wrt μ.

Denote by $\xi_x(t) \in \{S^m x, -\infty < m < \infty\}$ the position at time t of the moving particle which starts at x.

B) What is the limiting distribution of $\xi_x(n)$ on the space M as $n \to \infty$.

If $\xi_x(t) = S^{k_x(t)}x$, then $k_x(t+1) - k_x(t) = \pm 1$, i.e. $k_x(t)$ is a trajectory of a simple random walk, $k_x(0) = 0$. The transition probabilities of this walk are determited by the dynamics of S and the random media function p. The density π (wrt μ) of an invariant measure for our Markov chain satisfies the equation

$$\pi(x) = p(S^{-1}x)\pi(S^{-1}x) + (1 - p(Sx))\pi(Sx). \tag{2}$$

Therefore, the question A) is, in fact, the question of solvability of (2).

There are several cases where the answers to questions A) and B) are known. The case of a Diophantine rotation on the d-dimensional torus $\mathbb{T}^d$ was considered in [S1]. It was shown that the equation (2) always has a solution provided that the function $p(x)$ is smooth enough.

Moreover, for every x the distribution of $\xi_x(t)$ on $\mathbb{T}^d$ converges weakly to $\pi(y)dy$ as $t \to \infty$ for any x.

The non-symmetric case for an arbitrary ergodic measure preserving automorphism S was considered in [KS]. Non-symmetry implies a non-zero drift along trajectories of S. In this case the expression for π can be explicitly written (see [KS]). Namely, consider $r(x) = (1 - p(x))/p(x)$, $r_n(x) = \prod_{i=1}^{n} r(S^i x)$, and $\Lambda(x) = 1 + \sum_{n=1}^{\infty} r_n(x)$. Under the condition $\Lambda(x) \in L^1(M, \mu)$ the density $\pi(x) = \Lambda(x)/\int \Lambda(x)d\mu(x)$ and the answer to the question A) is positive. If some other functions defined with help of similar formulas as $\Lambda(x)$ belong to $L^1(M, \mu)$, then the answer to the question B) is also positive. We don't know what happens if $\Lambda(x) \notin L^1(M, \mu)$.

In this paper we consider the case of a transitive C^2-Anosov diffeomorphism S of a smooth compact manifold M (see [Bo]). The invariant measure μ is assumed to be Gibbs measure (more exact formulation of our assumtion will be given later) and the function $p \in C^2(M)$. The main result of this paper says that for a generic $p(x)$ (see below) the homological-like equation (2) has no measurable solution and if $P_x^{(t)}$ is the distribution of $\xi_x(t)$, then it has no weak limit as $t \to \infty$. Actually we shall prove that as t grows $P_x^{(t)}$ is concentrated on a small set which wanders around M. This situation was observed earlier in the case of random walks on $\mathbb{Z}^1$ in random enviroments [S2], [G], and [K] and this paper can be considered as a natural extension of [S2] and [G].

Let us give more precise formulation of our results. We begin with the invariance principle which we need. Consider $\eta(x) = \log[(1 - p(x))/p(x)]$. The symmetry condition means that $E\eta(x) = \int \eta(x)d\mu(x) = 0$. Define

$$
\begin{aligned}
\eta_x(n) &= \sum_{m=0}^{n-1} \eta(S^m x),\ n > 0,\ \eta_x(0) = 0, \\
\eta_x(n) &= -\sum_{m=-n}^{-1} \eta(S^m x),\ n < 0.
\end{aligned}
\tag{3}
$$

Interpolate $\eta_x(t)$ linearly for non-integer $t \in \mathbb{R} \setminus \mathbb{Z}$, $\eta_x(t) = (t - n)\eta_x(n) + (n + 1 - t)\eta_x(n + 1)$ for $n \le t \le n + 1$.

Invariance principle. *The probability distributions induced by μ and S on the set of functions $\omega_t(u) = \frac{1}{\sqrt{\sigma t}}\eta_x(tu)$, where $\sigma = \sigma(p)$ is a constant depending on the random media function p, $\omega_t(0) = 0$, converge weakly as $t \to \infty$ to the standard independent Wiener measures for both $u > 0$ and $u < 0$.*

It is known that invariance principle holds for a wide class of invariant measures of Anosov diffeomorphisms (see [Bo], [S3]).

Theorem 1. *If a measure μ on M, a map S, and a random media function p are such that the invariance principle holds, then the simple random walk along orbits of S has no absolutely continuous invariant measure with respect to μ.*

Theorem 1 follows from Theorem 2 which describes properties of localization of the distribution $P_x^{(t)}$ of particle starting from x for large values of time t. Let $A(x) = \{p(S^m x), -\infty < m < +\infty\}$. We shall denote by $P\{\ \cdot \mid A(x)\}$ probabilities of events related to simple random walks on $\mathbb{Z}^1$with these transition probabilities.

Theorem 2. *Under the same condition as in Theorem 1 for any $\epsilon > 0$ and any $0 < \rho < 1/2$ one can find $T = T(\epsilon, \rho) \in \mathbb{Z}^+$ and a constant $C = C(\epsilon, \rho)$ such that for $t > T$ one can find a subset $R_{\epsilon,\rho,t} \subset M, \mu(R_{\epsilon,\rho,t}) \geq 1 - \epsilon$ and functions $a_t^-(x,\epsilon), m_t(x), a_t^+(x,\epsilon)$ such that*

$$\begin{aligned}
&a_t^-(x,\epsilon) < 0 < a_t^+(x,\epsilon),\\
&a_t^-(x,\epsilon) < m_t(x) < a_t^+(x,\epsilon), \text{ and}\\
&C^{-1}(\epsilon)\ln^2 t < |a_t^+(x,\epsilon)|,\ |a_t^-(x,\epsilon)| < C(\epsilon)\ln^2 t, \qquad (4)\\
&\text{for each } k \in [a_t^-(x,\epsilon), a_t^+(x,\epsilon)] \text{ and } \xi_{S^k x} = S^m x\\
&P\{\ |m - m_t(x)| < \ C(\epsilon)\ (\ln\ln t)^{2+\rho} \mid A(x)\} > 1 - \epsilon\}
\end{aligned}$$

In other words, for a typical initial point x one can find a piece of the trajectory of x, containing x, whose length is not less that $C^{-1}(\epsilon)\ln^2 t$ and such that any point $\xi_{S^k x}(t)$ for $S^k x$ from this piece lies within a $C(\epsilon)\ln\ln t$-neighbourhood of some point $S^{m_x(t)}x$ with probability at least $1-\epsilon$.

Theorem 1 is derived from Theorem 2 in §2 and Theorem 2 is proven in §3. §4 contains proofs of auxiliary lemmas. The second author acknowledges the finantial support from RFFI, grant # 96-01-0037 and NSF, grant # DMS-9706794.

2. Derivation of Theorem 1 from Theorem 2

The following Theorem links Theorems 1 and 2.

Theorem 3. *For symmetric random walk in Theorem 1 for any $\epsilon > 0$ one can find $T = T(\epsilon)$ such that for any $t > T$ there exist subsets $F_t \subset M$ for which $\mu(F_t) < \epsilon$ and*

$$\int P_x^{(t)}(F_t)\,d\mu(x) > 1 - \epsilon, \qquad (5)$$

where $P_x^{(t)}(F_t) = P\{\xi_x(t) \in F_t| \ A(x)\}$.

Let us show that Theorem 3 implies Theorem 1. Assume that there exists an absolutely continuous measure ν with respect to μ which is invariant for our random walks. Consider a sequence $\epsilon_j > 0, \ \epsilon_j \to 0$ as $j \to \infty$. By Theorem 3 there is a sequence of subsets $R_{t_j}, \ F_{t_j} \subset M$ such that $\mu(R_{t_j}) > 1 - \epsilon_j$, $\mu(F_{t_j}) < \epsilon_j$, $t_j \geq t_j$, and

$$\int P_x^{(t_j)}(F_{t_j}) d\mu(x) \to 1 \text{ as } j \to \infty. \tag{6}$$

Let us show that

$$\mu(M \setminus F_{t_j}) \to 1 \text{ and } \nu(M \setminus F_{t_j}) \to 0 \text{ as } j \to \infty, \tag{7}$$

which clearly contradicts absolute continuity. The left limit is a direct consequence of the choice of subsets $\{F_{t_j}\}_j$. Now we prove the right limit.

Assuming absolute continuity one can find For any $\delta > 0$ a constant $c = c(\delta)$ such that $\int_{x:\ \frac{d\nu(x)}{d\mu(x)} \geq c} d\nu(x) \leq \delta$. Choose t_j to be sufficiently large. Then

$$\nu(M \setminus F_{t_j}) \leq \int_M P_x^{(t_j)}(M \setminus F_{t_j}) d\nu(x) + \delta \leq \delta +$$

$$\int_{\{x:\ \frac{d\nu(x)}{d\mu(x)} \geq c\}} P_x^{(t_j)}(M \setminus F_{t_j}) d\nu(x) + c \int_{\{x:\ \frac{d\nu(x)}{d\mu(x)} < c\}} P_x^{(t_j)}(M \setminus F_{t_j}) d\mu(x) \leq$$

$$2\delta + \int_{\{x:\ \frac{d\nu(x)}{d\mu(x)} < c\} \cap R_{t_j}} P_x^{(t_j)}(M \setminus F_{t_j}) d\mu(x) + c\mu(M \setminus R_{t_j}) \leq$$

$$3\delta + c \int_{R_{t_j}} (1 - P_x^{(t_j)}(F_{t_j})) d\mu(x) \leq 4\delta,$$

since one can take t_j arbitrary large independetly of δ. This proves Theorem 1.

Now we shall prove Theorem 3 using Theorem 2. Choose a large enough t. One can find a measurable partition ζ of M such that an a.e. element of ζ is an interval of a trajectory whose length lies between $\frac{b}{2} \ \ln^2 t$ and $b \ln^2 t$, where $b < C^{-1}/2$. Equivalently, one can find a set B_N such that for each $x \in B_N$ the first return time to B_N is between $\frac{b}{2} \ln^2 t$ and $b \ln^2 t$. Each element C_ζ of the partition η can be represented as $C_\zeta = \{S^k x : 0 \leq k \leq R(x)\}$, where $R(x)$ is the first return time, i.e. $S^k x \notin B_N$, $0 < k < R(x)$ and $b/2 \ln^2 t \leq R(x) \leq b \ln^2 t$. The conditional measure on C_η is the uniform measure, because

μ is invariant wrt T. If we want to stress the dependance of C_ζ on $x \in B_N$ we write $C_\zeta(x)$.

Consider the set I_t of elements C_ζ such that $\mu(R|C_\zeta) > 0$. Since $M \setminus I_t \subset M \setminus R$, we conclude that $\mu(I_t) \geq 1 - \epsilon$.

It follows from Theorem 2 that for each $C_\zeta(x) \in I_t$ one can find an interval on the trajectory of x

$$L(x) = \{x' = S^m x : \ |m - m_t(x)| < C(\ln\ln t)^{2+\rho}\}, \ x' \in C_\zeta(x) \cap R_{\epsilon,t},$$

such that

$$P\{\xi_{S^k x}(t) \in L(x) \mid A(x)\} > 1 - \epsilon/3 \tag{8}$$

for every k for which $S^k x \in C_\zeta(x)$. Put $F_t = \cup_{C_\zeta(x) \subset I_t} L(x)$. Then

$$\int_{I_t} P_x^{(t)}(F_t) d\mu(x) \geq (1 - \epsilon/3)\mu(I_t) \geq 1 - \epsilon \tag{9}$$

for sufficiently small ϵ.

Let us show that $\mu(F_t) \leq \frac{C(\ln\ln t)^{2+\rho}}{b\ln^2 t}$. Indeed, for a.e. y consider a long interval $\{S^m y, 0 \leq m \leq M\}$ for μ-a.e. y and let $M \to \infty$. The partition ζ decomposes this interval into subintervals except may be two boundary intervals. For a typical y the fraction of intervals belonging to I_t is greater than $1 - \epsilon/3$. It follows from Theorem 2 that for each of these intervals (recall that each has length in between $\frac{b}{2}\ln^2 t$ and $b\ln T$) there is an interval of the length less than $C(\ln\ln t)^{2+\rho}$ such that *all* particles starting from a $\approx b\ln^2 t$-interval will be concentrated with high probability in $C(\ln\ln t)^{2+\rho}$-interval. Therefore, the fraction of points belonging to F_t is less than $\frac{C(\ln\ln t)^{2+\rho}}{b\ln^2 t}$. This gives the required statement and proves Theorem 3.

3. Construction of wells

In this section we describe the construction of wells from [S1]. Our description relies on [K]. Construction of wells makes explicit functions $a_t^\pm(x,\epsilon)$, and $m_t(x)$ from the formulation of Theorem 2. Recall that $\eta(x) = \log\frac{1-p(x)}{p(x)}$,

$$\eta_x(n) = \begin{cases} \sum_{m=0}^{n-1} \eta(S^m x) & n > 0, \\ 0 & n = 0, \\ -\sum_{m=n}^{-1} \eta(S^m x) & n < 0. \end{cases} \tag{10}$$

For $t \notin \mathbb{R} \setminus \mathbb{Z}$ the value $\eta_x(t)$ is defined by the linear interpolation. Now we shall investigate the graph of the function $\eta_x(t)$ for a large t and a "generic" x and, in particular, its local maxima and minima.

Take $N \in \mathbb{Z}_+$. Define

$$
\begin{aligned}
\tau^+ =& \max\{k > 0,\ k \in \mathbb{Z} : \eta_x(k) \geq N\}, \\
\tau^- =& \max\{k < 0,\ k \in \mathbb{Z} : \eta_x(k) \geq N\}, \\
m =& \{t_0 :\ \eta_x(t_0) = \min_{\tau^- \leq t \leq \tau^+} \eta_x(t)\}.
\end{aligned} \tag{11}
$$

The triple $[\tau^-, m, \tau^+]$ is called a *well* of the path $\eta_x(t)$. The *depth of the well* $[\tau^-, m, \tau^+]$ is $d[\tau^-, m, \tau^+] = \min\{\eta_x(\tau^-) - \eta_x(m), \eta_x(\tau^+) - \eta_x(m)\}$.

If $[\tau^-, m, \tau^+]$ is a well and $\tau^- < m_1 < \tau_1 < m$ are such that

$$
\begin{aligned}
\eta_x(m_1) &= \min_{\tau^- \leq t \leq \tau_1} \eta_x(t), \\
\eta_x(\tau_1) - \eta_x(m_1) &= \max_{m_1 \leq t' \leq t'' \leq \tau_1} (\eta_x(\tau') - \eta_x(\tau'')),
\end{aligned} \tag{12}
$$

then $[\tau^-, m_1, \tau_1]$ and $[\tau_1, m, \tau^+]$ are also wells. This operation is called a *left refinement* of $[\tau^-, m, \tau^+]$. In the same way one can define a right refinement.

Now take $t \in \mathbb{Z}_+$. It follows from the Invariance Principle that for μ-a.e. $x \in M$ there exists

$$
\begin{aligned}
\tilde{a}_t^+(x) &= \{\tau \in \mathbb{Z}_+ :\ \eta_x(\tau) \geq \ln t + (\ln t)^{1/2}\}, \\
\tilde{a}_t^-(x) &= \{\tau \in \mathbb{Z}_- :\ \eta_x(\tau) \geq \ln t + (\ln t)^{1/2}\}, \\
\tilde{m}_t(x) &= \{\tau \in \mathbb{Z} :\ \eta_x(\tau) = \min_{s \in [\tilde{a}_t^-(x), \tilde{a}_t^+(x)]} \eta_x(s)\}.
\end{aligned} \tag{13}
$$

Moreover, as it will be shown later on there exists the smallest integer-valued well $[a_t^-(x), m_t(x), a_t^+(x)]$ obtainable by a finite number of refinements from $[\tilde{a}_t^-(x), \tilde{m}_t(x), \tilde{a}_t^+(x)]$ such that $a_t^-(x) < 0 < a_t^+(x)$ and the depth $d[a_t^-(x), m_t(x), a_t^+(x)] > \ln t + (\ln t)^{1/2}$. Let us call $m_t(x)$ a t-th bottom point of x.

We shall need the notion of zone of attraction of each t-th bottom $m_t(x)$. Let us give another description of the process of construction of $a_t^{\pm}(x)$ and $m_t(x)$. This construction will show uniqueness of the smallest integer-valued well $[a_t^-(x), m_t(x), a_t^+(x)]$ for μ-a.e. t. We rely on the exposition in [K] (in particular, on Lemma 2.1).

Consider the initial well $[\tilde{a}_t^-(x), \tilde{m}_t(x), \tilde{a}_t^+(x)]$ defined above. Put

$$
\begin{aligned}
S_t^+(x) &= \min\{\eta_x(\tau) : \ 0 \le \tau \le \tilde{a}_\tau^+(x)\}, \\
S_t^-(x) &= \min\{\eta_x(-\tau) : \ 0 \le \tau \le -\tilde{a}_\tau^-(x)\}, \\
T_t^+(x) &= \inf\{\tau : \ \eta_x(\tau) - S_\tau^+(x) = \ln t + (\ln t)^{1/2}\}, \\
T_t^-(x) &= -\inf\{\tau : \ \eta_x(-\tau) - S_\tau^-(x) = \ln t + (\ln t)^{1/2}\}, \\
m_t^+(x) &= (\text{time when } S^+(T^+) \text{ is achieved}), \\
m_t^-(x) &= \ -(\text{time when } S^-(T^-) \text{ is achieved}), \\
M_t^+(x) &= \max\{\eta_x(\tau) : \ 0 \le \tau \le m_t^+(x)\}, \\
M_t^-(x) &= \max\{\eta_x(\tau) : \ m_t^-(x) \le \tau \le 0\}, \\
\hat{m}_t^+(x) &= \min\{\tau : \ \eta_x(\tau) = M_t^+(x)\}, \\
\hat{m}_t^-(x) &= \min\{\tau : \ \eta_x(-\tau) = M_t^-(x)\}.
\end{aligned} \tag{14}
$$

Let us impose the condition that local minima/maxima, which we defined above, are Δ-separated one from another, where $\Delta > 0$ is a constant. The smaller is Δ, the bigger is the probability that this happens (see Lemma 1 below).

Definition 4. *A point $x \in M$ is called (Δ, t)-separated ($\Delta > 0$) if the following inequalities are satisfied*

$$
\begin{aligned}
&|\hat{m}_t^+(x)|, |\hat{m}_t^-(x)| > \Delta \ln^2 t, \ |m_t^\pm(x) - \hat{m}_t^\pm(x)| > \Delta \ln^2 t, \\
&\left| \frac{\eta_x(m_t^+(x))}{\eta_x(m_t^-(x))} - 1 \right| > \Delta, \ |M_t^\pm(x)| > 2\Delta(\ln t + (\ln t)^{1/2}), \\
&\left| \frac{M_t^\pm(x)}{\max\{M_t^\mp(x), \ \eta_x(m_t^\mp(x)) + \ln t + (\ln t)^{1/2}\}} - 1 \right| > \Delta.
\end{aligned} \tag{15}
$$

Lemma 1. *For any $\delta > 0$ there exists $T = T(\delta)$ and $\Delta = \Delta(\delta)$ such that for any $t > T$ one can find a set $R_{\delta,t} \subset M$, $\mu(R_{\delta,t}) > 1 - \delta$ and each point $x \in R_{\delta,t}$ is (Δ, t)-separated. Moreover, if $\xi_x(t) = S^m x$, then one of the two inequalities is satisfied:*

$$
P\left\{ \frac{|m - m_t^+(x)|}{\ln^2 t} < \Delta | \ A(x) \right\} > 1 - \delta
$$

$$
P\left\{ \frac{|m - m_t^-(x)|}{\ln^2 t} < \Delta | \ A(x) \right\} > 1 - \delta.
$$

The first inequality is satisfied if and only if one of the following conditions hold:

$$
\begin{gathered}
\eta_x(m_t^-(x)) > \eta_x(m_t^+(x)) \text{ and} \\
M_t^+(x) < \max\{M_t^-(x), \eta_x(m_t^-(x)) + \ln t + (\ln t)^{1/2}\};
\end{gathered} \tag{16}
$$

PSfrag replacements

$M_t^-(x)$
$M_t^+(x)$
$\hat{m}_t^+(x)$
$\hat{m}_t^-(x)$
$m_t^+(x)$
$m_t^-(x)$
$T_t^-(x)$
$T_t^+(x)$
$\eta_x(t)$
$\ln t + (\ln t)^{1/2}$
t

FIGURE 1. Local minima and maxima in a well

$$
\begin{aligned}
&\eta_x(m_t^-(x)) < \eta_x(m_t^+(x)) \text{ and} \\
&M_t^-(x) < \max\{M_t^+(x), \eta_x(m_t^+(x)) + \ln t + (\ln t)^{1/2}\}.
\end{aligned} \tag{17}
$$

Otherwise, the second inequality is satisfied.

The Proof of this Lemma is almost the same as the proof of Lemma 2.1 in [K]. It is easy to see that this Lemma implies uniqueness of the smallest integer-valued well. Indeed, an application of Lemma 1 gives the t-th bottom point $m_t(x)$ and the ends of the smallest well can be defined as

$$
\begin{aligned}
a_t^+(x) &= \min\{\tau \in \mathbb{Z}_+ : \ \eta_x(\tau) - \eta_x(m_t(x)) \geq \ln t + (\ln t)^{1/2}\}, \\
a_t^-(x) &= \min\{\tau \in \mathbb{Z}_- : \ \eta_x(\tau) - \eta_x(m_t(x)) \geq \ln t + (\ln t)^{1/2}\}.
\end{aligned} \tag{18}
$$

Definition 5. *A point $x \in M$ is in the t-zone of attraction of the t-th bottom point $S^{m_t(x)}x$ and $m_t(x) = m_t^+(x)$ if one of the conditions (16) is satisfied. Otherwise, $m_t(x) = m_t^-(x)$.*

This is an alternative way to define the t-th bottom point $m_t(x)$. Lemma 1 shows that $m_t(x)$ is defined in a unique way. Denote $y_t(x) = S^{m_t(x)}x$

Suppose x is in the t-zone of attraction of $y_t(x) = S^{m_t(x)}x$ and x is (Δ, t)-separated for some $\Delta = \Delta(\delta)$. Define

$$a_t^-(x,\delta) = \min\{\hat{m}_t^-(x) \le \tau < 0 : \ \eta_x(t) \le M_t^-(x) - \Delta(\ln t + (\ln t)^{1/2})\},$$

where the smaller is δ, the smaller is Δ. In a similar way one can define $a_t^+(x,\delta)$. It is easy to see from Lemma 1 that for each $k \in [a_t^-(x,\delta), a_t^+(x,\delta)]$ the point $y = S^k x$ is in the t-zone of attraction of the same point $y_t(x)$.

Define a random variable (stopping time)

$$\tau_k(x) = \min\{\tau : \xi_{S^k x}(\tau) = y_t(x) \mid A(x)\}. \tag{19}$$

Lemma 2. *With the above notations and notations of Lemma 1 let $x \in R_{\delta,t}$ and belong to the t-zone of attraction of $y_x(t)$. Then*

$$P\left\{ \max_{a_t^-(x,\delta) \le k \le a_t^+(x,\delta)} \tau_k(x) \ge \Delta t \mid A(x) \right\} < \delta \tag{20}$$

and the average with respect to x

$$P\left\{ \max_{a_t^-(x,\delta) \le k \le a_t^+(x,\delta)} \tau_k(x) \ge \Delta t \right\} \to 0 \text{ as } t \to \infty. \tag{21}$$

Proof of the Lemma is given in §4.

Lemma 3. *In notations of Lemma 2 if $\xi_{y_t(x)}(\tau) = S^{m(\tau)}y_t(x)$, then*

$$\begin{aligned} &P\ \{ \max_{0\le\tau\le t} m(\tau) < a_t^+(x,\delta) - m_t(x) \text{ or} \\ &\min_{0\le\tau\le t} m(\tau) < a_t^-(x,\delta) - m_t(x) \mid A(x)\} < \frac{\delta}{|a_t^+(x,\delta) - a_t^-(x,\delta)|}. \end{aligned} \tag{22}$$

Proof of the Lemma is given in §4.

Lemma 2 implies that with high probability all particles starting from the interval $[a_t^-(x,\delta), a_t^+(x,\delta)]$ will hit the t-th bottom point earlier than Δt. Lemma 3 implies that with high probability a particle starting from the t-bottom point $y_t(x)$ will stay inside the interval $[a_t^-(x,\delta), a_t^+(x,\delta)]$ for a time at least t.

Lemma 4. *In notations of Lemma 3 for any $\delta > 0$ and $0 < \rho < 1/2$ one can find a set $\tilde{R}_{\delta,\rho} \subset M$, $\mu(\tilde{R}_{\delta,\rho}) > 1 - \delta/2$ and constants $C = C(\delta,\rho), n = n(\delta,\rho)$ such that if $x \in \tilde{R}_{\delta,\rho}$ and $S^{m_t(k,x)}x = \xi_{S^k x}(t)$, then for any $N > n$*

$$P\{|m_t(k,x) - m_t(x)| > CN \mid A(x)\} < \exp(-N^{1/2-\rho/5}). \tag{23}$$

Since the size of each t-zone of attraction of $x \in \tilde{R}_\delta$ is at most $C \ln^2 t$ we get

Corollary 1. *In notations of Lemma 4*

$$P\left\{\max_{a_t^-(x,\delta)\le k\le a_t^+(x,\delta)} |m_t(k,x)-m_t(x)| > C(\ln\ln t)^{2+\rho}|\ A(x)\right\} < \delta.$$

4. Proof of the Lemmas 1–4.

Proof of the Lemma 1. Let us show that the condition that x is (Δ,t)-separated follows from the weak convergence to the Wiener measure. Let us write down explicitly the continuous functionals of a Brownian motion corresponding to the limits of $\frac{M_t^\pm(x)}{\ln t}, \frac{T_t^\pm(x)}{\ln^2 t}, \frac{m_t^\pm(x)}{\ln^2 t}, \frac{\hat m_t^\pm(x)}{\ln^2 t}$ as $t\to\infty$.

Let $W(t)$ be a standard Brownian motion. Define the following random variables following [K]:

$$\begin{aligned}
S^+(t) &= \min\{W(\tau):\ 0\le\tau\le t\},\\
S^-(t) &= \min\{W(-\tau):\ 0\le\tau\le t\},\\
T^+ &= \ \inf\{\tau:\ W(\tau)-S^+(\tau)=1\},\\
T^- &= -\inf\{\tau:\ W(-\tau)-S^-(\tau)=1\},\\
m^+ &= \ (\text{time when } S^+(T^+) \text{ is achieved}),\\
m^- &= -(\text{time when } S^-(T^-) \text{ is achieved}),\\
M^+ &= \max\{W(\tau):\ 0\le\tau\le m^+\},\\
M^- &= \max\{W(\tau):\ m^-\le\tau\le 0\},\\
\hat m^+ &= \min\{\tau:\ W(\tau)=M^+\},\\
\hat m^- &= \min\{\tau:\ W(-\tau)=M^-\}.
\end{aligned}\tag{24}$$

It is clear these functionals are the limits of $\frac{T_x^\pm(t)}{\ln^2 t}, \frac{m_x^\pm(t)}{\ln^2 t}, \frac{M_x^\pm(t)}{\ln^2 t}$, and $\frac{\hat m_x^\pm(t)}{\ln^2 t}$ introduced above. Therefore, the distributions of the functionals $\frac{T_x^\pm(t)}{\ln^2 t}, \frac{m_x^\pm(t)}{\ln^2 t}, \frac{M_x^\pm(t)}{\ln^2 t}, \frac{\hat m_x^\pm(t)}{\ln^2 t}$ converge to the distribution of $T^\pm, m^\pm, M^\pm$, and $\hat m^\pm$ (see Theorem 5.1 in [Bi], p.30). The explicit expressions of these distributions are of no importance to us, we shall only use the fact that they are absolutely continuous and concentrated on the appropriate segments in $\mathbb{R}$. Thus choosing δ small enough one can satisfy conditions (5) with probability $> 1-\delta$. In order to prove the second part of the Lemma 1 one should use Lemma 2.1 from [K], where the similar statement is proven. This completes the proof of Lemma 1.

Proof of Lemma 2 is almost the same as the proof of Lemma 1 in [G], where a similar statement is proven.

Proof of the Lemma 3. Let us define a stopping time

$$\tau_t(x) = \min\{\tau : \xi_x(\tau) = S^{a_t^+(x,\delta)-m_t(x)} y_t(x) \text{ or } \xi_x(\tau) = S^{a_t^-(x,\delta)-m_t(x)} y_t(x)\}.$$

Since, $|a_x^+(t,\delta) - a_x^-(t,\delta)| < C \ln^2 t$ it is enough to prove that $P\{\tau_x(t) < 3t|\ A(x)\} \leq 3exp(-(\ln t)^{1/2})$ is small compared with $\frac{b}{Cln^2 t}$. By Lemma 7 in [G] for any $z \geq 1$

$$\begin{aligned} P\{\min\{\tau : \xi_{y_t(x)}(\tau) = S^z y_t(x)\} < N|A(x)\} \geq \\ N \exp\{-\max_{0 \geq j \geq z} (\eta_{y_t(x)}(z-1) - \eta_{y_t(x)}(j))\}. \end{aligned} \tag{25}$$

Thus, $P\{\tau_x(t) < 3t|\ A(x)\} \leq 3t \exp(-\ln t - (\ln t)^{1/2}) = 3\exp(-(\ln t)^{1/2})$. This proves Lemma 3.

Proof of the Lemma 4. Let us consider the random environment A_x inside the t-th well. Define

$$\theta(z)_{x,t} = \eta_{y_t(x)}(z) = \eta_x(m_t(x) + z) - \eta_x(m_t(x)).$$

It follows from the Invariance Principle and the properties of the ladder epochs and the ladder heights (see [Fe], pp 395, 575) that for any $\rho > 0$ and z sufficiently large, but $|z| < \Delta \ln^2 t$, $\theta_{x,t}(z)|z|^{-1/2+\rho/5} \to +\infty$. Therefore, the environment around a t-th bottom point $m_t(x)$ is steep. Since the probability of a particle starting at $y_t(x) = S^{m_t(x)}x$ to leave $[a_t^-(x,\delta), a_t^+(x,\delta)]$ is negligibly small, put a reflecting barriers at $a_t^-(x,\delta)$ and $a_t^+(x,\delta)$ and consider the Markov chain inside this well. This Markov chain is stationary and ergodic. Let us compute the invariant distribution for this Markov chain and show that for this invariant distribution for the probability being CN apart from the bottom $m_t(x)$ decays at least as $\exp(-N^{1/2-\rho/5})$.

Denote $a_x^{\pm}(x,\delta) - m_x(t)$ by $l_x^{\pm}$, $l_x^- < 0 < l_x^+$, $p_i = p(S^i y_x(t))$, $i \in [l_x^-, l_x^+]$. The invariant distribution $\{\pi_i\}_{i \in [l_x^-, l_x^+]}$ has the form

$$\begin{aligned} \pi_i = \frac{1}{C} \prod_{j=0}^{i-1} \frac{p_j}{1 - p_{j+1}} \text{ for } 1 \leq i < l_x^+, \pi_0 = \frac{1}{C}, \\ \pi_i = \frac{1}{C} \prod_{-j=0}^{-i} \frac{p_j}{1 - p_{j+1}} \text{ for } 1 \leq -i < l_x^-, \\ \pi_{l_x^+} = p_{l_x^+} \pi_{l_x^+ - 1},\ \pi_{l_x^+} = (1 - p_{l_x^- + 1}) \pi_{l_x^- + 1}, \end{aligned} \tag{26}$$

and C is such that $\sum_{i=l_x^-}^{l_x^+} \pi_i = 1$. Notice that in our notations $\pi_z = \frac{1-p_0}{Cp_z} \exp(-\theta_{x,t}(z))$ and for any $\rho > 0$ and sufficiently large z

$$\theta_{x,t}(z)|z|^{-1/2+\rho/5} \to +\infty.$$

Therefore, for any z large enough, but $|z| < \Delta \ln^2 t$

$$\pi_z \leq (\text{const})\ \exp(-|z|^{1/2-\rho/5}). \tag{27}$$

This completes the proof of Lemma 4.

Proof of the Corollary To prove the Corollary notice that for any large enough $z > 0$, but $z < \Delta \ln^2 t$ Lemma 4 implies that the probability of a particle to be z-apart from the t-bottom point decays as $\exp(-|z|^{1/2-\rho/5})|z|^{1/2+\rho/5}$. Then if $C > 0$ is sufficiently large and $z = C(\ln\ln t)^{2+\rho}$

$$\begin{aligned} &\exp\left\{-C\left((\ln\ln t)^{2+\rho}\right)^{1/2-\rho/5}\right\}\left|C(\ln\ln t)^{2+\rho}\right|^{1/2+\rho/5} > \\ &\qquad \exp(-C\ln\ln t)\ln\ln t > (\ln t)^{C/2} > C(\ln t)^2. \end{aligned} \tag{28}$$

This completes the proof of the Corollary. The Corollary in turn implies Theorem 2.

References

[Bo] R. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, LNM, v.470, Springer-Verlag, Berlin-New York, 1975.

[Bi] P. Billingsley, Convergence of Probability Measures, Wiley, 1968

[G] A. Golosov, Localization of Random Walks in one-dimensional Random Enviroment, Comm. Math. Physics, **92**, 491–506 (1984)

[Fe] W. Feller, Introduction to Probability Theory and its application, vol.**2**, New York, London, Wiley, 1966

[K] H. Kesten, The Limit Distribution of Sinai's Random walk in a Random Enviroment, Physica A 138 (1986), 299–306.

[KS] V. Kaloshin, Ya. Sinai, Nonsymmetric Random Walks along orbits of Ergodic Automorphisms, Publ. AMS (in press)

[S1] Ya. Sinai, The limit behaviour of a one-dimensional random walk in a random enviroment, Theory of Prob. and its Application, no.**2**, vol.27, (1982), 247–258

[S2] Ya. Sinai, Simple Random Walks on Tori, J.Stat. Physics, vol.**94**, # 3/4, 1999, 695–708

[S3] Ya. Sinai, Topics in Ergodic Theory. Princeton Mathematical Series, **44**. Princeton University Press, Princeton, NJ, 1994

E-mail address: `kaloshin@math.princeton.edu`

E-mail address: `sinai@math.princeton.edu`