SUMMARY OF PUBLICATIONS AND PREPRINTS

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Numerations of the papers according to the webpage.

1. Smooth dynamical systems: Growth of the number of periodic points and Newhouse phenomenon

1.1. **Smooth Dynamical Systems.** Let M be a compact manifold, $\text{Diff}^r(M)$ be the space of C^r diffeomorphisms of M. Define for any diffeomorphism $f \in \text{Diff}^r(M)$ number of isolated periodic points of period n by

$$P_n(f) = \#\{\text{isolated } x \in M : \ f^n(x) = x\}. \tag{1}$$

We investigate growth of the number of periodic points with n. Artin–Mazur'65, using Nash's approximations, proved that for a dense set of diffeomorphisms number of periodic points $P_n(f)$ growth at most exponentially fast, i.e. for some $C > 0$

$$P_n(f) \le \exp(Cn) \text{ for any } n \in \mathbb{Z}_+ \text{ and some } C > 0. \tag{2}$$

Call diffeomorphisms satisfying (2) *Artin-Mazur (A-M) diffeomorphisms.*

In [I.1] we give an elementary proof of an extension of Artin–Mazur's result, namely, we prove that A-M diffeomorphisms with only *hyperbolic* periodic points are dense in $\text{Diff}^r(M)$.

In [I.2] we construct Baire generic diffeomorphisms inside of an open set (a Newhouse domain) in $\text{Diff}^r(M)$ of with *superexponential* (even an arbitrary fast) growth of number of periodic points. This theorem is based on a theorem of Gonchenko-Shilnikov-Turaev, which is also rigorously proved in [I.2]. In [I.3] we outline ideas from the proofs in [I.1], [I.2], [I.5 -I.6], and prove some new results.

We call a set of diffeomorphisms *prevalent* if in a sense it has "probability one". In [I.5 -I.6] we prove that for a prevalent diffeomorphism (or with "probability 1") for any positive $\epsilon > 0$ there is $C = C(\epsilon, f)$ such that

$$P_n(f) < \exp(Cn^{1+\epsilon}) \text{ for any } n \in \mathbb{Z}_+. \tag{3}$$

This gives a partial solution of a problem posed by Arnold about prevalence of Artin-Mazur diffeomorphisms. A new machinery of perturbation by Newton Intepolation Polynomials is developed there. Announcement is in [I.4].

In [I.7] using Newton Intepolation Polynomials we give a partial solution to Palis' conjecture about infinitely many coexisting sinks, namely, we show that only a finite number of sinks with bounded cyclicity (a natural notion of combinatorial complexity) can coexist for typical diffeomorphism.

In [I.8] we construct a C^r smooth unimodal map of an interval having arbitrary fast growth of the number of periodic points.

2. Instabilities in Hamiltonian systems and Arnold diffusion

In [II.1] we give simple topological proof of well-known result of Mather (we call it Mather connecting theorem), which says that inside of Birkhoff Region of Instability there are trajectories connecting any pair of Aubry-Mather sets. Then we apply this to give an alternative prooof of Mather accelerating theorem. The latter says that for a generic time-periodic mechanical system on 2-torus there are trajectories whose speed is unbounded.

In [II.2] we give a short introduction into fascinating Mather theory, weak KAM, and viscosity solutions of Hamilton-Jacobi equation and show interconnections.

In [II.3] we present a new mechanism of Arnold diffusion for high dimensional nearly integrable Hamiltonian system: *diffusion along long resonances.* The mechanism is based on theory of normally hyperbolic invariant manifolds, methods of generating functions, Aubry-Mather theory, and Mather's variational methods. Using this mechanism we construct small, but not too small analytic time-periodic perturbation of integrable Hamiltonian system such that it has linearly diffusing trajectories. This is a counterpart of Nekhoroshev estimates, improving Herman-Sauzin-Marco results.

In [II.4] we apply a deep result of Mather about Arnold diffusion for a priori stable nearly integrable Hamiltonian systems of 2.5 degrees of freedom to prove that a generic resonant totally elliptic point of 4-dimensional symplectic map is Lyapunov unstable.

In [II.5] we apply Aubry-Mather theory and prove existence of outer periodic and quasiperiodic motions for Planar Circular Restricted 3 Body Problem. In particular, our results hold for practical range of energy and mass ratio. For example, assuming that orbit of Pluto is in the plane of Sun-Jupiter, we prove existence of periodic and quasiperiodic motions for the system whose parameters are close to those of Sun-Jupiter-Pluto system.

3. Bifurcation theory and the Hilbert 16-th problem

The main result of the author in the subject is in [III.3], there the author solves a particular case of so-called Local Hilbert-Arnold Problem, extending results of Ilyashenko-Yakovenko. The results are obtained using theory of stratified sets, Khovanski's method, and theory of normal forms. Hilbert-Arnold naturally arises while trying to prove the second part of the Hilbert 16th Problem. The result from [III.3] is announced in [III.1]. The proof is outlined in [III.2]. In [III.4] the author discusses ideas from the proof in [III.3]. Here is short outline of relation between Local Hilbert-Arnold Problem and the second part of the Hilbert 16th.

Consider a polynomial line field on the real (x, y)-plane

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}, \; P, Q - \text{ polynomials in } x, y. \tag{4}$$

Then denote by $H(n)$ the uniform bound for the number of limit cycles (isolated periodic solutions) of (4) taken over all P, Q such that $\deg P, Q \leq n$.

The Hilbert 16-th problem. *Find $H(n)$ or bound for it.*

Existential Hilbert 16-th problem (EHP). *Prove $H(n) < \infty$ for any $n \in \mathbf{N}$.*

Whether the number of limit cycles for an *individual* field (4) is finite or not is called *Dulac's problem.* There are two different solutions due to Ilyashenko and Ecalle. Both proofs *do not give* a uniform bound over all polynomial fields of fixed degree.

Finiteness Conjecture (FC). (see e.g. Roussari) *Any finite parameter family of analytic vector fields on S^2 has a uniformly bounded number of limit cycles.*

Call *polycycle* a polygon γ consisting of equilibrium points (as vertices) and arcs (integral curves). Define *cyclicity* of a polycycle γ, denoted by $\mu(\gamma)$, the maximal number of coexisting limit cycles bifurcating (appearing after a perturbation) in a tube neighbourhood of γ. One can show (see e.g. Roussari) that

$$\boxed{\mu(\gamma) < \infty \text{ for any } \gamma} \quad \Longrightarrow \quad \boxed{\text{F C}} \quad \Longrightarrow \quad \boxed{EHP}$$

In 1985 Arnold proposed an analog of EHP or FC, so called **Hilbert-Arnold Problem (HAP)**: *prove that a generic finite parameter family of C^∞ vector field $\dot{x} = v(x, \epsilon),\ x \in S^2,\ \epsilon \in B$ on the sphere S^2 with a compact base B has a uniformly bounded number of limit cycles.* Similarly to the analytic case HAP follows from so-called **Local Hilbert-Arnold Problem (LHAP)**: *prove that cyclicity of any polycycle of C^∞ vector field occuring in a gereric finite parameter family has finite cyclicity.* A partial solution to this problem is obtained.

An equilibrium point point is called *elementary* if has at least one nonzero eigenvalue. A polycycle is called *elementary* if all its vertices are elementary. Codimension of a polycycle γ is minimal k so that γ occurs in a generic k-parameter family. "Loosely speaking" codimension is number of vertices.

Using theory of stratified sets, Khovanski's method, and normal forms the author [III.3] proves that $E(k)$, which is the maximal possible cyclicity of a elementary polycycle of codimension k, is bounded by

$$E(k) \leq 2^{25k^2} \text{ for any } k \in \mathbb{Z}_+.$$

This gives a partial solution to Local Hilbert-Arnold Problem.

4. Random flows and random walks along orbits of dynamical systems

In series of papers [IV.1 -IV.4] we consider an oil spill Ω on the surface of ocean $\mathbb{R}^2$. Ocean evolves in time and the ultimate goal is to remove the spill Ω_t after time $t > 0$. The problem gives rise to various questions about size and shape of Ω_t.

We model motion of an oil spot by *a stochastic differential equation* driven by a finite-dimensional Brownian motion. Let $\Omega \subset \mathbb{R}^2$ be an open set and let Ω_t evolves according to that equation. We prove the following results:

Central Limit Theorem [IV.2] for distribution of the spill holds true. This, in particular, says that for any $t > 0$ and some large $C > 0$ with probability 99% the ball of

radius $C\sqrt{t}$ contains 99% of the spill Ω_t.
Shape Theorem [IV.3]: Define the set of poison points by time t by $W_t = \{x \in R^2 : d(x, \Omega_s) < 1 \text{ for } 0 < s < t\}$. Then there is a nonrandom convex set $B \subset \mathbb{R}^2$ such that for any $\epsilon > 0$ almost surely we have

$$(1-\epsilon)tB \subset W_t \subset (1+\epsilon)tB.$$

In [IV.1] we show that Hausdorff Dimension of ballistic points, i.e. those escaping linearly in time, is maximal possible. We summarize the results in [IV.4].

In [IV.5-IV.6] we analyze the following question: Consider an automorphism S of probability space $(M, \mathcal{M}, \mu)$. Any measurable function $p : M \to (0, 1)$ generates a Markov chain whose phase space is M and a particle at $x \in M$ jumps to Sx with probability $p(x)$ and to $S^{-1}x$ with probability $1 - p(x)$. We call such a Markov chain a *simple random walk* along orbits of S. Random walk is called *symmetric* if $\int \ln \frac{1-p(x)}{p(x)} d\mu(x) = 0$ and non-symmetric otherwise. We implicitly assume that both integrals $\int \ln p(x)\, d\mu(x)$ and $\int \ln(1-p(x))\, d\mu(x)$ are finite. It is natural to raise the following questions:

A) Does simple random walk have an invariant measure absolutely continuous wrt μ?

Denote by $\xi_x(t) \in \{S^m x, -\infty < m < \infty\}$ the position at time t of the moving particle which starts at x.

B) Does the limiting distribution of $\xi_x(n)$ on the space M converges to such a measure as $n \to \infty$?

In [IV.5] we show that for diophantine rotations of the torus $\mathbb{T}^d$ both questions have positive answer. However, in [IV.6] we show that for Anosov diffeomorphisms both questions have a negative answer, moreover, a strong form of localization holds. This is closely related to random walks in random environment.

5. Surprising properties of fractal sets under projections

Consider a compact set X in $\mathbb{R}^N$ of Hausdorff dimension D. The classical result of Martrand–Matilla says that for an almost every linear projection of X into a subspace of dimension at least D Hausdorff dimension is preserved, i.e. Hausdorff dimension of X and its image are the same.

Now consider a compact set X in a Banach space B and its image under linear projection into a finite dimensional space. Wide range of dissipative PDE's proved to have an attractor in a Banach space of solutions. To investigate attractors numerically they are embedded into a finite-dimensional space. It turns out that preservation of Hausdorff dimension is *no longer true*. In [V.2] we construct a compact set of Hausdorff dimension D in the real Hilbert space l^2 such that for *all* linear projections π of B into $\mathbb{R}^n$, no matter how large n is, the Hausdorff dimension of $\pi(X)$ is less than 2 no matter how large D is. In [V.2] we also introduce a new characteristic of infinite-dimensional compact set called *thickness*. It characterizes how infinite dimensional a set is. We prove asymptotically sharp preservation results with respect to thickness. This generalizes the Martrand–Matilla result to infinite-dimensional setting.

In fractal geometry there is a useful potential-theoretic formula for Hausdorff dimension introduced by Frostman

$$D_2(\mu) = \sup\left\{s : \int\int \frac{d\mu(x)\ d\mu(y)}{|x-y|^s} < \infty.\right\}. \tag{5}$$

In [V.1] we found the generalization of this formula to the part $(q > 1)$ of the Renyi-Grassenberg-Procaccia dimension spectrum of fractal measures

$$D_q(\mu) = \sup\left\{s : \int \left(\int \frac{d\mu(x)}{|x-y|^s}\right)^{q-1} d\mu(y) < \infty.\right\}. \tag{6}$$

Using this formula we extended classical results of Marstrand–Mattila about preservation of Hausdorff dimension under a.e. projection to preservation of a part of the dimension spectrum $(1 < q \leq 2)$ of fractal measures under a.e. projection. For the other q's counterexamples of nonpreservation of $D_q(\mu)$ are presented.

In [V.3] we improve certain exponents in preservation results from [V.2]

6. Notion of Prevalence or Probability one in nonlinear infinite-dimensional spaces

In [VI.1] the author proposes a way to define notion of *prevalence* or "*with probability one*" in nonlinear infinite-dimensional space so that natural properties hold. With this definition the author proves that various properties from dynamical systems and singularity theory are prevalent. Main examples are Thom transversality theorem, multi-jet transversality theorem, Mather stability theorem, and Kupka-Smale theorem. In [VI.2 -VI.3] a preliminary version of this definition is given.

7. Diophantine properties of SO(3)

In [VII.1] we discuss properties of elements of noncompact group $SO(3)$: consider Haar random pair $A, B \in SO(3)$ and define the closest distance of irreducible words of length n to the indentity Id as follows: for a multiindex $\mathcal{I} = (i_1, j_1, \dots, i_m, j_m)$ let $|\mathcal{I}| = \sum(|i_k| + |j_k|)$ and $W_{\mathcal{I}}(A, B) = A^{i_1}B^{j_1} \dots A^{i_m}B^{j_m}$. Define $s_{A,B}(n) = \min_{|\mathcal{I}|} \operatorname{dist}(W_{\mathcal{I}}(A, B), Id)$. Gamburd-Jakobson-Sarnak conjecture says that $s_{A,B}(n)$ decays exponentially. We partially solve that by proving that for Haar almost every pair $A, B \in SO(3)$ for some $D = D(A, B) > 0$ we have $s_{A,B} \geq D^{-n^2}$.

8. Whitney stratification

In [VIII.1] we give an elementary proof of a classical result of Whitney, which says that any analytic set has a (Whitney) stratification, partition into smooth parts so that these parts fit together regularly.

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