# Semi-infinite linear algebra

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## 1 Determinants and traces of infinite matrices

Let $V$ be a vector space over a field $k$. We can make sense of the trace of a linear endomorphism $f \in \mathrm{End}(V)$ as long as $f$ has finite rank. Namely, the finite rank operators are precisely the image of the canonical injection

$$V^* \otimes V \longrightarrow \mathrm{End}(V),$$

and under this identification trace corresponds to the evaluation pairing $V^* \otimes V \to k$.

The Lie algebra $\mathfrak{gl}(V)$ has underlying vector space $\mathrm{End}(V)$ with the Lie bracket given by the commutator

$$[f, g] := f \circ g - g \circ f.$$

The subspace $\mathfrak{gl}_f(V) \subset \mathfrak{gl}(V)$ consisting of finite rank maps forms a Lie ideal. It is not hard to see that the trace map

$$\mathrm{tr} : \mathfrak{gl}_f(V) \longrightarrow k$$

has the same cyclic invariance property as the usual trace, and in particular tr is a Lie algebra homomorphism when $k$ is given the trivial Lie bracket.

What about determinants? We will use the following formula as motivation.

**Proposition 1.1.** *If $V$ is a finite-dimensional vector space and $f \in \mathrm{End}(V)$, then*

$$\det(f) = \sum_{r=0}^{\dim V} \mathrm{tr}(f - \mathrm{id}_V; \textstyle\bigwedge^r V).$$

Note that if $V$ is infinite-dimensional and $f - \mathrm{id}_V$ has finite rank, then the expression

$$\sum_{r=0}^{\infty} \mathrm{tr}(f - \mathrm{id}_V; \textstyle\bigwedge^r V) \tag{1.1}$$

still makes sense, because $f - \mathrm{id}_V$ acts by zero on $\bigwedge^r V$ for $r$ greater than its rank. So we define the determinant of an operator $f$ such that $f - \mathrm{id}_V$ has finite rank by the formula (1.1). This determines a group homomorphism

$$\det : \mathrm{GL}_f(V) \longrightarrow k^\times,$$

where $\mathrm{GL}_f(V) \subset \mathrm{GL}(V)$ is the subgroup consisting of invertible operators $f$ such that $f - \mathrm{id}_V$ has finite rank.

## 2 The finite-dimensional setting

A finite-dimensional vector space $V$ over $k$ has two invariants: its dimension and determinant line. We combine them into the datum of a graded line, so that

$$\det V := \textstyle\bigwedge^{\dim V} V$$

is placed in degree $\dim V$. Any isomorphism $f : V \xrightarrow{\sim} W$ of finite-dimensional vector spaces determines an isomorphism of graded lines $\det f : \det V \xrightarrow{\sim} \det W$. Alternatively, we can view $\det f$ as a trivialization of (i.e. nonzero vector in) the degree zero line $(\det V)^{-1} \otimes \det W$. In particular, when $V = W$ then $\det f \in k$ is the determinant of $f$ in the traditional sense.

The determinant construction has following additional feature: a short exact sequence

$$0 \longrightarrow V \longrightarrow W \longrightarrow U \longrightarrow 0$$

of finite-dimensional vector spaces gives rise to an isomorphism of graded lines

$$\det W \xrightarrow{\sim} \det V \otimes \det U. \tag{2.1}$$

If $V = 0$ or $U = 0$ then this reduces to the aforementioned determinant of an isomorphism. Using induction we see that if $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ is a filtration, then there is a canonical isomorphism

$$\det V \xrightarrow{\sim} \det(\operatorname{gr} V),$$

where $\operatorname{gr} V := \bigoplus_i V_i/V_{i-1}$.

Observe that from the two short exact sequences

$$0 \longrightarrow V \longrightarrow V \oplus W \longrightarrow W \longrightarrow 0$$

and

$$0 \longrightarrow W \longrightarrow V \oplus W \longrightarrow V \longrightarrow 0,$$

we can use (2.1) to produce isomorphisms of graded lines

$$\det(V \oplus W) \xrightarrow{\sim} \det V \otimes \det W \text{ and } \det(V \oplus W) \xrightarrow{\sim} \det V \otimes \det W.$$

In fact, these fit into a triangle

$$\begin{array}{ccc} & \det(V \oplus W) & \\ \swarrow & & \searrow \\ \det V \otimes \det W & \longrightarrow & \det W \otimes \det V, \end{array}$$

where the horizontal isomorphism sends $\omega \otimes \eta \mapsto (-1)^{\dim V \cdot \dim W} \eta \otimes \omega$. This implies that for any direct sum $\bigoplus_i V_i$ indexed by a finite set $I$, there is a canonical isomorphism

$$\det(\textstyle\bigoplus_i V_i) \xrightarrow{\sim} \textstyle\bigotimes_i \det V_i$$

which does not depend on a choice of ordering for $I$.

If $f : V \to W$ is any linear map, then the short exact sequences

$$0 \longrightarrow \ker f \longrightarrow V \longrightarrow \operatorname{im} f \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{im} f \longrightarrow W \longrightarrow \operatorname{coker} f \longrightarrow 0$$

give rise to an isomorphism of graded lines

$$(\det \ker f)^{-1} \otimes \det \operatorname{coker} f \xrightarrow{\sim} (\det V)^{-1} \otimes \det W. \tag{2.2}$$

Note that this isomorphism asserts in particular that

$$\dim W - \dim V = \dim \ker f - \dim \operatorname{coker} f,$$

a version of the rank-nullity theorem. If $f$ is an isomorphism, so that the line on the left is canonically trivial, then (2.2) is just $\det f$.

# 3 Fredholm operators

The following definition is borrowed from functional analysis.

**Definition 3.1.** A linear map $f : V \to W$ is called *Fredholm* if $\ker f$ and $\operatorname{coker} f$ are finite-dimensional. Its *index* is the integer

$$\operatorname{ind} f := \dim \operatorname{coker} f - \dim \ker f.$$

It is not hard to see that $f : V \to W$ is Fredholm if and only if there exists a linear map $g : W \to V$ such that $\mathrm{id}_V - g \circ f$ and $\mathrm{id}_W - f \circ g$ have finite rank.

In particular, any linear map between finite-dimensional vector spaces is Fredholm. Note that a composition of Fredholm maps is Fredholm.

We define the *relative determinant line* attached to a Fredholm map $f : V \to W$ to be the graded line

$$\det(V, W, f) := (\det \ker f)^{-1} \otimes \det \operatorname{coker} f,$$

which lies in degree $\operatorname{ind} f$. If $V$ and $W$ are finite-dimensional, then the isomorphism (2.2) shows that $\det(V, W, f) \tilde{\to} (\det V)^{-1} \otimes \det W$.

The relative determinant line is compatible with composition in the following sense: for any Fredholm maps $f : V \to W$ and $g : W \to U$ there is a canonical isomorphism of graded lines

$$\det(V, U, g \circ f) \xrightarrow{\sim} \det(W, U, g) \otimes \det(V, W, f). \tag{3.1}$$

This is constructed using the isomorphisms (2.1) associated with the short exact sequences of finite-dimensional vector spaces

$$0 \longrightarrow \ker f \longrightarrow \ker(g \circ f) \xrightarrow{f} \operatorname{im} f \cap \ker g \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{im} f \cap \ker g \longrightarrow \ker g \longrightarrow (\operatorname{im} f + \ker g)/ \operatorname{im} f \longrightarrow 0,$$
$$0 \longrightarrow (\operatorname{im} f + \ker g)/ \operatorname{im} f \longrightarrow \operatorname{coker} f \xrightarrow{g} \operatorname{im} g / \operatorname{im}(g \circ f) \longrightarrow 0,$$

and

$$0 \longrightarrow \operatorname{im} g / \operatorname{im}(g \circ f) \longrightarrow \operatorname{coker}(g \circ f) \longrightarrow \operatorname{coker} g \longrightarrow 0.$$

Notice also that if $f$ is an isomorphism, then we have a canonical trivialization

$$\det f : k \xrightarrow{\sim} \det(V, W, f).$$

Suppose $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ are Fredholm. Then it is not hard to see that $f_1 \oplus f_2$ is Fredholm, and that there is a canonical isomorphism of graded lines

$$\det(V_1 \oplus V_2, W_1 \oplus W_2, f_1 \oplus f_2) \xrightarrow{\sim} \det(V_1, W_1, f_1) \otimes \det(V_2, W_2, f_2).$$

Moreover, the triangle

$$\begin{array}{ccc} & \det(V_1 \oplus V_2, W_1 \oplus W_2, f_1 \oplus f_2) & \\ \swarrow & & \searrow \\ \det(V_1, W_1, f_1) \otimes \det(V_2, W_2, f_2) & \longrightarrow & \det(V_2, W_2, f_2) \otimes \det(V_1, W_1, f_1) \end{array}$$

commutes, where the horizontal isomorphism sends $\omega \otimes \eta \mapsto (-1)^{\operatorname{ind} f_1 \cdot \operatorname{ind} f_2} \eta \otimes \omega$. This can be interpreted as follows: for any collection of Fredholm maps $\{f_i : V_i \to W_i\}$ indexed by a finite set $I$, then we have an isomorphism

$$\det(\textstyle\bigoplus_i V_i, \bigoplus_i W_i, \bigoplus_i f_i) \xrightarrow{\sim} \bigotimes_i \det(V_i, W_i, f_i)$$

which does not depend on a choice of ordering for $I$.

## 4 Asymptotic linear maps

For any vector spaces $V$ and $W$, let

$$\operatorname{Hom}_f(V,W) \subset \operatorname{Hom}(V,W)$$

denote the space of finite rank linear maps $V \to W$. Observe that the composition of a finite rank map with an arbitrary map has finite rank, i.e. the finite rank maps form an ideal in the category Vect of vector spaces. This means that we can define a category $\mathrm{Vect}_\infty$ whose objects are vector spaces, with morphisms given by *asymptotic linear maps*

$$\operatorname{Hom}_\infty(V,W) := \operatorname{Hom}(V,W)/\operatorname{Hom}_f(V,W),$$

and composition is well-defined. Given a linear map $f$, we write $f^\infty$ for the corresponding asymptotic linear map. In particular $\mathrm{id}_V^\infty = 0$ if and only if $V$ is finite-dimensional.

Observe that a linear map $f$ is Fredholm if and only if $f^\infty$ is an isomorphism. The relative determinant line also has the property that if $f, g : V \rightrightarrows W$ are Fredholm maps such that $f - g$ has finite rank, then there is a canonical isomorphism of graded lines

$$\det(V,W,f) \xrightarrow{\sim} \det(V,W,g).$$

To see this, use the isomorphisms (2.1) for the short exact sequences

$$0 \longrightarrow \ker f \longrightarrow f^{-1}(\operatorname{im}(f-g)) \xrightarrow{f} \operatorname{im}(f-g) \cap \operatorname{im} f \longrightarrow 0,$$
$$0 \longrightarrow \ker g \longrightarrow g^{-1}(\operatorname{im}(f-g)) \xrightarrow{g} \operatorname{im}(f-g) \cap \operatorname{im} g \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{im}(f-g) \cap \operatorname{im} f \longrightarrow \operatorname{coker} f \longrightarrow W/(\operatorname{im}(f-g) + \operatorname{im} f) \longrightarrow 0,$$
$$\text{and}$$
$$0 \longrightarrow \operatorname{im}(f-g) \cap \operatorname{im} g \longrightarrow \operatorname{coker} g \longrightarrow W/(\operatorname{im}(f-g) + \operatorname{im} g) \longrightarrow 0,$$

observing that $f^{-1}(\operatorname{im}(f-g)) = g^{-1}(\operatorname{im}(f-g))$ and that

$$\operatorname{im}(f-g) + \operatorname{im} f = \operatorname{im} f + \operatorname{im} g = \operatorname{im}(f-g) + \operatorname{im} g.$$

Thus the relative determinant line is well-defined on $\mathrm{Iso}_\infty(V,W)$, the set of asymptotic isomorphisms $V \to W$, and we can write $\det(V,W,f^\infty)$ without ambiguity. In particular, we can assign to any element $f^\infty$ of the group

$$\mathrm{GL}_\infty(V) := \mathrm{Iso}_\infty(V,V)$$

the line $\det(V,V,f^\infty)$, and because of (3.1) we have isomorphisms

$$\det(V,V,g^\infty \circ f^\infty) \xrightarrow{\sim} \det(V,V,g^\infty) \otimes \det(V,V,f^\infty),$$

which satisfy natural unitality and associativity conditions. That is, to each element of $\mathrm{GL}_\infty(V)$ we attach a $k^\times$-torsor, in a manner compatible with composition. This is precisely the data needed to define a central extension

$$1 \longrightarrow k^\times \longrightarrow \mathrm{GL}_\infty(V)^\flat \longrightarrow \mathrm{GL}_\infty(V) \longrightarrow 1,$$

called the *Tate extension*. Namely, an element of $\mathrm{GL}_\infty(V)^\flat$ consists of an element $f^\infty \in \mathrm{GL}_\infty(V)$ and a trivialization $\alpha : k \xrightarrow{\sim} \det(V,V,f^\infty)$. The group operation is defined by

$$(f^\infty, \alpha) \cdot (g^\infty, \beta) := (f^\infty \circ g^\infty, \alpha \otimes \beta),$$

where in the second component we mean

$$k = k \otimes k \xrightarrow[\sim]{\alpha\otimes\beta} \det(V,V,f^\infty) \otimes \det(V,V,g^\infty) \xrightarrow{\sim} \det(V,V,f^\infty \circ g^\infty).$$

We remark that the homomorphism $\pi : \mathrm{GL}(V) \to \mathrm{GL}_\infty(V)$ lifts canonically to $\mathrm{GL}(V) \to \mathrm{GL}_\infty(V)^\flat$, because a choice of $f \in \mathrm{GL}(V)$ trivializes $\det(V, V, f^\infty)$. Notice also that the index determines a homomorphism

$$\mathrm{ind} : \mathrm{GL}_\infty(V) \longrightarrow \mathbb{Z}.$$

This can be viewed as part of the data of the super extension $\mathrm{GL}_\infty(V)^\flat$.

The index zero subgroup of the Tate extension $\mathrm{GL}_\infty(V)^\flat$ can also be described as the pushout

$$\mathrm{GL}(V) \coprod_{\mathrm{GL}_f(V)} k^\times,$$

using $\det : \mathrm{GL}_f(V) \to k^\times$ introduced in Section 1. To see this, it suffices to show that the restriction of $\pi$ to $\mathrm{GL}_f(V)$ is det. As previously noted, if $f \in \mathrm{GL}(V)$ then $\det(V, V, f^\infty)$ is canonically trivialized, and this datum defines $\pi(f)$. On the other hand, if $f \in \mathrm{GL}_f(V)$ then since $\mathrm{id}_V - f$ has finite rank we also have a trivialization coming from

$$k = \det(V, V, \mathrm{id}_V) \xrightarrow{\sim} \det(V, V, f) = \det(V, V, f^\infty).$$

The ratio of these two trivializations of $\det(V, V, f^\infty)$ is $\det(f) \in k^\times$, as one can show by a direct calculation.

One defines the Tate extension of the Lie algebra

$$\mathfrak{gl}_\infty(V) := \mathfrak{gl}(V)/\mathfrak{gl}_f(V)$$

in a similar manner, by putting

$$\mathfrak{gl}_\infty(V)^\flat := \mathfrak{gl}(V) \coprod_{\mathfrak{gl}_f(V)} k$$

where we use $\mathrm{tr} : \mathfrak{gl}_f(V) \to k$ introduced in Section 1. This yields a central extension

$$0 \longrightarrow k \longrightarrow \mathfrak{gl}_\infty(V)^\flat \longrightarrow \mathfrak{gl}_\infty(V) \longrightarrow 0.$$

# 5 Tate vector spaces

Any vector space $V$ can be canonically realized as a filtered colimit $V = \mathrm{colim}\, V_i$ of finite-dimensional vector spaces $V_i$. Therefore its dual $V^* = \lim V_i^*$ is canonically a cofiltered limit of the finite-dimensional vector spaces $V_i^*$. Notice that the map

$$\mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(W^*, V^*)$$

is not an isomorphism in general. However, if we give $V^*$ the topology with base at $0 \in V^*$ consisting of the kernels of the projections $V^* \to V_i^*$, then

$$\mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}_{\mathrm{cts}}(W^*, V^*)$$

is an isomorphism.

We define $\mathrm{Pro\,Vect}_f$ to be the category of topological vector spaces which are isomorphic to a cofiltered limit of (always discrete) finite-dimensional vector spaces. Thus duality is an equivalence

$$\mathrm{Vect}^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Pro\,Vect}_f\,.$$

**Definition 5.1.** A *Tate vector space* is a topological vector space isomorphic to $V \oplus W$, where $V$ has the discrete topology and $W$ belongs to $\mathrm{Pro\,Vect}_f$. We denote the category of Tate vector spaces by TateVect.

Thus TateVect is the category of topological vector spaces generated by Vect and $\mathrm{Pro\,Vect}_f$ under finite direct sums. If $V$ is a Tate vector space, then its dual $V^*$ has a natural topology: if $V = W \oplus U^*$ where $W$ and $U$ are discrete, then the topology on $V^*$ determined by $V^* = W^* \oplus U$ does not depend on the choice of decomposition. It follows that duality is an equivalence

$$\mathrm{TateVect}^{\mathrm{op}} \xrightarrow{\sim} \mathrm{TateVect},$$

so we have a true duality on Tate vector spaces.

**Remark 5.2.** It is possible to avoid the use of topological algebras, at the expense of introducing more category theory. Namely, for any category $\mathscr{C}$ with finite limits, one can define

$$\operatorname{Pro}\mathscr{C} \subset \operatorname{Fun}(\mathscr{C}, \mathrm{Set})^{\mathrm{op}}$$

as the subcategory consisting of functors $F : \mathscr{C} \to \mathrm{Set}$ which preserve finite limits. The $\operatorname{Pro}\mathscr{C}$ is complete, and the Yoneda embedding $\mathscr{C} \to \operatorname{Pro}\mathscr{C}$ preserves finite limits. In fact $\operatorname{Pro}\mathscr{C}$ is the category obtained from $\mathscr{C}$ by freely adjoining cofiltered limits.

Our previous notation $\operatorname{Pro}\mathrm{Vect}_f$ is consistent with the new, and in particular one can check that the inclusion $\mathrm{Vect}_f \subset \mathrm{Vect}$ induces a fully faithful functor $\operatorname{Pro}\mathrm{Vect}_f \to \operatorname{Pro}\mathrm{Vect}$. Then TateVect is the full subcategory of Pro Vect generated by Vect and $\operatorname{Pro}\mathrm{Vect}_f$ under finite direct sums.

A *lattice* in a Tate vector space is an open linear subspace which belongs to $\operatorname{Pro}\mathrm{Vect}_f$. A *colattice* is a subspace complementary to a lattice (in particular, discrete).

**Example 5.3.** Let $X$ be a smooth algebraic curve over $k$ and $x \in X(k)$. Write $K_x$ for the fraction field of the completed local ring $\mathscr{O}_x$ of $X$ at $x$; the former is isomorphic as a topological algebra to the field of Laurent series $k((t))$ and the latter to the power series algebra $k[[t]]$. Then any finite-dimensional vector space over $K_x$ has a natural structure of Tate vector space over $k$. An open $k$-subspace is a lattice if and only if it is finitely generated over $\mathscr{O}_x$.

The topological $k$-dual $K_x^*$ is canonically isomorphic to the the space of 1-forms $\Omega^1_{K_x}$ on the punctured formal disk around $x$, itself a one-dimensional vector space over $K_x$. The pairing is given by

$$f \otimes dg \mapsto \operatorname{Res}(f\, dg).$$

For any two lattices $L_1$ and $L_2$ in a Tate vector space $V$, we define their *relative determinant line* to be the graded line

$$\begin{aligned} \det(L_1, L_2) &:= \det(L_1/(L_1 \cap L_2))^{-1} \otimes \det(L_2/(L_1 \cap L_2)) \\ &= \det((L_1 + L_2)/L_1) \otimes \det((L_1 + L_2)/L_2)^{-1}. \end{aligned}$$

In particular, if $L_1 \subset L_2$ then we have

$$\det(L_1, L_2) = \det(L_2/L_1).$$

This construction can also be made using asymptotic maps. Namely, if we choose a splitting of $V \to V/L_1$ then the resulting composition

$$V/L_1 \longrightarrow V \longrightarrow V/L_2$$

is Fredholm, and the resulting asymptotic isomorphism does not depend on the choice of splitting. Thus we have a canonically defined object $V_\infty$ of $\mathrm{Vect}_\infty$, given by $V/L$ for some lattice $L$. The construction $V \mapsto V_\infty$ extends to a functor

$$\mathrm{TateVect} \longrightarrow \mathrm{Vect}_\infty\,. \tag{5.1}$$

In these terms, we have

$$\det(L_1, L_2) \xrightarrow{\sim} \det(V/L_1, V/L_2, \mathrm{id}_{V_\infty}).$$

For any three lattices $L_1$, $L_2$, and $L_3$ in a Tate vector space $V$, we have an isomorphism of graded lines

$$\det(L_1, L_2) \otimes \det(L_2, L_3) \xrightarrow{\sim} \det(L_1, L_3) \tag{5.2}$$

satisfying natural unitality and associativity conditions. These conditions imply in particular that

$$\det(L_1, L_2) \xrightarrow{\sim} \det(L_2, L_1)^{-1}.$$

If $g \in \mathrm{GL}(V)$, the group of continuous automorphisms of $V$, observe that there is a canonical isomorphism

$$\det(L, L) \xrightarrow{\sim} \det(gL, gL) \tag{5.3}$$

for any lattice $L \subset V$.

# 6 The Tate extension redux

The functor (5.1) gives rise to a homomorphism

$$\mathrm{GL}(V) \longrightarrow \mathrm{GL}_\infty(V_\infty).$$

Thus we can define the *Tate extension* of $\mathrm{GL}(V)$ by

$$\mathrm{GL}(V)^\flat := \mathrm{GL}(V) \underset{\mathrm{GL}_\infty(V_\infty)}{\times} GL_\infty(V_\infty)^\flat.$$

Explicitly, if one chooses a lattice $L \subset V$, then an element of $\mathrm{GL}(V)^\flat$ is a pair consisting of an element $g \in \mathrm{GL}(V)$ and a trivialization $\alpha : k \tilde{\to} \det(L, gL)$. It follows from the definition of $V_\infty$ that $\det(L, gL)$ is independent of $L$, or one can show it directly. Multiplication is defined by $(g, \alpha) \cdot (h, \beta) := (gh, \alpha \otimes \beta)$, where we use

$$\begin{aligned} k = k \otimes k &\xrightarrow[\sim]{\alpha\otimes\beta} \det(L, gL) \otimes \det(L, hL) \\ &\xrightarrow{\sim} \det(L, gL) \otimes \det(gL, ghL) \\ &\xrightarrow{\sim} \det(L, ghL). \end{aligned}$$

To reiterate, this defines a central extension

$$1 \longrightarrow k^\times \longrightarrow \mathrm{GL}(V)^\flat \longrightarrow \mathrm{GL}(V) \longrightarrow 1.$$

Notice also that we have a homomorphism

$$\mathrm{GL}(V) \longrightarrow \mathrm{GL}_\infty(V_\infty) \xrightarrow{\mathrm{ind}} \mathbb{Z}.$$

Write $\mathrm{GL}(V, L) \subset \mathrm{GL}(V)$ for the subgroup consisting of $g \in \mathrm{GL}(V)$ such that $gL = L$. Such a $g$ induces an automorphism of $V/L$, which determines a trivialization of $\det(V/L, V/L, g^\infty)$. Thus the projection $\mathrm{GL}(V)^\flat \to \mathrm{GL}(V)$ splits canonically over $\mathrm{GL}(V, L)$. Similarly, the projection has a canonical splitting over $\mathrm{GL}(V, M)$ for $M \subset V$ a colattice.

Let $\mathrm{GL}_c(V), \mathrm{GL}_d(V) \subset \mathrm{GL}(V)$ be the subgroups consisting of $g \in \mathrm{GL}(V)$ such that $\mathrm{id}_V - g$ has bounded, respectively discrete, image. We have

$$\mathrm{GL}_c(V) \cap \mathrm{GL}_d(V) = \mathrm{GL}_f(V).$$

All three of these subgroups are normal. Observe that $\mathrm{GL}_c(V)$ is the kernel of

$$\mathrm{GL}(V) \longrightarrow \mathrm{GL}_\infty(V_\infty),$$

and that the index zero subgroup of $\mathrm{GL}(V)$ is generated by $\mathrm{GL}_c(V)$ and $\mathrm{GL}_d(V)$, which implies that

$$\mathrm{GL}_d(V)/\mathrm{GL}_f(V) \xrightarrow{\sim} \mathrm{GL}_\infty(V_\infty).$$

In these terms, the index zero subgroup of the extension $\mathrm{GL}_\infty(V_\infty)^\flat$ is isomorphic to

$$\mathrm{GL}_d(V) \coprod_{\mathrm{GL}_f(V)} k^\times,$$

using $\det : \mathrm{GL}_f(V) \to k^\times$.

The Tate extension of the Lie algebra $\mathfrak{gl}(V)$ of continuous endomorphisms of $V$ can be constructed similarly. Namely, we write $\mathfrak{gl}_c(V), \mathfrak{gl}_d(V) \subset \mathfrak{gl}(V)$ for the Lie ideals consisting of endomorphisms with bounded, respectively discrete, image, so that

$$\mathfrak{gl}_f(V) = \mathfrak{gl}_c(V) \cap \mathfrak{gl}_d(V).$$

Now we have

$$\mathfrak{gl}_\infty(V_\infty)^\flat \xrightarrow{\sim} \mathfrak{gl}_d(V) \coprod_{\mathfrak{gl}_f(V)} k$$

where we used $\mathrm{tr} : \mathfrak{gl}_f(V) \to k$, and

$$\mathfrak{gl}(V)^\flat := \mathfrak{gl}(V) \underset{\mathfrak{gl}_\infty(V_\infty)}{\times} \mathfrak{gl}_\infty(V_\infty)^\flat.$$

# 7 Some geometric representation theory

All of the above discussion can be redone in families. This means that instead of only working with vector spaces over the ground field $k$, we consider more generally projective modules over commutative $k$-algebras. Almost every object previously introduced is the $k$-points of a functor from commutative $k$-algebras to sets, which is often the functor of points of a variety or can be "approximated" by varieties. For example, the multiplicative group $\mathbb{G}_m$ is the functor which sends a commutative $k$-algebra $A$ to its group of units $A^\times$. The algebraic group $\mathbb{G}_m$ will appear where we wrote $k^\times$ before.

The group $\mathrm{GL}(V)$ for $V$ a Tate vector space can be upgraded to the functor which sends $A$ to the group of $A$-linear continuous automorphisms of $A \otimes_k V$. It acts on the Sato Grassmannian $\mathrm{Gr}^{\frac{\infty}{2}}(V)$, whose points are families of lattices in $V$. More precisely, the functor $\mathrm{Gr}^{\frac{\infty}{2}}(V)$ sends $A$ to the set of open $A$-submodules $M \subset A \otimes_k V$ which are *bounded* in the sense that for any discrete $A$-module $N$ and any $A$-linear map $f : M \to N$, the image of $f$ is contained in a finitely generated submodule of $N$.

This action is transitive, i.e. $\mathrm{Gr}^{\frac{\infty}{2}}(V)$ is a homogeneous space for $GL(V)$. The relative determinant line is best understood as a line bundle $\mathscr{L}_{\det}$ on

$$\mathrm{Gr}^{\frac{\infty}{2}}(V) \times \mathrm{Gr}^{\frac{\infty}{2}}(V).$$

The two projections make this product a groupoid over $\mathrm{Gr}^{\frac{\infty}{2}}(V)$, with the unit given by the diagonal embedding and composition given by

$$(L_1, L_2) \circ (L_2, L_3) = (L_1, L_3).$$

Thus the isomorphism (5.2) endows $\mathscr{L}_{\det}$ with a multiplicative structure over this groupoid, and the isomorphism (5.3) amounts to a $\mathrm{GL}(V)$-equivariant structure on $\mathscr{L}_{\det}$, where $\mathrm{GL}(V)$ acts on $\mathrm{Gr}^{\frac{\infty}{2}}(V) \times \mathrm{Gr}^{\frac{\infty}{2}}(V)$ by the diagonal action.

Now we define $\mathscr{L}_{\det,L}$ to be the restriction of $\mathscr{L}_{\det}$ to $\{L\} \times \mathrm{Gr}^{\frac{\infty}{2}}(V)$. Since $L$ is not fixed by $\mathrm{GL}(V)$, the $\mathrm{GL}(V)$-equivariant structure on $\mathscr{L}_{\det}$ does not restrict to one on $\mathscr{L}_{\det,L}$: instead we have an isomorphism $g^*(\mathscr{L}_{\det,gL}) \tilde{\to} \mathscr{L}_{\det,L}$ for any $g \in \mathrm{GL}(V)$. This lack of equivariance means that $\mathrm{GL}(V)$ does not act on the discrete vector space

$$\bigwedge\nolimits_L := \Gamma(\mathrm{Gr}^{\frac{\infty}{2}}(V), \mathscr{L}^*_{\det,L})^*,$$

but in fact it is still a *projective* representation of $\mathrm{GL}(V)$. Recall that this means that that there is a homomorphism

$$\mathrm{GL}(V) \longrightarrow \mathrm{PGL}(\bigwedge\nolimits_L) := \mathrm{GL}(\bigwedge\nolimits_L)/\mathbb{G}_m.$$

Note that for $s \in \Gamma(\mathrm{Gr}^{\frac{\infty}{2}}(V), \mathscr{L}^*_{\det,L})$, we have $s(L') \in \det(L, L')$ for any lattice $L' \subset V$. Fix $g \in \mathrm{GL}(V)$ and choose $\alpha_g : \det(gL, L) \tilde{\to} \det(L, L)$ arbitrarily. Then we can define $(g \cdot s)(L')$ to be the image of $s(g^{-1}L')$ under

$$\begin{aligned} \det(L, g^{-1}L') &\xrightarrow[\sim]{(5.3)} \det(gL, L') \\ &\xrightarrow[\sim]{(5.2)} \det(gL, L) \otimes \det(L, L') \\ &\xrightarrow[\sim]{\alpha_g} \det(L, L) \otimes \det(L, L') \\ &\xrightarrow{\sim} \det(L, L'). \end{aligned}$$

Since the $\alpha_g$, which are ambiguous up to scaling, cannot be chosen in a consistent way for all $g \in \mathrm{GL}(V)$, we see that this formula only defines a projective representation. This projective representation yields an equivalent construction of the Tate extension of $\mathrm{GL}(V)$, namely as the fiber product

$$\mathrm{GL}(V)^\flat = \mathrm{GL}(V) \underset{\mathrm{PGL}(\bigwedge_L)}{\times} \mathrm{GL}(\bigwedge\nolimits_L).$$

In fact, there is a $\mathrm{GL}(V)^\flat$-equivariant structure on the line bundle $\mathscr{L}_{\det,L}$ which makes $\bigwedge_L$ an algebraic $\mathrm{GL}(V)^\flat$-representation. Sometimes it is called the *fermionic Fock representation*. Note that the central $\mathbb{G}_m \subset \mathrm{GL}(V)^\flat$ acts by dilations on $\bigwedge_L$. There is a $\mathrm{GL}(V)^\flat$-equivariant isomorphism $\bigwedge_L \tilde{\to} \bigwedge'_L$ for any other lattice $L' \subset V$, and in fact

$$\mathrm{Hom}_{\mathrm{GL}(V)^\flat}(\bigwedge\nolimits_L, \bigwedge\nolimits'_L) = \det(L, L').$$

# 8 Clifford modules

Let $V$ be a Tate vector space, and equip $V^* \oplus V$ with the standard hyperbolic symmetric bilinear form. If we view $V$ as a graded Tate vector space placed in degree 1, so that $V^*$ lies in degree $-1$, then this hyperbolic form becomes skew-symmetric, and therefore defines a central extension of graded Lie algebras

$$0 \longrightarrow k \longrightarrow C \longrightarrow V \oplus V^* \longrightarrow 0$$

with $k$ placed in degree 0. The non-graded version, where one starts with a symplectic form, is usually called a Heisenberg Lie algebra.

The Clifford algebra $\mathrm{Cl}(V)$ is the twisted enveloping algebra attached to this central extension, meaning it is the graded universal enveloping algebra of $C$ modulo the relation which identifies $1 \in k \subset C$ with the unit. The category $\mathrm{Cliff}_V$ of Clifford modules is by definition the category of discrete graded $\mathrm{Cl}(V)$-modules (on which $\mathrm{Cl}(V)$ acts continuously), or equivalently discrete graded $C$-modules such that $1 \in k \subset C$ acts by the identity.

Given a lattice $L \subset V$, we obtain a Lagrangian subspace $L \oplus L^\perp \subset V \oplus V^*$, itself a lattice. This gives rise to a subalgebra $\mathrm{Cl}(L \oplus L^\perp) \subset \mathrm{Cl}(V)$, and we define

$$M_L := \mathrm{Cl}(V) \underset{\mathrm{Cl}(L \oplus L^\perp)}{\otimes} k$$

where $\mathrm{Cl}(L \oplus L^\perp)$ acts on $k$ by the augmentation (i.e. $L \oplus L^\perp$ acts by zero).

**Proposition 8.1.** *For any lattice $L \subset V$, the functor from graded vector spaces to $\mathrm{Cliff}_V$ which sends $W \mapsto W \otimes M_L$ is an equivalence.*

In fact, there is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Cl}(V)}(M_{L_1}, M_{L_2}) \xrightarrow{\sim} \det(L_1, L_2)$$

for any lattices $L_1, L_2 \subset V$.

Write $\mathrm{O}(V \oplus V^*) \subset \mathrm{GL}(V \oplus V^*)$ for the orthogonal group, consisting of continuous automorphisms which preserve the form. It acts on $\mathrm{Cl}(V)$ by automorphisms, hence acts by autoequivalences on $\mathrm{Cliff}_V$. Explicitly, for $g \in \mathrm{O}(V \oplus V^*)$ and $M$ in $\mathrm{Cliff}_V$, we denote by $M^g$ the module with the same underlying vector space and $\mathrm{Cl}(V)$-action defined by

$$(c, m) \mapsto (g^{-1} \cdot c)m.$$

Fix a lattice $L \subset V$. Proposition 8.1 implies that for any $g \in \mathrm{O}(V \oplus V^*)$, there exists an isomorphism $\beta_g : M_L \xrightarrow{\sim} M_L^g$. Since $M_L^g = M_L$ as a vector space, we can view $g \mapsto \beta_g$ as a homomorphism

$$\mathrm{O}(V \oplus V^*) \to \mathrm{PGL}(M_L).$$

In particular, we obtain a central extension

$$\mathrm{O}(V \oplus V^*)^\flat = \mathrm{O}(V \oplus V^*) \underset{\mathrm{PGL}(M_L)}{\times} \mathrm{GL}(M_L)$$

of $\mathrm{O}(V \oplus V^*)$.

Now observe that there is a canonical homomorphism

$$\mathrm{GL}(V) \longrightarrow \mathrm{O}(V \oplus V^*)$$

given by $g \mapsto g \oplus (g^*)^{-1}$. This identifies $\mathrm{GL}(V)$ with the group of continuous automorphisms of $V \oplus V^*$ which preserve the form and the grading. In fact, there is a canonical isomorphism

$$\mathrm{GL}(V)^\flat \longrightarrow \mathrm{GL}(V) \underset{\mathrm{O}(V \oplus V^*)}{\times} \mathrm{O}(V \oplus V^*)^\flat,$$

which follows from the existence of a canonical isomorphism

$$\textstyle\bigwedge_L \xrightarrow{\sim} M_L$$

which intertwines the projective actions of $\mathrm{GL}(V)$.