Reconstructing Dynamical Systems from Amplitude Measures of Spiky Time-Series

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ABSTRACT AND MOTIVATION

Many examples exist in nature where the output of a dynamic system is a spiky time series. Well known examples of such systems include the Rössler system [1], which was developed to model the Earth's magnetic field, the Hindmarsh-Rose [2, 3] system which has been proposed to model neuron membrane electrical activity, Chay-Keizer equations which model pancreatic cells, and many other. These hypothetical mathematical models of the underlying pulse generating systems were developed empirically by developing sets of equations that could account for all the different operating regimes of the system.

Of interest is to perform the inverse of such empirical modeling that is extract a model of the underlying dynamics from the output data. The data are typically single time series, for example a recording of neuronal impulses. The modeling from data would allow us:

- to verify the accuracy of the empirically derived models
- to identify alternate, simpler models, that are topologically equivalent to the existing empirical models of physical processes. Simplier models are desirable because they are easier to use in numerical simulations. Alternate models also have theoretical importance because they can be used to draw correspondence between the different physical systems and may also reveal physcial structures that are not obvious in more complex system representations.

Attempts to recover multidimensional models from spiky data have not been successful. Moreover, theoretical explanations have been offered for why such recovery is not possible.

In this paper, for the first time, a global 3-D model is derived directly from a spiky time series. We use the z-component of the Rössler system as an example to perform model recovery. The reasons for our choice of the system and its relevance to neuronal models are discussed.

METHODS

The method we use to recover a global model has been recently developed in our laboratory. We first recover a differential model from the data which is then transformed to the coordinates of the original Rössler system using an ansatz library [14] we have developed. We show that the topological structure of the recovered model is equivalent to the topological structure of the Rössler system.

We use a numerical trick which allows us to obtain a differential model. We derive the model by ignoring parts (50%) of each peak in the time series and use the remaining inter-spike intervals of the series. We also randomly draw and mix sections from the inter spike range of the series.

This method can be applied to modeling in general systems that generate spiky time series, not limited to the Rössler system. In particular, one can apply this approach to model the mechanisms underlying neuronal impulses.

MODELING SPIKY DYNAMICS

The Rössler system z-component is a spike train, examples of which are shown in Figures 1 and 2. This system provides a good introductory point for this research for the following reasons:

- The Rössler system is one of the most widely studied non-linear systems.
- The z-component of Rössler provides an example of a spiky time series we are interested in modeling.
- The Rössler system is a low-dimensional chaotic system for which we have a simple model representation with only quadratic non-linearities.
- Even though the Rössler system is fairly simple, the so-called coefficient of "observability" [6] of the dynamics is low for the z-component. It is therefore has been commonly accepted that one cannot recover a three-dimensional model of the Rössler system from its z-component, due to the presence of spikes.
- We have shown that the Rössler system and the Hindmarsh-Rose system are subsystems of the same differential model. The Hindmarsh-Rose is more general since its differential model contains all terms of the differential model of the Rössler system plus an additional one. We therefore believe that the method presented here for the Rössler system can be applied to model other spiky processes, in particular the neuronal outputs that are described by the Hindmarsh-Rose model.

PROBLEM DEFINITION

Global modeling reconstruction techniques allow us to obtain a set of differential equations for the phase portrait reconstructed from a limited set of measurements, e.g. when a single variable time series is recorded. Pioneering papers by Takens [4] and Packard *et al* [5] provide the theoretical background for reconstructing the phase portrait from recorded scalar time series. Crutchfield [7] used such reconstructed phase space to obtain global models built on delay or derivative coordinates. Principal components, introduced by Broomhead and King [8], may also be used. Gibson *et al* [9] showed that the relationships between delays, derivatives, and principal components consist of rotations and rescalings. These coordinate sets are therefore equivalent, although one set may ometimes be better than another for numerical reasons.

Practically, even noise-free, infinite scalar time series may not contain enough information to obtain all the dynamical properties [10, 11, 12]. More recently, an observability index has been introduced to quantify the "observability" of the dynamics from a scalar time series [6]. A small index value means that obtaining a global model is less likely. In this way, it has been possible to classify the observability of the Rössler system by its three variables. The order $y \triangleright x \triangleright z$, where $\triangleright$ means "provides a better observability of the underlying dynamics than" has been obtained. Thus, the y-variable is the best variable for obtaining a model while the z-variable is the worst one. No simple model has been obtained from the z-variable up to the 4D model proposed in [6]. An *ad hoc* structure selection, constructed by identifying the fixed point coordinates, was required for obtaining a successful 3D model [13]. Otherwise, no 3D successful model has been obtained.

Here, we derive a global model from the z-variable of the Rössler system using a structure selection method. An ansatz library is used to predefine putative differential models. Each ansatz for the original unknown system leads to a particular differential model. Candidate differential models are selected using the dispersion of the model coefficients over different windows [14]. The models are hereafter transformed into the ansatz form using a genetic algorithm.

ANSATZ LIBRARY [14]

Any system

$$\dot{x} = \eta_0 H_1(x) + \eta_1 x^n y$$

$$\eta_0, n \in \mathbb{N}_0 \ ; \ \eta_0, n \leq 1;$$

$$H_1(x) = a_0 + a_1 x + a_4 x^2$$

$$\dot{y} = H_2(x, y) + \eta_2 x^m y^p z$$

$$\begin{array}{lcl} \eta_0 = 0 & : & m, p \in \mathbb{N}_0 \ ; \ n + m \leq 1 \\ \eta_0 = 1 & : & p = 0 \ ; n, m \in \mathbb{N}_0 \ ; \ n, m \leq 1 \end{array}$$

$$H_2(x, y) = b_0 + b_1 x + b_2 y + b_4 x^2 + b_5 xy + b_7 y^2$$

$$\dot{z} = H_3(x, y, z)$$

$$\begin{aligned} H_3(x, y, z) = \ & c_0 + c_1 x + c_2 y + c_3 z + c_4 x^2 + \\ & c_5 xy + c_6 xz + c_7 y^2 + c_8 yz + c_9 z^2 \end{aligned}$$

can be written as differential model $\Downarrow \varphi$

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= Z \\ \dot{Z} &= F(X, Y, Z, \alpha_n) = \sum_{n=1}^{N_\alpha} \alpha_n P_n \end{aligned}$$

Only 6 possible general systems exist and form our Ansatz library [14], which constraints our search procedure thus making it possible to reconstruct the original system of ODE's from a single variable time series.

Rössler system

$$
\begin{aligned}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + \mathcal{C}z + xz
\end{aligned}
$$

Examples:

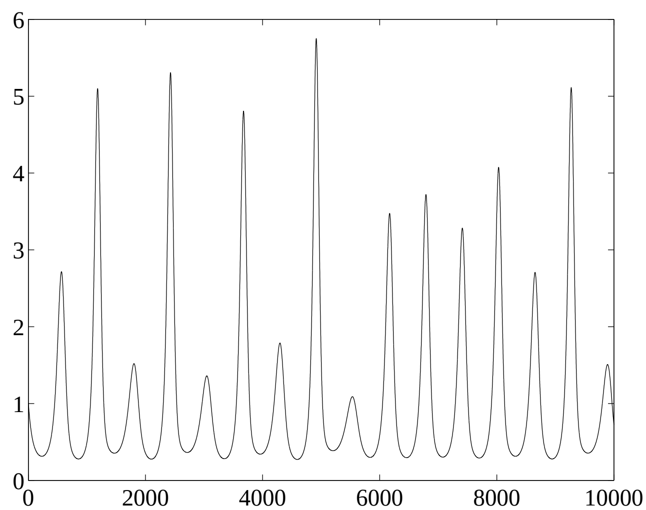


Figure 1: Time series of the z-component of the Rössler system with $(a, b, \mathcal{C}) = (0.398, 2.0, -4.0)$

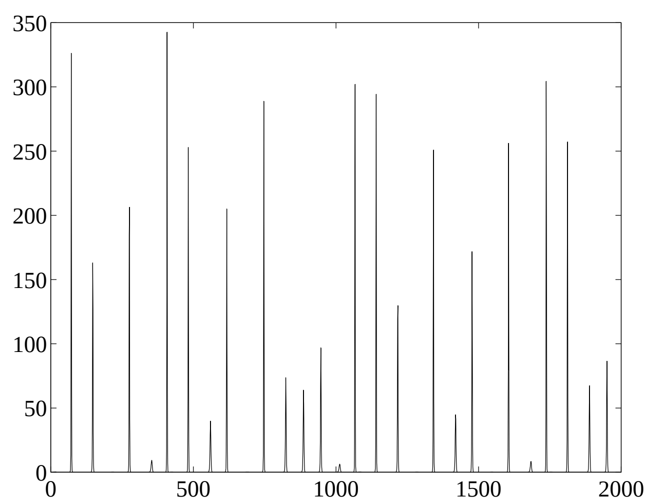


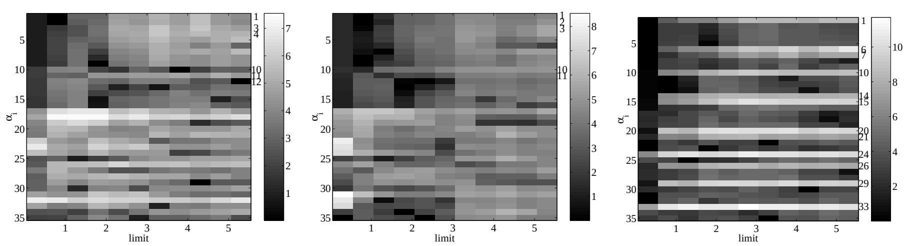
Figure 2: Time series of the z-component of the Rössler system with $(a, b, \mathcal{C}) = (0.2, 0.2, -35.0)$

Model from the z-component [15]:

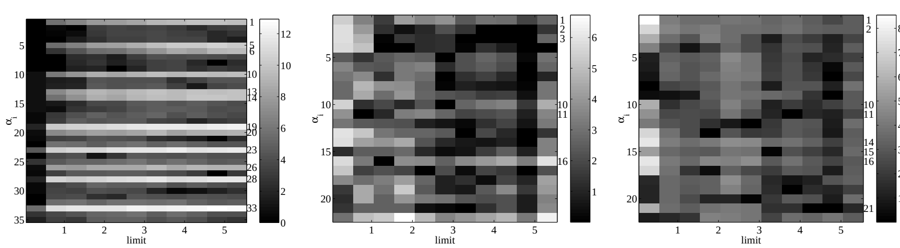
1. Ansatz selection from the library $\Rightarrow$ possible differential models
2. selection of maximal 3d models from the possible differential models
3. inversion of the transformation φ

1 Ansatz selection from the library

We use the z-component with coefficients $(a, b, \mathcal{C}) = (0.398, 2.0, -4.0)$ shown in Figure 1. For each ansatz $A_{1,2,\ldots,6}$ we plot significance of the model coefficients α_n's derived from inter-spike intervals as we disregard a progressively smaller area under each spike.



(a) Ansatz A_1 (b) Ansatz A_2 (c) Ansatz A_3



(d) Ansatz A_4 (e) Ansatz A_5 (f) Ansatz A_6

We select the ansatz which have the most stable coefficients α_i across all the interspike intervals. Such ansatz are A_3 and A_4 from which we identify the 12 monomials

$$P = \{1, X, X^2, Y, \frac{Y}{X}, XY, \frac{Y^2}{X^2}, \frac{Y^2}{X}, \frac{Y^3}{X^2}, Z, \frac{Z}{X}, \frac{YZ}{X}\}$$

of a possible model.

2. Maximal 3d model from A_3 and A_4

A_4:

$$
\begin{aligned}
\dot{x} &= a_0 + a_1 x + a_5 xy \\
\dot{y} &= b_0 + b_1 x + b_2 y + b_3 z \\
\dot{z} &= c_0 + c_1 x + c_2 y + c_3 z + c_5 xy
\end{aligned}
$$

A_3:

$$
\begin{aligned}
\dot{x} &= a_0 + a_1 x + a_5 xy\,, \\
\dot{y} &= b_0 + b_1 x + b_2 y + b_6 xz\,, \\
\dot{z} &= c_0 + c_2 y + c_3 z\,,
\end{aligned}
$$

with the inverse transformations:

$$
\varphi_{A_4}\left\{
\begin{array}{lcl}
\alpha_1 &=& -a_0\, b_3\, c_2 + a_0\, b_2\, c_3 \\
\alpha_5 &=& a_5\, b_3\, c_0 - a_1\, b_3\, c_2 - a_5\, b_0\, c_3 + a_1\, b_2\, c_3 - a_0\, b_3\, c_5 \\
\alpha_6 &=& a_5\, b_3\, c_1 - a_5\, b_1\, c_3 - a_1\, b_3\, c_5 \\
\alpha_{10} &=& (b_3\, c_2 - b_2\, c_3) \\
\alpha_{13} &=& (a_0\, b_2 + a_0\, c_3) \\
\alpha_{14} &=& (a_5\, b_1 + b_3\, c_5) \\
\alpha_{19} &=& 2\, a_0 \\
\alpha_{20} &=& (-b_2 - c_3) \\
\alpha_{23} &=& -2 \\
\alpha_{26} &=& (b_2 + c_3) \\
\alpha_{28} &=& -a_0 \\
\alpha_{33} &=& 3
\end{array}
\right.
$$

and

$$
\varphi_{A_3}\left\{
\begin{array}{lcl}
\alpha_1 &=& a_0\, b_2\, c_3 \\
\alpha_6 &=& -a_0\, b_6\, c_2 - a_5\, b_0\, c_3 + a_1\, b_2\, c_3 \\
\alpha_7 &=& a_5\, b_6\, c_0 - a_1\, b_6\, c_2 - a_5\, b_1\, c_3 \\
\alpha_{10} &=& (-a_5\, b_0 + a_1\, b_2 - b_2\, c_3) \\
\alpha_{14} &=& (2\, a_0\, b_2 + a_0\, c_3) \\
\alpha_{15} &=& b_6\, c_2 \\
\alpha_{20} &=& 3\, a_0 \\
\alpha_{21} &=& (-2\, b_2 - c_3) \\
\alpha_{24} &=& -3 \\
\alpha_{26} &=& (b_2 + c_3) \\
\alpha_{29} &=& -a_0 \\
\alpha_{33} &=& 4
\end{array}
\right.
$$

3. Inversion of the transformation φ

We attempt to carry out the inverse transformations $\varphi_{A_3}^{-1}$ and $\varphi_{A_4}^{-1}$ in order to get a model in the original phase space

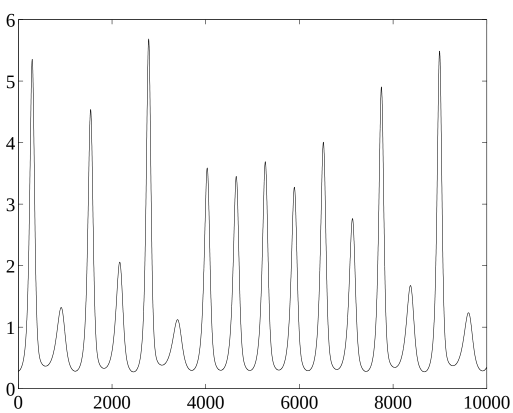
$$\Downarrow$$

only $\varphi_{A_4}^{-1}$ successful leading to

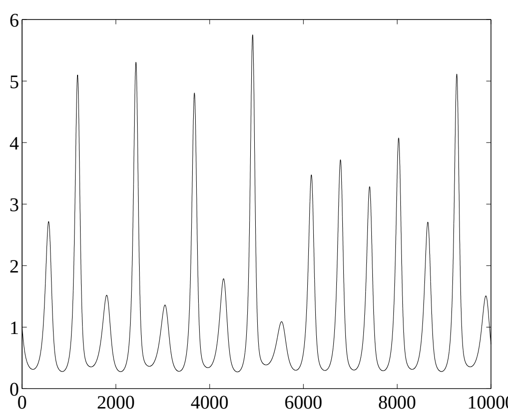
$$\Downarrow$$

$$\begin{aligned}
\dot{x} &= 1.998 - 1.007x - 0.272xy\,, \\
\dot{y} &= 4.115 + 0.238x - 1.455y + 2.435z\,, \\
\dot{z} &= -9.714 + 0.661x + 0.158y + 1.595z + 0.235xy\,.
\end{aligned}$$

RESULTS

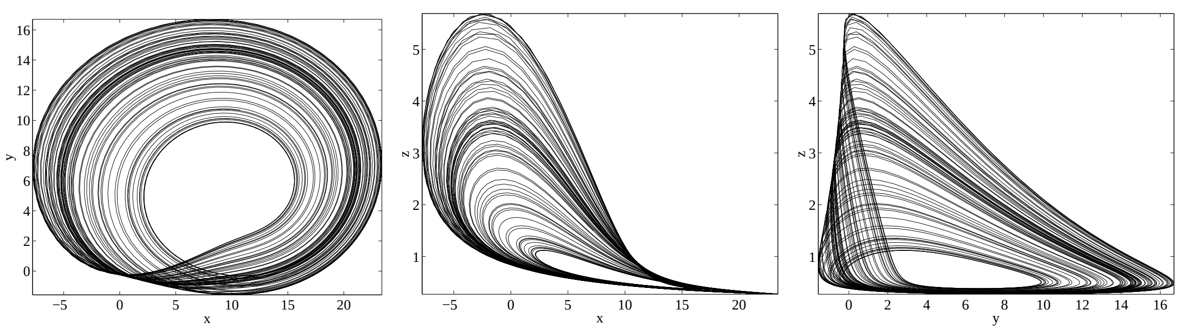


(a) model

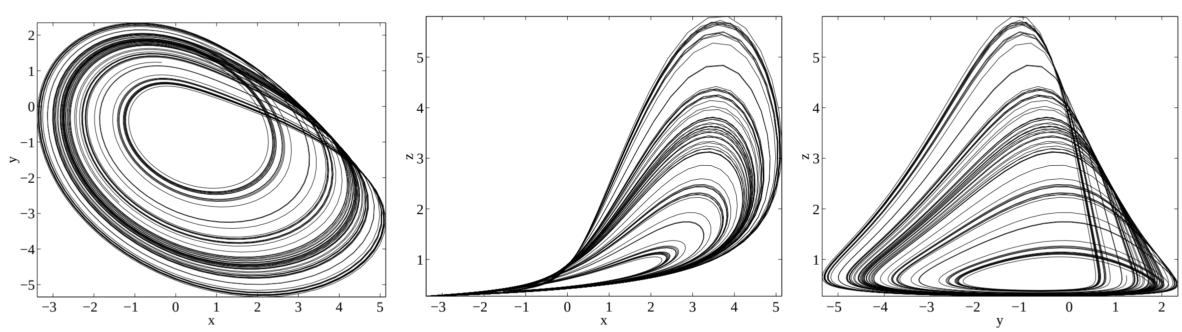


(b) original Rössler system (see also Fig. 1)

ORIGINAL PHASE SPACE PROJECTIONS



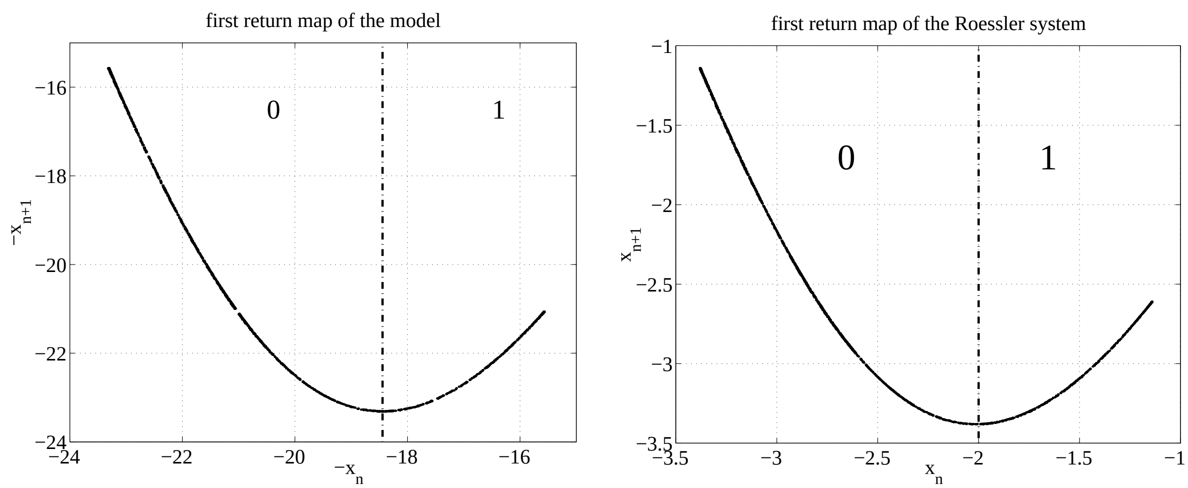
(a) model



(b) original Rössler system

TOPOLOGICAL EQUIVALENCE

The first return maps of the obtained model and the original Rössler system are



(a) model

(b) original Rössler system

Figure 3: First return maps of the original Rössler system and the model. The first return map of the model is plotted with reversed signs in order to make the topological encodings more obvious.

We use method described in [16] to also compute the unstable periodic orbits from the first return maps. From this we find that the model and the original Rössler system ([17]) have the same unstable periodic orbits, namely `1`, `10`, `1011`, `101110`, and `101111`, when we take into account that the model has opposite signs for the x- and y-coordinate. This means that both systems are topologically equivalent.

SUMMARY

We demonstrate that we can identify a global dynamical model from the spiky scalar data of the z-component of the Rössler system. The approach presented can be expanded beyond the z-component of Rössler to analyze other time series that contain spiking and bursting dynamics. In particular, we believe it can be a useful tool for identifying parameters of the Hindmarsh-Rose model that correspond to the various dynamical behaviors observed in neuron outputs.

Acknowledgements

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References

[1] Otto E. Roessler. An equation for continous chaos. *Phys. Lett.*, 57A:397, 1976.

[2] J.L. Hindmarsh and R.M. Rose. A model of the nerve impulse using two first-order differential equations. *Nature*, 296:162, 1982.

[3] J.L. Hindmarsh and R.M. Rose. A model of neuronal bursting using three coupled first order differential equations. *Proc. R. Soc. Lond.*, 221B:87, 1984.

[4] Floris Takens. Detecting strange attractors in turbulence. *Lecture Notes in Mathematics*, 898:366, 1981.

[5] N. H. Packard, J. P. Crutchfield, J. D. Farmer, and R. S. Shaw. Geometry from a time series. *Phys. Rev. Lett.*, 45:712, 1980.

[6] C. Letellier, J. Maquet, L. Le Sceller, G. Gouesbet, and L.A. Aguirre. On the non-equivalence of observables in phase-space reconstructions from recorded time series. *Journal of Physics A*, 31:7913–7927, 1998.

[7] J.P. Crutchfield and B.S. McNamara. Equations of motion from a data series. *Complex Systems*, 1(3):417–52, 1987.

[8] D.S. Broomhead and G.P. King. Extracting qualitative dynamics from experimental data. *Physica D*, 20:217, 1986.

[9] John F. Gibson, J. Doyne Farmer, Martin Casdagli, and Stephen Eubank. An analytic approach to practical state space reconstruction. *Physica D*, 57:1, 1992.

[10] G.P. King and I. Stewart. Phase space reconstruction for symmetric dynamical systems. *Physica D*, 58:216–228, 1992.

[11] C. Letellier and G. Gouesbet. Topological characterization of reconstructed attractors modding out symmetries. *Journal de Physique II*, 6:1615–1638, 1996.

[12] D. Kugiumtzis. State space reconstruction parameters in the analysis of chaotic time series - the role of the time window length. *Physica D*, 95:13–28, 1996.

[13] L. Le Sceller, C. Letellier, and G. Gouesbet. Structure selection for global vector field reconstruction by using the identification of fixed points. *Physical Review E*, 60 (2):1600, 1999.

[14] C. Lainscsek, C. Letellier, and F. Schürrer. Ansatz library for global modeling using a structure selection. *Physical Review E, in press*, 2001.

[15] C. Lainscsek, C. Letellier, and I. Gorodnitsky. Phase space reconstruction of the rössler system from a single variable time series of the z-variable. *in preparation*, 2001.

[16] C. Letellier, P. Dutertre, and G. Gouesbet. Characterization of the lorenz system, taking into account the equivariance of the vector field. *Phys. Rev. E*, 49:3492, 1994.

[17] C. Letellier, P. Dutertre, and B. Maheu. Unstable periodic orbits and templates of the rossler system: toward a systematic topological characterization. *Chaos*, 5(1):271, 1995.