Problem 10.13

At time t the charge is at $\mathbf{r}(t) = a[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}]$, so $\mathbf{v}(t) = \omega a[-\sin(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t)\,\hat{\mathbf{y}}]$. Therefore $\boldsymbol{\mathscr{r}} = z\,\hat{\mathbf{z}} - a[\cos(\omega t_r)\,\hat{\mathbf{x}} + \sin(\omega t_r)\,\hat{\mathbf{y}}]$, and hence $\mathscr{r}^2 = z^2 + a^2$ (of course), and $\mathscr{r} = \sqrt{z^2 + a^2}$.

$$\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{v} = \frac{1}{\mathscr{r}}(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) = \frac{1}{\mathscr{r}}\left\{-\omega a^2[-\sin(\omega t_r)\cos(\omega t_r) + \sin(\omega t_r)\cos(\omega t_r)]\right\} = 0, \text{ so } \left(1 - \frac{\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{v}}{c}\right) = 1.$$

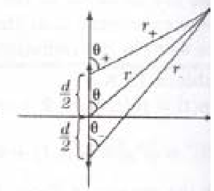
Problem 11.9

At $t = 0$ the dipole moment of the ring is

$$\begin{aligned}\mathbf{p}_0 &= \int \lambda \mathbf{r}\, dl = \int (\lambda_0 \sin\phi)(b\sin\phi\,\hat{\mathbf{y}} + b\cos\phi\,\hat{\mathbf{x}})b\,d\phi = \lambda_0 b^2\left(\hat{\mathbf{y}}\int_0^{2\pi}\sin^2\phi\,d\phi + \hat{\mathbf{x}}\int_0^{2\pi}\sin\phi\cos\phi\,d\phi\right)\\ &= \lambda b^2(\pi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{x}}) = \pi b^2\lambda_0\,\hat{\mathbf{y}}.\end{aligned}$$

As it rotates (counterclockwise, say) $\mathbf{p}(t) = p_0[\cos(\omega t)\,\hat{\mathbf{y}} - \sin(\omega t)\,\hat{\mathbf{x}}]$, so $\ddot{\mathbf{p}} = -\omega^2\mathbf{p}$, and hence $(\ddot{\mathbf{p}})^2 = \omega^4 p_0^2$. Therefore (Eq. 11.60) $P = \dfrac{\mu_0}{6\pi c}\omega^4(\pi b^2\lambda_0)^2 = \boxed{\dfrac{\pi\mu_0\omega^4 b^4\lambda_0^2}{6c}}.$

Problem 11.11



(a) $$\begin{aligned}V_\pm &= \mp\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta_\pm}{r_\pm}\right)\sin[\omega(t - r_\pm/c)]. \quad V_{\text{tot}} = V_+ + V_-.\\ r_\pm &= \sqrt{r^2 + (d/2)^2 \mp 2r(d/2)\cos\theta} \cong r\sqrt{1 \mp (d/r)\cos\theta} \cong r\left(1 \mp \frac{d}{2r}\cos\theta\right).\\ \frac{1}{r_\pm} &\cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right).\end{aligned}$$

$$\begin{aligned}\cos\theta_\pm &= \frac{r\cos\theta \mp (d/2)}{r_\pm} = r\left(\cos\theta \mp \frac{d}{2r}\right)\frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) = \cos\theta \pm \frac{d}{2r}\cos^2\theta \mp \frac{d}{2r}\\ &= \cos\theta \mp \frac{d}{2r}(1 - \cos^2\theta) = \cos\theta \mp \frac{d}{2r}\sin^2\theta.\\ \sin[\omega(t - r_\pm/c)] &= \sin\left\{\omega\left[t - \frac{r}{c}\left(1 \mp \frac{d}{2r}\cos\theta\right)\right]\right\} = \sin\left(\omega t_0 \pm \frac{\omega d}{2c}\cos\theta\right), \text{ where } t_0 \equiv t - r/c.\\ &= \sin(\omega t_0)\cos\left(\frac{\omega d}{2c}\cos\theta\right) \pm \cos(\omega t_0)\sin\left(\frac{\omega d}{2c}\cos\theta\right) \cong \sin(\omega t_0) \pm \frac{\omega d}{2c}\cos\theta\cos(\omega t_0).\end{aligned}$$

$$\begin{aligned}V_\pm &= \mp\frac{p_0\omega}{4\pi\epsilon_0 cr}\left\{\left(1 \pm \frac{d}{2r}\cos\theta\right)\left(\cos\theta \mp \frac{d}{2r}\sin^2\theta\right)\left[\sin(\omega t_0) \pm \frac{\omega d}{2c}\cos\theta\cos(\omega t_0)\right]\right\}\\ &= \mp\frac{p_0\omega}{4\pi\epsilon_0 cr}\left\{\left(\cos\theta \mp \frac{d}{2r}\sin^2\theta \pm \frac{d}{2r}\cos^2\theta\right)\left[\sin(\omega t_0) \pm \frac{\omega d}{2c}\cos\theta\cos(\omega t_0)\right]\right\}\\ &= \mp\frac{p_0\omega}{4\pi\epsilon_0 cr}\left[\cos\theta\sin(\omega t_0) \pm \frac{\omega d}{2c}\cos^2\theta\cos(\omega t_0) \pm \frac{d}{2r}\left(\cos^2\theta - \sin^2\theta\right)\sin(\omega t_0)\right].\\ V_{\text{tot}} &= -\frac{p_0\omega}{4\pi\epsilon_0 cr}\left[\frac{\omega d}{c}\cos^2\theta\cos(\omega t_0) + \frac{d}{r}\left(\cos^2\theta - \sin^2\theta\right)\sin(\omega t_0)\right]\\ &= \boxed{-\frac{p_0\omega^2 d}{4\pi\epsilon_0 c^2 r}\left[\cos^2\theta\cos(\omega t_0) + \frac{c}{\omega r}\left(\cos^2\theta - \sin^2\theta\right)\sin(\omega t_0)\right].}\end{aligned}$$

In the radiation zone ($r \gg \omega/c$) the second term is negligible, so $\boxed{V = -\dfrac{p_0\omega^2 d}{4\pi\epsilon_0 c^2 r}\cos^2\theta\cos[\omega(t - r/c)].}$

Meanwhile

$$\begin{aligned}\mathbf{A}_\pm &= \mp\frac{\mu_0 p_0\omega}{4\pi r_\pm}\sin[\omega(t - r_\pm/c)]\,\hat{\mathbf{z}}\\ &= \mp\frac{\mu_0 p_0\omega}{4\pi r}\left\{\left(1 \pm \frac{d}{2r}\cos\theta\right)\left[\sin(\omega t_0) \pm \frac{\omega d}{2c}\cos\theta\cos(\omega t_0)\right]\right\}\hat{\mathbf{z}}\\ &= \mp\frac{\mu_0 p_0\omega}{4\pi r}\left[\sin(\omega t_0) \pm \frac{\omega d}{2c}\cos\theta\cos(\omega t_0) \pm \frac{d}{2r}\cos\theta\sin(\omega t_0)\right]\hat{\mathbf{z}}.\\ \mathbf{A}_{\text{tot}} &= \mathbf{A}_+ + \mathbf{A}_- = -\frac{\mu_0 p_0\omega}{4\pi r}\left[\frac{\omega d}{c}\cos\theta\cos(\omega t_0) + \frac{d}{r}\cos\theta\sin(\omega t_0)\right]\hat{\mathbf{z}}\\ &= \boxed{-\frac{\mu_0 p_0\omega^2 d}{4\pi c r}\cos\theta\left[\cos(\omega t_0) + \frac{c}{\omega r}\sin(\omega t_0)\right]\hat{\mathbf{z}}.}\end{aligned}$$

In the radiation zone, $\boxed{\mathbf{A} = -\dfrac{\mu_0 p_0 \omega^2 d}{4\pi c r}\cos\theta\cos[\omega(t-r/c)]\,\hat{\mathbf{z}}.}$

(b) To simplify the notation, let $\alpha \equiv -\dfrac{\mu_0 p_0 \omega^2 d}{4\pi}$. Then

$$
\begin{aligned}
V &= \alpha\frac{\cos^2\theta}{r}\cos[\omega(t-r/c)];\\
\nabla V &= \frac{\partial V}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\,\hat{\boldsymbol{\theta}} = \alpha\cos^2\theta\left\{-\frac{1}{r^2}\cos[\omega(t-r/c)] + \frac{\omega}{rc}\sin[\omega(t-r/c)]\right\}\hat{\mathbf{r}}\\
&\quad + \alpha\frac{-2\cos\theta\sin\theta}{r^2}\cos[\omega(t-r/c)]\,\hat{\boldsymbol{\theta}} = \alpha\frac{\omega}{c}\frac{\cos^2\theta}{r}\sin[\omega(t-r/c)]\,\hat{\mathbf{r}} \quad \text{(in the radiation zone).}\\
\mathbf{A} &= \frac{\alpha}{c}\frac{\cos\theta}{r}\cos[\omega(t-r/c)]\left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right). \quad \frac{\partial\mathbf{A}}{\partial t} = -\frac{\alpha\omega}{c}\frac{\cos\theta}{r}\sin[\omega(t-r/c)]\left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right).
\end{aligned}
$$

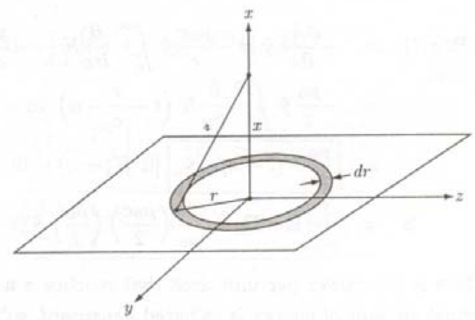
$$
\begin{aligned}
\mathbf{E} &= -\nabla V - \frac{\partial\mathbf{A}}{\partial t} = -\frac{\alpha\omega}{cr}\sin[\omega(t-r/c)]\left(\cos^2\theta\,\hat{\mathbf{r}} - \cos^2\theta\,\hat{\mathbf{r}} + \sin\theta\cos\theta\,\hat{\boldsymbol{\theta}}\right)\\
&= \boxed{-\frac{\alpha\omega}{cr}\sin\theta\cos\theta\sin[\omega(t-r/c)]\,\hat{\boldsymbol{\theta}}.}\\
\mathbf{B} &= \nabla\times\mathbf{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}\\
&= \frac{\alpha}{cr}\left\{\frac{\partial}{\partial r}\left(\cos\theta\cos[\omega(t-r/c)](-\sin\theta)\right) - \frac{\partial}{\partial\theta}\left[\frac{\cos^2\theta}{r}\cos[\omega(t-r/c)]\right]\right\}\hat{\boldsymbol{\phi}}\\
&= \frac{\alpha}{cr}(-\sin\theta\cos\theta)\frac{\omega}{c}\sin[\omega(t-r/c)]\,\hat{\boldsymbol{\phi}} \text{ (in the radiation zone)} = \boxed{-\frac{\alpha\omega}{c^2 r}\sin\theta\cos\theta\sin[\omega(t-r/c)]\,\hat{\boldsymbol{\phi}}.}
\end{aligned}
$$

Notice that $\mathbf{B} = \dfrac{1}{c}(\hat{\mathbf{r}}\times\mathbf{E})$ and $\mathbf{E}\cdot\hat{\mathbf{r}} = 0$.

$$
\begin{aligned}
\mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0 c}\mathbf{E}\times(\hat{\mathbf{r}}\times\mathbf{E}) = \frac{1}{\mu_0 c}\left[E^2\,\hat{\mathbf{r}} - (\mathbf{E}\cdot\hat{\mathbf{r}})\mathbf{E}\right] = \frac{E^2}{\mu_0 c}\,\hat{\mathbf{r}}\\
&= \boxed{\frac{1}{\mu_0 c}\left\{\frac{\alpha\omega}{rc}\sin\theta\cos\theta\sin[\omega(t-r/c)]\right\}^2\hat{\mathbf{r}}.} \quad \boxed{I = \frac{1}{2\mu_0 c}\left(\frac{\alpha\omega}{rc}\sin\theta\cos\theta\right)^2.}\\
P &= \int\langle\mathbf{S}\rangle\cdot d\mathbf{a} = \frac{1}{\mu_0 c}\left(\frac{\alpha\omega}{c}\right)^2\int\sin^2\theta\cos^2\theta\sin\theta\,d\theta\,d\phi = \frac{1}{2\mu_0 c}\left(\frac{\alpha\omega}{c}\right)^2 2\pi\int_0^\pi(1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta.\\
&\qquad \text{The integral is}: -\left.\frac{\cos^3\theta}{3}\right|_0^\pi + \left.\frac{\cos^5\theta}{5}\right|_0^\pi = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}.\\
&= \frac{1}{2\mu_0 c}\frac{\omega^2}{c^2}\frac{\mu_0^2}{16\pi^2}(p_0 d)^2\omega^4 2\pi\frac{4}{15} = \boxed{\frac{\mu_0}{60\pi c^3}(p_0 d)^2\omega^6.}
\end{aligned}
$$

Notice that it goes like ω^6, whereas *dipole* radiation goes like ω^4.

Problem 11.24



(a) $\mathbf{A}(x,t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(t_r)}{\mathscr{r}}\,da$

$= \frac{\mu_0\hat{\mathbf{z}}}{4\pi}\int \frac{K(t_r)}{\sqrt{r^2+x^2}}2\pi r\,dr$

$= \frac{\mu_0\hat{\mathbf{z}}}{2}\int \frac{K(t-\sqrt{r^2+x^2}/c)}{\sqrt{r^2+x^2}}r\,dr.$

The maximum r is given by $t-\sqrt{r^2+x^2}/c = 0$;

$r_{\text{max}} = \sqrt{c^2t^2-x^2}$ (since $K(t)=0$ for $t<0$).

(i)

$$\mathbf{A}(x,t) = \frac{\mu_0K_0\hat{\mathbf{z}}}{2}\int_0^{r_m}\frac{r}{\sqrt{r^2+x^2}}\,dr = \frac{\mu_0K_0\hat{\mathbf{z}}}{2}\sqrt{r^2+x^2}\Big|_0^{r_m} = \frac{\mu_0K_0\hat{\mathbf{z}}}{2}\left(\sqrt{r_m^2-x^2}-x\right) = \frac{\mu_0K_0(ct-x)}{2}\,\hat{\mathbf{z}}.$$

$$\mathbf{E}(x,t) = -\frac{\partial\mathbf{A}}{\partial t} = \boxed{-\frac{\mu_0K_0c}{2}\,\hat{\mathbf{z}},} \text{ for } ct>x, \text{ and } 0, \text{ for } ct<x.$$

$$\mathbf{B}(x,t) = \nabla\times\mathbf{A} = -\frac{\partial A_z}{\partial x}\,\hat{\mathbf{y}} = \boxed{\frac{\mu_0K_0}{2}\,\hat{\mathbf{y}},} \text{ for } ct>x, \text{ and } 0, \text{ for } ct<x.$$

(ii)

$$\mathbf{A}(x,t) = \frac{\mu_0\alpha\,\hat{\mathbf{z}}}{2}\int_0^{r_m}\frac{(t-\sqrt{r^2+x^2}/c)}{\sqrt{r^2+x^2}}r\,dr = \frac{\mu_0\alpha\hat{\mathbf{z}}}{2}\left[t\int_0^{r_m}\frac{r}{\sqrt{r^2+x^2}}\,dr - \frac{1}{c}\int_0^{r_m}r\,dr\right]$$

$$= \frac{\mu_0\alpha\,\hat{\mathbf{z}}}{2}\left[t(ct-x)-\frac{1}{2c}(c^2t^2-x^2)\right] = \frac{\mu_0\alpha\,\hat{\mathbf{z}}}{4c}(x^2-2ctx+c^2t^2) = \frac{\mu_0\alpha(x-ct)^2}{4c}\,\hat{\mathbf{z}}.$$

$$\mathbf{E}(x,t) = -\frac{\partial\mathbf{A}}{\partial t} = \boxed{\frac{\mu_0\alpha(x-ct)}{2}\,\hat{\mathbf{z}},} \text{ for } ct>x, \text{ and } 0, \text{ for } ct<x.$$

$$\mathbf{B}(x,t) = \nabla\times\mathbf{A} = -\frac{\partial A_z}{\partial x}\,\hat{\mathbf{y}} = \boxed{-\frac{\mu_0\alpha}{2c}(x-ct)\,\hat{\mathbf{y}},} \text{ for } ct>x, \text{ and } 0, \text{ for } ct<x.$$

(b) Let $u \equiv \frac{1}{c}\left(\sqrt{r^2+x^2}-x\right)$, so $du = \frac{1}{c}\left[\frac{1}{2}\frac{1}{\sqrt{r^2+x^2}}2r\,dr\right] = \frac{1}{c}\frac{r}{\sqrt{r^2+x^2}}\,dr$, and

$t-\frac{\sqrt{r^2+x^2}}{c} = t-\frac{x}{c}-u$, and as $r: 0\to\infty$, $u: 0\to\infty$. Then $\mathbf{A}(x,t) = \frac{\mu_0 c\,\hat{\mathbf{z}}}{2}\int_0^\infty K\left(t-\frac{x}{c}-u\right)du.$ qed

$$\mathbf{E}(x,t) = -\frac{\partial\mathbf{A}}{\partial t} = -\frac{\mu_0c\,\hat{\mathbf{z}}}{2}\int_0^\infty\frac{\partial}{\partial t}K\left(t-\frac{x}{c}-u\right)du. \text{ But } \frac{\partial}{\partial t}K\left(t-\frac{x}{c}-u\right) = -\frac{\partial}{\partial u}K\left(t-\frac{x}{c}-u\right).$$

$$= \frac{\mu_0c}{2}\,\hat{\mathbf{z}}\int_0^\infty\frac{\partial}{\partial u}K\left(t-\frac{x}{c}-u\right)du = \frac{\mu_0c}{2}\,\hat{\mathbf{z}}\left[K\left(t-\frac{x}{c}-u\right)\right]\Big|_0^\infty = -\frac{\mu_0c}{2}\left[K(t-x/c)-K(-\infty)\right]\hat{\mathbf{z}}$$

$$= \boxed{-\frac{\mu_0c}{2}K(t-x/c)\,\hat{\mathbf{z}},} \text{ [if } K(-\infty)=0].$$

Note that (i) and (ii) are consistent with this result. Meanwhile

$$\mathbf{B}(x,t) = -\frac{\partial A_z}{\partial x}\,\hat{\mathbf{y}} = -\frac{\mu_0c}{c}\,\hat{\mathbf{y}}\int_0^\infty\frac{\partial}{\partial x}K\left(t-\frac{x}{c}-u\right)du. \text{ But } \frac{\partial}{\partial x}K\left(t-\frac{x}{c}-u\right) = \frac{1}{c}\frac{\partial}{\partial u}K\left(t-\frac{x}{c}-u\right).$$

$$= -\frac{\mu_0}{2}\,\hat{\mathbf{y}}\int_0^\infty\frac{\partial}{\partial u}K\left(t-\frac{x}{c}-u\right)du = -\frac{\mu_0}{2}\,\hat{\mathbf{y}}\left[K\left(t-\frac{x}{c}-u\right)\right]\Big|_0^\infty = \frac{\mu_0}{2}\left[K(t-x/c)-K(-\infty)\right]\hat{\mathbf{y}}$$

$$= \boxed{\frac{\mu_0}{2}K(t-x/c)\,\hat{\mathbf{y}},} \text{ [if } K(-\infty)=0].$$

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}\left(\frac{\mu_0c}{2}\right)\left(\frac{\mu_0}{2}\right)K(t-x/c)\left[-\hat{\mathbf{z}}\times\hat{\mathbf{y}}\right] = \frac{\mu_0c}{4}\left[K(t-x/c)\right]^2\hat{\mathbf{x}}.$$

This is the power per unit area that reaches x at time t; it left the surface at time $(t-x/c)$. Moreover, an equal amount of energy is radiated *downward*, so the total power leaving the surface at time t is $\frac{\mu_0c}{2}[K(t)]^2$.

Physics 106c – Problem Set 7 – Due May 29th, 2009
Solutions

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May 28, 2009

Problem 5

(a) From Eq. (9.48), we can get the electromagnetic wave, propagating in z-direction with an electric field of amplitude E_0 polarized along the x-axis.

$$\mathbf{E}(z,t) = E_0 \cos(kz - \omega t + \delta)\hat{\mathbf{x}}$$

Because the velocity of the charge remains very small compared to that of light, the Lorentz force is dominated by the electric field. Note that the point charge is sitting at the origin.

$$\begin{aligned}
\mathbf{F} &= q\mathbf{E}(0,t) = qE_0 \cos(-\omega t + \delta)\hat{\mathbf{x}} = m\mathbf{a} \\
\Rightarrow \mathbf{a} &= \frac{qE_0}{m}\cos(-\omega t + \delta)\hat{\mathbf{x}} \\
\therefore \mathbf{x}(t) &= \frac{qE_0}{m\omega^2}(\cos\delta - \cos(-\omega t + \delta))\hat{\mathbf{x}}
\end{aligned}$$

if we take the initial conditions into consideration.

(b) From Eq. (11.69), we know

$$\begin{aligned}
\mathbf{S}_{rad} &= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{\mathbf{r}} = \frac{\mu_0 q^4 E_0^2}{16\pi^2 m^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2(-\omega t + \delta)\hat{\mathbf{r}} \\
\Rightarrow < \mathbf{S}_{rad} > &= \frac{\mu_0 q^4 E_0^2}{32\pi^2 m^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{\mathbf{r}}
\end{aligned}$$

(c) From Eq. (9.57), the total power over the area σ in the incoming electromagnetic wave is $c\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\sigma\hat{\mathbf{z}}$. Averaging over time, we can get $\frac{1}{2}c\epsilon_0 E_0^2 \sigma\hat{\mathbf{z}}$. Furthermore, the total power radiated by the point charge is

$$\int < \mathbf{S}_{rad} > \cdot d\mathbf{a} = \frac{\mu_0 q^4 E_0^2}{32\pi^2 m^2 c}\int\left(\frac{\sin^2\theta}{r^2}\right) r^2 \sin\theta d\theta d\phi = \frac{\mu_0 q^4 E_0^2}{12\pi m^2 c}$$

Therefore, the total cross-section is

$$\sigma = \frac{\mu_0^2 q^4}{6\pi m^2}$$