problem 1

Let's start by assuming a harmonic wave is incident: $V_{inc} = \cos(k(vt - x))$

The reflected wave we call $V_{ref} = R\cos(k(vt+x)+\phi)$

Total voltage wave $= V_{inc} + R\,V_{ref}$.

Total current wave $= \frac{1}{R_c}(V_{inc} - R\,V_{ref})$ where R_c is the characteristic impedance.

We have a capacitive termination at $x=0 \Rightarrow$
$I(x=0,t) = C\left.\frac{\partial V}{\partial t}\right|_{x=0}$

$$\frac{\partial V}{\partial t} = -kv\sin(k(vt-x)) - RkV\sin(k(vt+x)+\phi)$$

$\Rightarrow$ at $x=0$ $\quad \frac{\partial V}{\partial t} = -kv\sin kvt - Rkv\sin(kvt+\phi)$

$$I(x=0,t) = \frac{1}{R_c}\left\{\cos kvt - R\cos(kvt+\phi)\right\}$$

Define $\alpha = R\cos\phi$, $\beta = R\sin\phi$, $\gamma = R_c C k v$

Boundary condition $\Rightarrow$

$$-\gamma\sin kvt - \gamma\alpha\sin kvt - \gamma\beta\cos kvt$$
$$= \cos kvt - \alpha\cos kvt + \beta\sin kvt$$

Equating $\sin kvt$ & $\cos kvt$ terms gives:

$$-\gamma(1+\alpha) = \beta$$
$$-\gamma\beta = 1-\alpha$$

Solving,

$$\boxed{\alpha = \frac{1-\gamma^2}{1+\gamma^2} \qquad \beta = \frac{-2\gamma}{1+\gamma^2}}$$

Note that when the cap is removed, $\gamma=0 \Rightarrow \alpha=1, \beta=0$ which is correct for an open ended coax. Likewise, when $C=\infty$ the coax is effectively shorted and so $\alpha=-1, \beta=0$ which is also correct.

Now we actually have a square pulse incident. Forget the capacitor for a moment & assume the coax is ∞-long. We can write our incident pulse as a Fourier cosine transform:

$$V_{inc}(x,t) = \int_0^\infty g(K) \cos(K(vt-x))\, dK$$

For simplicity, suppose at $t=0$ this pulse is centered at $X=0$ (the coax is ∞ & there is no capacitor). For that case

$$V(x,0) = \begin{cases} V_0 & |x| < vT/2 \\ 0 & |x| > vT/2 \end{cases} \qquad v = \text{velocity}$$

where T is the width of the pulse in time.

$$V(x,0) = \int_0^\infty g(K) \cos Kx \, dK$$

You should know that for this case,

$$g(K) = \text{const.} \frac{\sin Ka}{Ka} \qquad a = \frac{vT}{2}$$

Since we are ultimately interested only in the shape of the reflected pulse we will henceforth ignore the constant. {I think the constant is $\frac{vTV_0}{3}$}

Now that we know $g(K)$ we can construct our reflected pulse. Since the wave-equation is linear, it must be

$$V_{ref} = \text{const}\cdot \int_0^\infty \frac{\sin ka}{ka}\, R \cos(k(vt+x)+\phi)\, dk$$

$$= \text{const}\cdot \int_0^\infty \frac{\sin ka}{ka}\left\{ \alpha \cos k(vt+x) - \beta \sin k(vt+x) \right\} dk$$

Once again we can ignore the termination and assume this wave is travelling along an ∞ coax. At $t=0$ then, the pulse is

$$V_{ref}(x) = \text{const}\cdot \int_0^\infty dk\, \frac{\sin ka}{ka}\left(\alpha \cos kx - \beta \sin kx \right)$$

with $\alpha = \frac{1-\gamma^2}{1+\gamma^2}$ $\beta = \frac{-2\gamma}{1+\gamma^2}$ $\gamma = k R_c C v$

and $a = vT/2$.

I don't know how to do this integral analytically, so I'll resort to numerics.

Define $u = ka$

$$V_{ref} = \frac{\text{const}}{a} \int_0^\infty \frac{\sin u}{u}\left\{ \alpha \cos bu - \beta \sin bu \right\} du$$

where $b = x/a$. Note also that $\gamma = kR_c C v = 2u\,(R_c C/T)$. The factor $R_c C/T$ is the ratio of the RC time created from the coax characteristic impedance and T, the width of the pulse. For our problem

$$\frac{R_c C}{T} = \frac{50\Omega \times 10^{-10} F}{20 \times 10^{-9} s} = \frac{1}{4} \Rightarrow \boxed{\gamma = \frac{u}{2}}$$

here

Lets stay general though, and set $\gamma = u\delta$ where $\delta = 2R_cC/T = 1/2$ in our problem.

$$V_{ref} = \frac{\text{const}}{a} \underbrace{\int_0^\infty \frac{du \sin u}{u(1+\delta^2u^2)} \left\{(1-\delta^2u^2)\cos bu + 2\delta u \sin bu\right\}}_{I(b)}$$

I performed the integral numerically.
The upper limit on u was set to be 1000. I did a straight forward box integration of 10^6 steps, i.e. $du = 10^{-3}$. I computed the integral for 400 values of b, ranging from -4 to $+4$.

Since $b = x/a$ and $a = vT/2$, the actual initial pulse ran from $b = -1$ to $+1$

The constant out infront, $\frac{\text{const}}{a}$, is really

$\frac{\text{const}}{a} = vT\,\frac{V_0}{3} \cdot \frac{1}{a} = \frac{2}{3} V_0$. The problem as stated gave $V_0 = 1\,\text{volt}$, $T = 20\,\text{ns}$, $R_c = 50\,\Omega$, $C = 100\,\text{pF}$.

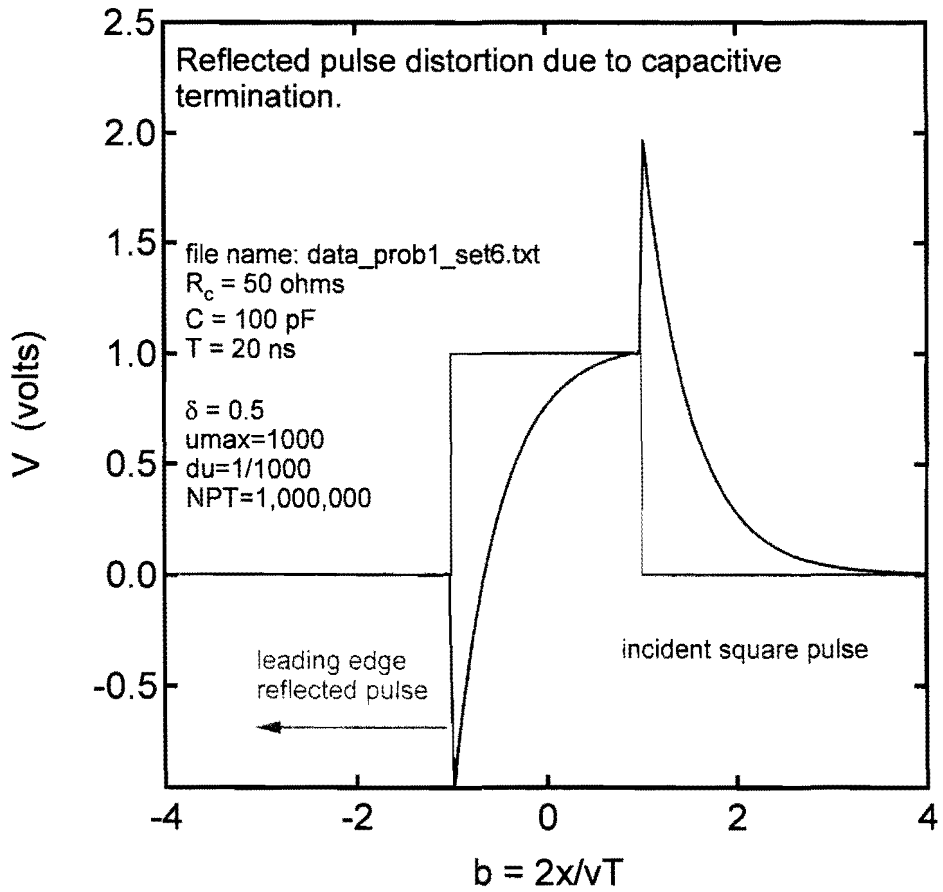
Reflected pulse shape calculated
by numerical integration using
HT Basic program prob1_set6.txt

50 ohm coax terminated by 100 pF capacitor.

20 ns wide, 1 volt high square pulse incident from left.

The strange shape makes sense. When the very sharp leading edge of the incident pulse hits the capacitor, the very sharpness of it comes from very high frequencies. For these high frequencies the capacitor acts like a shorted termination. Hence, the initial reflection is *inverted*. As time goes by (while the pulse is still at the capacitor) the much lower frequencies of the pulse are most important. For these the capacitor behaves like an open circuit. Hence, the reflection attempts to rise back up to being in-phase with the incident voltage (i.e. the inversion is inverted!). Finally, when the tailing edge of the pulse hits the capacitor, the voltage drop if the incident pulse is inverted and added to the existing level of the reflected pulse. This accounts for the sharp rise at the tailing edge. After this, the voltage falls back to zero. The time constant for the decay is RC, where R=50 ohms and C=100pf, so RC=5ns.

In the figure, v is the velocity of waves in the coax and T is the pulse width. b is a dimensionless distance parameter.


Reflected pulse distortion due to capacitive termination.
file name: data_prob1_set6.txt
R_c = 50 ohms
C = 100 pF
T = 20 ns
δ = 0.5
umax=1000
du=1/1000
NPT=1,000,000
leading edge
reflected pulse
incident square pulse
V (volts)
b = 2x/vT
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-4
-2
0
2
4

```
3    CREATE "temp.txt",500
4    ASSIGN @P1 TO "temp.txt";FORMAT ON
10 Lp: FOR K=-200 TO 200
11    B=K/50
13    Umax=1000
20    N=1000
21    Npt=N*Umax
23    Du=1/N
30    D=.5
50    V=0
60    FOR J=0 TO Npt
70    U=J*Du
71    X=D*D*U*U
72    IF J=0 THEN
73      F=1
74    ELSE
80      F=SIN(U)/U/(1+X)
81    END IF
90    G=(1-X)*COS(B*U)+2*D*U*SIN(B*U)
100    V=V+Du*F*G
110    NEXT J
111    V=V*2/3
113    PRINT B,V
120    OUTPUT @P1;Umax,N,B,V
130    NEXT K
131    ASSIGN @P1 TO *
145    END
```

BASIC program to do numerical integration

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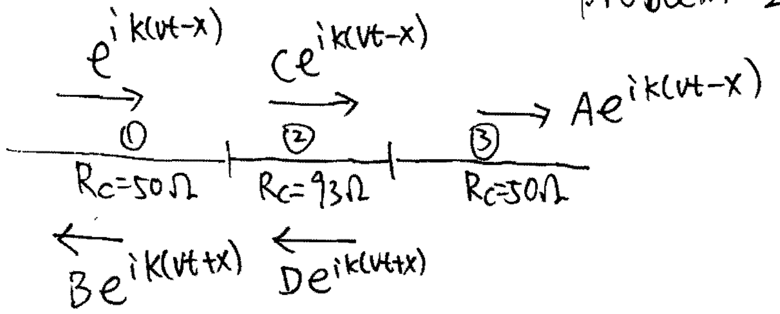
Problem 2



So In steady state there are waves in different regiom

In the boundary between ① and ② $\quad x=0$

Voltage continous: $\quad 1+B=C+D$

Current : $\quad \frac{1}{R_1}-\frac{B}{R_2}=\frac{C}{R_1}-\frac{D}{R_2}$

In the boundary between ② and ③ $\quad x=l$

Voltage : $\quad Ce^{-ikl}+De^{ikl}=Ae^{-ikl}$

Current : $\quad \frac{Ce^{-ikl}}{R_1}-\frac{De^{ikl}}{R_2}=\frac{Ae^{-ikl}}{R_1}$

Solve the above equations we can get

$$T=|A|^2=\left|\frac{2}{1+\frac{R_1^2+R_2^2}{2R_1R_2}+e^{i2kl}\left(1-\frac{R_1^2+R_2^2}{2R_1R_2}\right)}\right|^2=\frac{1}{1.22-0.22\cos(9.5\times10^{-10}\omega)}$$

When $2kl=2n\pi$ resonance $T=1$

$2kl=(2n+1)\pi$ anti-resonance T minimun

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Problem 3

For TE modes we have $\omega_{mn} = c\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}$

For the lowest mode, $\omega_{10} = \frac{c\pi}{a} = 2\pi \cdot 10\text{GHz} \Rightarrow a = 1.5\text{cm}$

2nd mode, $\omega_{01} = \frac{c\pi}{b} = 2\pi \cdot 18\text{GHz} \Rightarrow b = 0.8\text{cm}$

3rd mode. $\omega_{20} = \frac{2\pi}{a} = 20\text{GHz} \cdot 2\pi$

4th mode. $\omega_{11} = c\sqrt{\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}} = 20.6\text{ GHz} \cdot 2\pi$

5th mode $\omega_{02} = c\sqrt{\frac{4\pi^2}{b^2}} = 36\text{GHz} \cdot 2\pi$

For TM modes $\omega_{11} = \omega_{11}(\text{TE}) = 20.6\text{GHz} \cdot 2\pi$

So between 18–22 GHz there are 3 TE modes and 1 TM mode.

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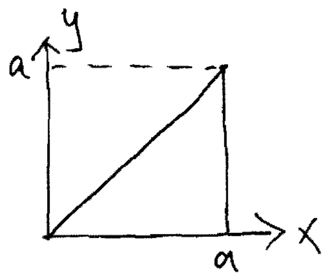
Problem 4



For this problem, we can think of the triangle as "half" of a square. The key step is to note that there is a Z_2 reflection on the coordinates of the form.

$$Z_2 : x \to y , y \to x$$

Eigenfunction $\psi(x,y)$ can be classifield as either Z_2-odd or Z_2-even

$$Z_2 : \psi(x,y) \to \pm \psi(x,y)$$

On the diagonal, the odd function satisfy Dirichlet condition automatically. (TM)

The even function satisfy Neumann condition $\to$ TE

More specifically, for TE mode, ψ is B_z, we have Z_2-even eigenfunction:

$$B_z = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} + \cos\frac{n\pi x}{a}\cos\frac{m\pi y}{b} \quad (m=n=0 \text{ not allowed})$$

It's easy to check $B_x = B_y$ on the diagnal.

Then it's trivial to get the cut-off frequency