Physics 106c – Problem Set 5 — Solutions

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Problem 1

Plane E&M wave incident on thin metal plate, conductivity σ, thickness t. Skin depth $\delta \gg t$.
Perpendicular incidence.
Incident wave $\vec{E}$ field linearly polarized along $\hat{x}$

$$\vec{E}_{inc} = E_0 \hat{x} e^{i(kz-\omega t)}$$
$$\vec{B}_{inc} = \frac{E_0}{c} \hat{y} e^{i(kz-\omega t)}$$

a) Plate is so thin that it can be treated as a sheet.

Boundary conditions:

$\vec{E}_{\parallel}$ continuous

$\vec{H}_{\parallel}$ DISCONTINUOUS, determined by currents flowing in sheet.

Specifically, $\Delta \vec{H}_{\parallel} = \vec{K} \times \hat{n}$ where $\vec{K}$ is the sheet current density

$$\vec{K} = \vec{J} t = \sigma t \vec{E} \quad \text{by Ohm's Law.}$$

Since vacuum is on both sides of the sheet $\vec{H} = \vec{B}/\mu_0$

$\Longrightarrow$

$$\Delta \vec{B}_{\parallel} = \mu_0 \sigma t \, \vec{E} \times \hat{n}$$

normal to surface

Reflected + Transmitted Beams:

$$\vec{E}_r = E_{or}\hat{x}\, e^{i(-kz-\omega t)}$$
$$\vec{B}_r = -\frac{E_{or}}{c}\hat{y}\, e^{i(-kz-\omega t)}$$

$$\vec{E}_t = E_{ot}\hat{x}\, e^{i(+kz-\omega t)}$$
$$\vec{B}_t = \frac{B_{ot}}{c}\hat{y}\, e^{i(kz-\omega t)}$$

Continuity of $E_{\parallel}$ $\Rightarrow$ $\boxed{E_o + E_{or} = E_{ot}}$ ①

Discontinuity of $H_{\parallel}$ $\Rightarrow$

$$\hat{y}(B_o + B_{or} - B_{ot}) = \mu_o \sigma t (E_{ot}\hat{x} \times \hat{n})$$

where $\hat{n} = -\hat{z}$ in this case. [E_{ot} will be the electric field in the plane — could use $E_o + E_{or}$ just as well) Since $\hat{x} \times \hat{n} = \hat{y}$ we get

$$B_o + B_{or} - B_{ot} = \mu_o \sigma t\, E_{ot}$$

or:

$$E_o - E_{or} - E_{ot} = \mu_o \sigma t c\, E_{ot}$$

let $\xi \equiv \mu_o \sigma t c$. We then have

$$\boxed{E_o = E_{or} + (1+\xi) E_{ot}}$$ ②

Combining (1) & (2) Gives:

$$\boxed{\begin{aligned} E_{or} &= -E_o \frac{\xi}{2+\xi} \\ E_{ot} &= E_o \frac{2}{2+\xi} \end{aligned}}$$

$\xi \equiv \mu_o c \sigma t$

Similarly:

$$\boxed{\begin{aligned} B_{or} &= +\frac{E_o}{c} \frac{\xi}{2+\xi} \\ B_{ot} &= \frac{E_o}{c} \frac{2}{2+\xi} \end{aligned}}$$

Note 2 limits:

$\sigma = 0 \Rightarrow E_{or} = 0 \quad E_{ot} = 1$

no reflection at all.

$\sigma = \infty \Rightarrow E_{or} = -E_o \quad E_{ot} = 0$

perfect reflection (w/180° phase reversal)

4) b Energy considerations.

Poynting flux $\vec{S} = \frac{1}{\mu_o} \vec{E} \times \vec{B}$

$$|S_{inc}| = \frac{1}{\mu_o c} E_o^2$$

$$|S_{refl}| = \frac{1}{\mu_o c} E_{or}^2 = |S_{inc}| \frac{\xi^2}{(2+\xi)^2}$$

$$|S_{trans}| = \frac{1}{\mu_o c} E_{ot}^2 = |S_{inc}| \frac{4}{(2+\xi)^2}$$

note that $|S_{ref}| + |S_{trans}| = |S_{inc}| \dfrac{4+\xi^2}{(2+\xi)^2}$

$< |S_{inc}|$

So, energy must be being absorbed by the plate.

$$\text{Power absorption fraction} = \frac{|S_{inc}| - |S_{ref}| - |S_{trans}|}{|S_{inc}|}$$

$$= 1 - \frac{(\xi^2+4)}{(2+\xi)^2}$$

$$\boxed{\text{Absorption Fraction} = \frac{4\xi}{(2+\xi)^2}}$$

4c) To maximize absorption, take derivative w.r.t. ξ:

$$\frac{d}{d\xi} \frac{4\xi}{(2+\xi)^2} = \frac{8-4\xi}{(2+\xi)^3}$$

$$= 0 \quad \text{when} \quad \xi = 2$$

$$\boxed{\text{Max absorption at } \mu_0 c \sigma t = 2}$$

$$\boxed{\text{At maximum, the fraction absorbed} = \tfrac{1}{2}}$$

Note that $\mu_0 c = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 377\,\Omega$, the so called "impedance of free space"

The absorption is maximized when

$$\frac{1}{\sigma t} = \text{resistivity of sheet}$$

$$= \frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}} = 189\,\Omega$$

Problem 2

2. ① For TE_{10} mode in rectangular waveguide:

$B_z = B_0 \cos\left(\frac{\pi x}{a}\right)$, $E_z = 0$.

Then using (9.180) $\Rightarrow$ $\begin{cases} E_x = 0, & E_y = \frac{i\omega}{(\omega/c)^2 - k^2} B_0 \sin\left(\frac{\pi x}{a}\right) \cdot \frac{\pi}{a} \\ B_x = \frac{-ik}{(\omega/c)^2 - k^2} B_0 \sin\left(\frac{\pi x}{a}\right) \cdot \frac{\pi}{a}, & B_y = 0. \end{cases}$

② (9.181) $\Rightarrow$ $k = \sqrt{(\omega/c)^2 - \frac{\pi^2}{a^2}}$ $\Rightarrow$ $\left(\frac{\omega}{c}\right)^2 = k^2 + \frac{\pi^2}{a^2}$

$\therefore$ $E_y = i\omega \frac{a}{\pi} B_0 \sin\left(\frac{\pi x}{a}\right)$, $B_x = -ik\frac{a}{\pi} B_0 \sin\left(\frac{\pi x}{a}\right)$.

$S_z = \frac{1}{\mu_0}(\vec{E}\times\vec{B})_z = \frac{-1}{\mu_0} E_y^* B_x = + \frac{\omega k}{\mu_0}\left(\frac{a}{\pi}\right)^2 B_0^2 \sin^2\left(\frac{\pi x}{a}\right)$

$\Rightarrow$ total amount of energy per unit time flowing down the guide

$$= \int_0^b \int_0^a \langle S_z \rangle \, dx\, dy = \frac{\omega k}{2\mu_0} \frac{a^3 b}{\pi^2} B_0^2 = \frac{a^3 b}{2\mu_0 \pi^2} B_0^2 \frac{\omega}{c} \sqrt{\omega^2 - \omega_{10}^2}$$

③ stored energy per unit length in the guide

$$= \int_0^b \int_0^a \left(\frac{\epsilon_0}{2}|E|^2 + \frac{|B|^2}{2\mu_0} \right) dx\, dy$$

$$= \int_0^b \int_0^a \left[\frac{\epsilon_0}{2}\omega^2 \frac{a^2}{\pi^2} B_0^2 \sin^2\left(\frac{\pi x}{a}\right) + \frac{1}{2\mu_0} k^2 \frac{a^2}{\pi^2} B_0^2 \sin^2\left(\frac{\pi x}{a}\right) + \frac{1}{2\mu_0} B_0^2 \cos^2\left(\frac{\pi x}{a}\right) \right] dx\,dy$$

$$= \frac{\epsilon_0}{4}\omega^2 \frac{a^3 b}{\pi^2} B_0^2 + \frac{1}{4\mu_0} k^2 \frac{a^3 b}{\pi^2} B_0^2 + \frac{B_0^2}{4\mu_0} \cdot ab$$

$$\underset{c^2 = \frac{1}{\mu_0 \epsilon_0}}{=} \frac{1}{4\mu_0} \frac{a^3 b}{\pi^2} B_0^2 \left(\frac{\omega^2}{c^2} + k^2 + \frac{\pi^2}{a^2} \right)$$

$$\underset{\left(\frac{\omega}{c}\right)^2 = k^2 + \frac{\pi^2}{a^2}}{=} \frac{a^3 b}{2\mu_0 \pi^2} B_0^2 \frac{\omega^2}{c^2}$$

$$= \frac{a^3 b}{2\mu_0 \pi^2} B_0^2 \frac{\omega}{c} \sqrt{\omega^2 - \omega_{10}^2} \left(\frac{1}{c\sqrt{1 - \left(\frac{\omega_{10}}{\omega}\right)^2}} \right)$$

$$= \frac{a^3 b}{2\mu_0 \pi^2} B_0^2 \frac{\omega}{c} \sqrt{\omega^2 - \omega_{10}^2} \cdot \frac{1}{v_g} .$$ ※

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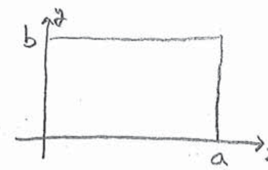
Problem 3

Problem 2 (20 points)



TE waves: $E_z = 0$, $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] B_z = 0$

$B_x = \frac{i}{(\omega/c)^2 - k^2} k \frac{\partial B_z}{\partial x}$ $\quad B_y = \frac{i}{(\omega/c)^2 - k^2} k \frac{\partial B_z}{\partial y}$ $\quad (a \geq b)$

TM waves: $B_z = 0$, $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] E_z = 0$

Boundary conditions. $\vec{E}^{\parallel} = 0 \Rightarrow E_z = 0$ at $x = 0$ and a, $y = 0$ and b

$\vec{B}^{\perp} = 0 \Rightarrow B_x = 0$ at $x = 0$ and a, $B_y = 0$ $y = 0$ and b

For TE waves, $B_z = B_0 \cos(m\pi x/a)\cos(n\pi y/b)$ and $\omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2}$

For TM waves, $E_z = E_0 \sin(m\pi x/a)\sin(n\pi y/b)$ and $\omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2}$,

a) Without loss of generality $a \geq b$.

$f_{mn} = \omega_{mn}/2\pi = \frac{c}{2}\sqrt{(m/a)^2 + (n/b)^2}$

The lowest frequency is f_{10}. $f_{10} = \frac{c}{2a} = \frac{3\times10^8}{2a} = 10\,\text{GHz} = 10^{10}\,\text{Hz}$

$\Rightarrow a = 1.5\times10^{-2}\,\text{m} = 1.5\,\text{cm}$

$f_{20} = 20\,\text{GHz} > 18\,\text{GHz}$, so the second lowest frequency is f_{01}.

$f_{01} = \frac{c}{2b} = \frac{3\times10^8}{2b} = 18\times10^9\,\text{Hz} \Rightarrow b = \frac{1}{120}\,\text{m} = \frac{1}{1.2}\,\text{cm} \cong 0.83\,\text{cm}$

b) When at least m or n is zero, there are no TM_{m0}, TM_{0n}, TM_{00} modes. So these modes are TE modes.

c) We found that $18\,\text{GHz} \leq f_{01}, f_{20} \leq 22\,\text{GHz}$

$f_{11} = \frac{c}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} = \sqrt{100 + 18\times18} = 20.5913$, and others are larger than 22 GHz.

$\therefore$ 3 TE modes: TE_{01}, TE_{20}, TE_{11}

1 TM mode: TM_{11}