

Physics 106c – Problem Set 3 — Solutions

April 28, 2012

Problem 1

$$B = \gamma t$$

$$L = 30 \text{ cm}$$

$$R = 3 \text{ mm}$$

$$\gamma = .001 \text{ T/s}$$

$$\rho = 1.6 \times 10^{-6} \text{ ohm-meter}$$

Wrong -- see below

First, split the copper rod into rings, with width dr and height dz . A given ring is located at a length z along the rod and at a radius r , with z and r varying from $0 \leq r \leq R$ and $0 \leq z \leq L$. The flux through the ring is

$$\Phi = B\pi r^2$$

Due to the symmetry of the problem, the electrical field induced by the varying magnetic field should be constant around the ring. Thus, the line integral of the electrical field around the ring will just be the product of the field strength and ring circumference. Note for later that this line integral is literally the voltage around the ring. This gives

$$d\Phi/dt = \frac{dB}{dt} \pi r^2 = \gamma \pi r^2 = E 2\pi r = V$$

$$\vec{E} = (-r\gamma/2)\hat{\theta}$$

Using the previous week's derivation of the relation between σ , $\vec{E}$, and $\vec{B}$, and if we recognize that only the θ portion of the $\vec{J}$ will be relevant, we find that

$$J = \sigma / (1 + (\frac{tB}{ne})^2) (-\frac{\gamma}{2}\hat{\theta})$$

$$\sigma = 1/\rho$$

Using the relation that the power dissipated P is $P = VI$, and integrating θ from 0 to 2π , we find

$$P_{loop} = V \int \vec{J} \cdot \hat{\theta} dA = \gamma^2 \pi r^3 dr dz / 2\rho (1 + (\frac{tB}{ne})^2)$$

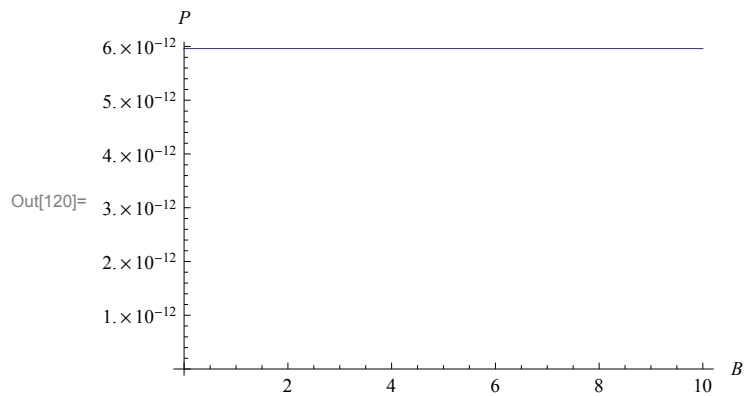
The power dissipated in the entire rod is simply the integral of this quantity over all applicable z and r values. This gives

$$P = \gamma^2 \pi L r^4 / 8\rho (1 + (\frac{tB}{ne})^2)$$

```

γ = 0.001;
R = 0.003;
L = 0.3;
ρ = 1.6 * 10 ^ (-6);
n = 8.5 * 10 ^ (28);
e = 1.6 * 10 ^ (-19);
P[B_] := Pi * γ ^ 2 * R ^ 4 * L / 8 / ρ / (1 + (B / n / e / ρ) ^ 2);
Plot[P[B], {B, 0, 10}, AxesLabel → {B, P}, AxesOrigin → {0, 0}]

```



NOTE: I gave you the wrong value of the resistivity!! At low temperatures ($T < 10\text{K}$) the typical resistivity of copper is about 10^4 times smaller than the value I gave. For this reason, the heating calculation showed extremely little dependence on B . In reality, the heating drops off quickly at high fields due to the B^2 term in the denominator. Mea culpa!

Problem 2

a) $\nabla \times \vec{E} = d\vec{B}/dt$

Inside a perfect conductor, E is the 0 function on the microscopic level, so dB/dt is as well.

b) $\nabla \times \vec{E} = d\vec{B}/dt$

Consider a loop of perfectly conducting wire of radius a . The line integral of the electrical field around the loop is equal to the time derivative of the magnetic flux in the loop. Since we have a perfect conductor, $E = 0$, the line integral is 0, which requires the derivative of the flux to be 0, i.e. it is constant.

c) Since a superconductor is among other things a perfect conductor, the conductivity σ is infinite. A current of any magnitude can flow within the conductor with 0 $\vec{E}$ field, hence the electrical field must be 0.

d) As the sphere goes through its superconducting transition, the superconductor must generate a current to cancel the $B = B_0 \hat{z}$ field at each point inside the sphere, and that current must be confined to the surface of the sphere due to the arguments of the previous parts of the problem. Note from Ex 5.11 the magnetic field of a sphere with charge density σ , radius R , and rotating at an angular velocity ω generates a uniform magnetic field with the sphere of strength

$$B = 2\mu_0 \sigma R \omega / 3$$

directed in the $+\hat{z}$ direction.

$$\sigma = 3B_0 / (2\mu_0 R \omega)$$

The current at an angle θ on the surface of the sphere is

$$J = \sigma v = \sigma R \sin \theta \omega = 3B_0 \sin \theta / (2\mu_0)$$

Problem 3

Φ - magnetic flux through one loop.

1 - primary coil

2 - secondary coil

$$\Phi_1 = N_1 \Phi, \Phi_2 = N_2 \Phi.$$

$$\mathcal{E}_1 = - \frac{d\Phi_1}{dt} = -N_1 \frac{d\Phi}{dt}$$

$$\mathcal{E}_2 = - \frac{d\Phi_2}{dt} = -N_2 \frac{d\Phi}{dt} \Rightarrow \frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}.$$

Problem 4

a)

For the upper plate only, $\mathbf{B} = -(\mu_0/2)K\hat{x}$ above the plane, and $\mathbf{B} = +(\mu_0/2)K\hat{x}$ below the plane. For the lower plate only, $\mathbf{B} = +(\mu_0/2)K\hat{x}$ above the plane, and $\mathbf{B} = -(\mu_0/2)K\hat{x}$ below the plane. So together we have

$$\mathbf{B} = \begin{cases} \mu_0 K \hat{x} & \text{between the plates} \\ 0 & \text{everywhere else} \end{cases}$$

b)

First choose our coordinate system such that the upper plate is at $y = d/2$ and the lower plate is at $y = -d/2$. The induced electric field satisfies:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \alpha \hat{x}$$

In analogy with Maxwell's eqn $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, we can take $-\alpha \hat{x}$ to be the "body current" that generate $\mathbf{E}$. Furthermore, if we think of this "body current" as composed of stack of parallel infinite planes with surface current $\mathbf{K}'$, we can conclude that $\mathbf{E}$ has only $\hat{z}$ component, is antisymmetric for $z = 0$ plane, is in $-\hat{z}$ direction for $z > 0$ and in $\hat{z}$ direction for $z < 0$.

To calculate the electric field between the plates, draw a rectangle perpendicular to $\mathbf{B}$ and symmetrically cross $z = 0$ plane, we have:

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{l} &= \int \int -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \\ \Rightarrow 2Ez &= \mu_0 \alpha z 2y \\ \Rightarrow E &= \mu_0 \alpha y \end{aligned} \tag{1}$$

To calculate the electric field outside the plates, let the rectangle cross the two plates, we then have:

$$\begin{aligned} 2Ez &= \mu_0 \alpha z d \\ \Rightarrow E &= \mu_0 \alpha d/2 \end{aligned} \tag{2}$$

c)

Outside the plates the Poynting vector is zero because there is no magnetic field. Between the plates:

$$\begin{aligned}\mathbf{S} &= \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \\ &= -\mu_0 \alpha^2 y t \hat{y}\end{aligned}\tag{3}$$

The flux into the region between the plates is then

$$-S(z = d/2) + S(z = -d/2) = \mu_0 \alpha^2 dt$$

The magnetic energy per unit area stored between the plates are:

$$\begin{aligned}E &= \frac{1}{2\mu_0} B^2 d \\ &= \mu_0 \alpha^2 t^2 d/2\end{aligned}\tag{4}$$

$\mathbf{E}$ is constant, so it won't contribute to the energy change rate. The energy change rate is then:

$$\frac{dE}{dt} = \mu_0 \alpha^2 t d\tag{5}$$

We see the flux matches the energy change rate.

d)

Here we can simply change $\alpha \rightarrow \partial K/\partial t$ and get:

$$\begin{aligned}\mathbf{B} &= \begin{cases} \mu_0 K \hat{x} & \text{between the plates} \\ 0 & \text{everywhere else} \end{cases} \\ \mathbf{E} &= -\mu_0 \frac{\partial K}{\partial t} y \hat{z} \quad \text{between the plates} \\ \mathbf{E} &= \pm \frac{\mu_0 d}{2} \frac{\partial K}{\partial t} \hat{z} \quad \text{outside the plates, - for } z > 0 \\ \mathbf{S} &= \begin{cases} -\mu_0 K \frac{\partial K}{\partial t} y \hat{z} & \text{between the plates} \\ 0 & \text{everywhere else} \end{cases}\end{aligned}$$

The flux into the region between the plates is then

$$-S(z = d/2) + S(z = -d/2) = \mu_0 K \frac{\partial K}{\partial t} d$$

The magnetic energy per unit area stored between the plates are:

$$\begin{aligned}E &= \frac{1}{2\mu_0} B^2 d \\ &= \mu_0 K^2 d/2\end{aligned}\tag{6}$$

The magnetic energy change rate is then:

$$\frac{dE}{dt} = \mu_0 K \frac{\partial K}{\partial t} d \quad (7)$$

We see the flux matches the magnetic energy change rate only.

This does not mean that total energy is not conserved, it's just the consequence of quasistatic approximation, where we can ignore the electric energy compared to the magnetic energy if the current variation frequency is very low. Total energy is always conserved. When the current variation is so fast that the electric energy is comparable to the magnetic energy, the quasistatic approximation breaks down and we need to take into account the displacement current, which in fact reduces the original magnetic field.