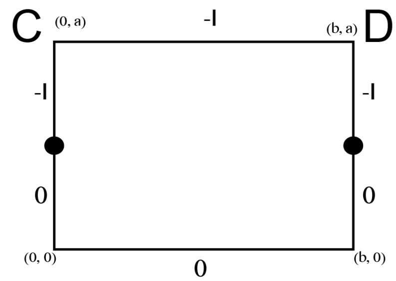
Physics 106c – Problem Set 1 — Solutions

April 26, 2012

Problem 1



With charge conservation and Maxwell equation, we can write down $\nabla\times\vec{K} = \frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y} = 0$ and $\nabla\vec{E} = 0$ and introduce ϕ here.

$$K_x = -\frac{\partial\phi}{\partial y}, \quad K_y = \frac{\partial\phi}{\partial y} \tag{1}$$

Plug K_x and K_y into the $\nabla\vec{E} = \nabla(\rho\vec{K}) = 0$ and one will get $\nabla\phi^2 = 0$. The boundary condition (as in the figure) can be obtained by $I = \int_0^a K_x \, \mathrm{d}y$. The problem is solved in last term HW2. After some algebra, the ϕ can be solved as the following. For $\frac{a}{2} < y < a$

$$\phi(x,y) = -I + \frac{2I}{\pi}\sum_{n\text{ odd}} \frac{\sin(\frac{n\pi x}{b})\sinh(\frac{n\pi(a-y)}{b})}{n\sinh(\frac{n\pi a}{2b})} \tag{2}$$

for $0 < y < a/2$,

$$\phi(x,y) = -\frac{2I}{\pi}\sum_{n\text{ odd}} \frac{\sin(\frac{n\pi x}{b})\sinh(\frac{n\pi y}{b})}{n\sinh(\frac{n\pi a}{2b})} \tag{3}$$

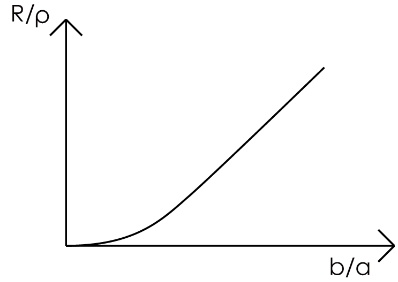
So the voltage $V_{cd} = \int_0^b E_x \, \mathrm{d}x$ is

$$V_{cd} = \rho\frac{4I}{\pi}\sum_{n\, odd}\frac{1}{n\sinh(\frac{n\pi a)}{2b}} \tag{4}$$

and the resistance R equals

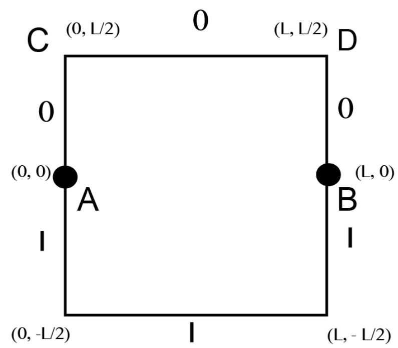
$$R_{cd} = \frac{4\rho}{\pi}\sum_{n\, odd}\frac{1}{n\sinh(\frac{n\pi a}{2b})} \tag{5}$$

The figure below is the plot of $R_c d$ vs b/a. In the limit $b/a \to \infty$, $R_{cd} \to \rho\frac{4}{\pi}\frac{2b}{\pi a}\sum_{n\, odd}\frac{1}{n^2} = \rho\frac{8b}{\pi^2 a}\frac{\pi^2}{8} = \rho b/a$.



Problem 2

To determine the boundary condition for ϕ, one can start from the boundary condition of K. (i) $K_x(0, y) = K_x(L, y) = 0$, for $y \neq 0$, $y \in [L/2, -L/2]$. (ii) $K_y(x, L/2) = K_x(x, -L/2) = 0$, for $x \in [0, L]$. (iii) $I = \int_{-L/2}^{L/2} K_x dy$. With $K_x = -\frac{\partial \phi}{\partial y}$ and $K_y = \frac{\partial \phi}{\partial x}$, one can set $\phi = 0$ in upper plane and $\phi = I$ in lower plane.



Since there's no time-varying B field, and $E_x = \rho_{xx} K_x$, $E_y = \rho_{yy} K_y$, one will get

$$\frac{\partial K_y}{\partial x} - \frac{\rho_{xx}}{\rho_{yy}} \frac{\partial K_x}{\partial y} = 0 \tag{6}$$

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\rho_{xx}}{\rho_{yy}} \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{7}$$

Here we can rescale the $y' = \sqrt{\frac{\rho_{yy}}{\rho_{xx}}} y = \eta y$. From Problem 1, we can immediate have the ϕ which is.

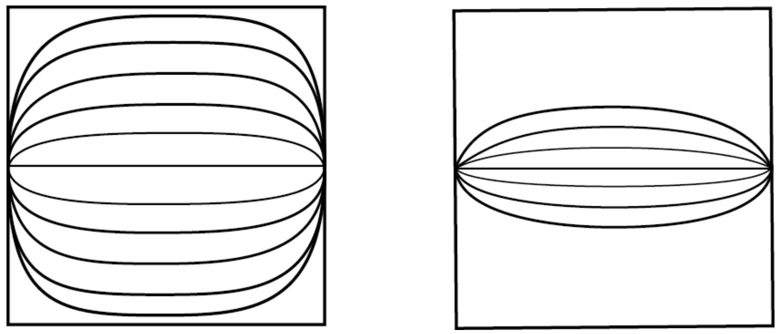
$$\phi(x, y) = -\frac{I}{\pi} \sum_{n \text{ odd}} \frac{\sin(\frac{n\pi x}{L}) \sinh(\frac{n\pi\eta(L/2-y)}{L})}{n \sinh(\frac{n\pi\eta}{2})} \tag{8}$$

The potential differences V_{CD} and R are

$$V_{CD} = \sqrt{\rho_{xx}\rho_{yy}} \frac{4I}{\pi} \sum_{n\, odd} \frac{1}{n \sinh(\frac{n\pi\eta)}{2}} \tag{9}$$

$$R_{CD} = \sqrt{\rho_{xx}\rho_{yy}} \sum_{n\, odd} \frac{4}{n\pi \sinh(\frac{n\pi\eta)}{2}} \tag{10}$$

When $\rho_{xx} = 1$, $\rho_{yy} = 0.2$, $R_1 = 0.802658$, In another case, $\rho_{xx} = 0.2$, $\rho_{yy} = 1$, $R_1 = 0.0340057$. So there is a difference between R_1/R_2, ρ^1_{xx}/ρ^2_{xx} and ρ^1_{yy}/ρ^2_{yy}. For the first case (the left figure), since ρ_{xx} is larger than ρ_{yy} current wants to flow to y direction more than in the second case(the right figure). So the current which entered in plate goes longer path, and as a result effectively it feels more resistance. Therefore, considering such phenomena, R_1/R_2 is bigger than ρ^1_{xx}/ρ^2_{xx} which we just consider x-direction.



Problem 3

$$\vec{j} = \sigma(\vec{E} + \vec{v}\times\vec{B}), \quad \vec{B} = (0, 0, B)$$

$$j_i = \sigma(E_i + \epsilon_{ijk} v_j B_k), \quad \vec{j} = ne\vec{v} \Rightarrow v_i = \frac{j_i}{ne}$$

$$\begin{cases} j_1 = \sigma E_1 + \sigma\epsilon_{123} v_2 B \\ j_2 = \sigma(E_2 + \epsilon_{213} v_1 B) \end{cases}$$

$$\downarrow$$

$$\begin{cases} j_1 = \sigma E_1 + \frac{\sigma B}{ne} j_2 \\ j_2 = \sigma E_2 - \frac{\sigma B}{ne} j_1 \end{cases}$$

$$\downarrow$$

$$j_1 = \sigma E_1 + \frac{\sigma B}{ne}\left(\sigma E_2 - \frac{\sigma B}{ne} j_1\right)$$

$$j_1\left(1 + \left(\frac{\sigma B}{ne}\right)^2\right) = \sigma\left(E_1 + \frac{\sigma B}{ne} E_2\right)$$

$$j_1 = \frac{1\cdot\sigma}{1 + \left(\frac{\sigma B}{ne}\right)^2}\left(E_1 + \frac{\sigma B}{ne} E_2\right)$$

$$j_2 = \sigma E_2 - \frac{\sigma B}{ne} j_1 = \sigma E_2 - \frac{\sigma}{1 + \left(\frac{\sigma B}{ne}\right)^2}\left(\frac{\sigma B}{ne} E_1 + \frac{\sigma B}{ne} E_2\right) =$$

$$= \frac{\sigma}{1 + \left(\frac{\sigma B}{ne}\right)^2}\left(E_2 - \frac{\sigma B}{ne} E_1\right)$$

$$\Rightarrow \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} = \frac{\sigma}{1 + \left(\frac{\sigma B}{ne}\right)^2}\begin{pmatrix} 1 & \frac{\sigma B}{ne} \\ -\frac{\sigma B}{ne} & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = \frac{\sigma}{1 + \left(\frac{\sigma B}{ne}\right)^2}\begin{pmatrix} 1 & \frac{\sigma B}{ne} \\ -\frac{\sigma B}{ne} & 1 \end{pmatrix}.$$

3. $\sigma_{xx} = \dfrac{\sigma}{1+(\frac{\sigma B}{ne})^2}$

$$\begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{xy} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}^{-1} = \begin{pmatrix} 1/\sigma & -\frac{B}{ne} \\ \frac{B}{ne} & 1/\sigma \end{pmatrix}$$

$\rho_{xx} = 1/\sigma$, If we fix B (finite), let $\sigma \to \infty$

ρ_{xx}, σ_{xx} both $\to 0$

$\rho_{xx} \to 0$ just because resistivity is inverse of conductivity

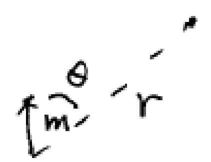
$\sigma_{xx} \to 0$ because when $\sigma \to 0$, from $\vec{J} = \sigma(\vec{E} + \frac{\vec{J}}{ne} \times \vec{B})$

$\Rightarrow \vec{E} + \frac{\vec{J} \times \vec{B}}{ne} \to 0 \Rightarrow \vec{J} \perp \vec{E} \Rightarrow$ diagonal components of

$\overleftrightarrow{\sigma}$ must vanish.

Problem 4

In free space, we have



$$\vec{B}_0 = \frac{\mu_0 \vec{m}}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

For linear magnetic materials, μ doesn't change

$$\vec{B} = \frac{\vec{B}_0\,\mu}{\mu_0} = \vec{B}_0(1+\chi_m)$$

$$= \frac{\mu_0(1+\chi_m)}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$