Physics 106c – Problem Set 1 — Solutions

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Problem 1

If we assume $\phi(z=0)=0$, then on the symmetry axis we have

$$\mathbf{B} = \nabla\phi \quad \Rightarrow \phi = \int \mathbf{B}\cdot d\mathbf{l} = \int_0^z \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}dz = \frac{\mu_0 I}{2}\frac{z}{\sqrt{R^2+z^2}}$$

When $z \gg R$

$$\begin{aligned}
\phi(\text{on the symmetry axis}) &= \frac{\mu_0 I}{2}\frac{z}{\sqrt{R^2+z^2}} = \frac{\mu_0 I}{2}\left(1 - \frac{1}{2}\frac{R^2}{z^2} + \frac{3}{8}\frac{R^4}{z^4} - \frac{5}{16}\frac{R^6}{z^6} + \cdots\right) \\
\Rightarrow \phi(\mathbf{r}) &= \frac{\mu_0 I}{2}\left(1 - \frac{1}{2}\frac{R^2}{r^2}P_1(\cos\theta) + \frac{3}{8}\frac{R^4}{r^4}P_3(\cos\theta) - \frac{5}{16}\frac{R^6}{r^6}P_5(\cos\theta) + \cdots\right) \\
\therefore \mathbf{B} &= \nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} \\
&= \frac{\mu_0 I}{2}\left(\frac{R^2}{r^3}P_1(\cos\theta) - \frac{3}{2}\frac{R^4}{r^5}P_3(\cos\theta) + \frac{15}{8}\frac{R^6}{r^7}P_5(\cos\theta) + \cdots\right)\hat{\mathbf{r}} \\
&+ \frac{\mu_0 I}{2}\left(-\frac{1}{2}\frac{R^2}{r^3}\frac{dP_1(\cos\theta)}{d\theta} + \frac{3}{8}\frac{R^4}{r^5}\frac{dP_3(\cos\theta)}{d\theta} - \frac{5}{16}\frac{R^6}{r^7}\frac{dP_5(\cos\theta)}{d\theta} + \cdots\right)\hat{\theta}
\end{aligned}$$

Therefore, at very large distance, the first order term of the magnetic field is

$$\mathbf{B} = \frac{\mu_0 I}{2}\left(\frac{R^2}{r^3}P_1(\cos\theta)\hat{\mathbf{r}} - \frac{1}{2}\frac{R^2}{r^3}\frac{dP_1(\cos\theta)}{d\theta}\hat{\theta}\right) = \frac{\mu_0}{4\pi}\frac{I\pi R^2}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right)$$

,which is exactly the magnetic field of a simple dipole with dipole moment $m = I\pi R^2$.

Problem 2

$$
\begin{aligned}
\mathbf{A} &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau' \\
\Rightarrow \nabla\cdot\mathbf{A} &= \frac{\mu_0}{4\pi}\int \nabla\cdot\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)d\tau' \\
&= \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\cdot\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d\tau' \\
&= -\frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d\tau' \\
&= -\frac{\mu_0}{4\pi}\int \left(\nabla'\cdot\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)-\frac{\nabla'\cdot\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)d\tau' \\
&= -\frac{\mu_0}{4\pi}\left(\oint \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\cdot d\mathbf{a} - \int \frac{\nabla'\cdot\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau'\right)
\end{aligned}
$$

Because the current distribution is localized, the surface integral of the first term is zero. Furthermore, the second term is also zero because the magnetostatics condition.($\nabla\cdot\mathbf{J}=0$)

Problem 3

Use Eq. (5.38), we have

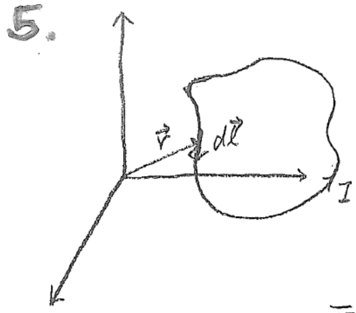
$$B(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$

$$
\begin{aligned}
dB =& \frac{\mu_0 dI}{2}\frac{R^2}{(R^2+z^2)^{3/2}} \\
=& \frac{\mu_0 nIdz}{2}\frac{R^2}{(R^2+z^2)^{3/2}} \\
\therefore B =& \int \frac{\mu_0 nIdz}{2}\frac{R^2}{(R^2+z^2)^{3/2}} \\
=& \frac{\mu_0 In}{2}(\cos\theta_1-\cos\theta_2)
\end{aligned}
$$

(Ps. See the figure 5.25)

Problem 4

5.



$$\vec{\tau} = \oint d\vec{\tau} = \oint \vec{r} \times d\vec{F} = \oint \vec{r} \times (I d\vec{l} \times \vec{B})$$

$$= I\left(\oint d\vec{l}\,(\vec{r}\cdot\vec{B}) - \oint \vec{B}\,(\vec{r}\cdot d\vec{l})\right)$$

$$= I \oint d\vec{l}\,(\vec{r}\cdot\vec{B}) \qquad \because \vec{B} \text{ is const. } \oint \vec{r}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{r})\cdot d\vec{a} = 0$$

$$\underset{(*)}{=} I \int d\vec{a} \times \nabla(\vec{r}\cdot\vec{B}) = I \int d\vec{a} \times \vec{B} \qquad \because \nabla(\vec{r}\cdot\vec{B}) = \hat{x}^i \partial_i (x_j B^j) = \hat{x}^i \delta_{ij} B^j = \vec{B}$$

$$= \vec{m} \times \vec{B}$$

(*) $\oint T d\vec{l} = \int d\vec{a} \times (\vec{\nabla} T)$

$$\int (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint \vec{v}\cdot d\vec{l} \underset{\substack{\vec{v} = \vec{c}T \\ c \text{ constant}}}{\Rightarrow} \int \left(T \cancelto{0}{(\vec{\nabla}\times\vec{c})} - \vec{c}\times\vec{\nabla}T\right)\cdot d\vec{a} = \oint T\vec{c}\cdot d\vec{l}$$

$$\underset{(\vec{A}\times\vec{B})\cdot\vec{C} = (\vec{C}\times\vec{A})\cdot\vec{B}}{\Rightarrow} \vec{c}\cdot\int d\vec{a}\times(\vec{\nabla}T) = \vec{c}\cdot\oint T d\vec{l} \Rightarrow \int d\vec{a}\times\vec{\nabla}T = \oint T d\vec{l}$$

Problem 5

(a)

$$
\begin{aligned}
\mathbf{B}_{ave} =& \frac{1}{\frac{4}{3}\pi R^3}\int \mathbf{B}\ d\tau = \frac{1}{\frac{4}{3}\pi R^3}\int (\nabla\times\mathbf{A})\ d\tau \\
=& -\frac{1}{\frac{4}{3}\pi R^3}\int_s \mathbf{A}\times d\mathbf{a} = -\frac{1}{\frac{4}{3}\pi R^3}\int_s \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\ \mathbf{d}\tau'}{|\mathbf{r}-\mathbf{r}'|}\times d\mathbf{a}
\end{aligned}
$$

We have to do the surface integral first. By symmetry, we can easily argue that $\int_s \frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{a}$ is proportional to (or in the same direction of) $\mathbf{r}'$, which means

$$
\int_s \frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{a} = \text{const }\ \cdot\mathbf{r}'
$$

Therefore,

$$
\begin{aligned}
&\int_s \frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{a} = \int_s \frac{1}{|\mathbf{r}-\mathbf{r}'|}(\hat{\mathbf{r}}'\cdot d\mathbf{a})\hat{\mathbf{r}}' \\
&= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \frac{R^2\sin\theta\ \cos\theta}{\sqrt{r'^2+R^2-2Rr'\cos\theta}}\hat{\mathbf{r}}' \\
&= 2\pi R^2\frac{(r'+R)(r'^2-r'R+R^2)-(|r'-R|)(r'^2+r'R+R^2)}{3r'^2R^2}\hat{\mathbf{r}}' \\
&= \begin{cases} \frac{4\pi}{3}\mathbf{r}' \text{ if } r'<R \\ \frac{4\pi}{3}\frac{R^3}{r'^3}\mathbf{r}' \text{ if } r'>R \end{cases}
\end{aligned}
$$

For currents are all within the sphere, $r'<R$

$$
\begin{aligned}
\Rightarrow \mathbf{B}_{ave} &= -\frac{1}{\frac{4}{3}\pi R^3}\int_s \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\ \mathbf{d}\tau'}{|\mathbf{r}-\mathbf{r}'|}\times d\mathbf{a} \\
&= -\frac{1}{\frac{4}{3}\pi R^3}\frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\times\frac{4\pi}{3}\mathbf{r}'\ d\tau' \\
&= \frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3}
\end{aligned}
$$

(b) For currents are all outside the sphere ($r'>R$), we have

$$
\begin{aligned}
\mathbf{B}_{ave} &= -\frac{1}{\frac{4}{3}\pi R^3}\int_s \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\ \mathbf{d}\tau'}{|\mathbf{r}-\mathbf{r}'|}\times d\mathbf{a} \\
&= -\frac{1}{\frac{4}{3}\pi R^3}\frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\times\frac{4\pi}{3}\frac{R^3}{r'^3}\mathbf{r}'\ d\tau' \\
&= -\frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\times\left(\frac{\hat{\mathbf{r}}'}{r'^2}\right)d\tau' \\
&= \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\times\nabla'\left(\frac{1}{r'}\right)d\tau'
\end{aligned}
$$

The field which the steady currents produce is

$$
\begin{aligned}
\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A} &= \frac{\mu_0}{4\pi}\int \nabla\times\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)d\tau' \\
&= -\frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\times\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d\tau' \\
&= \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\times\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d\tau' \\
\Rightarrow \mathbf{B}(\mathbf{0}) &= \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\times\nabla'\left(\frac{1}{r'}\right)d\tau'
\end{aligned}
$$