Physics 106b: Electrodynamics - Assignment 4

Problems

1. The potential can be expanded in terms of multipole moments as follows:

$$V(\vec{x}) = \frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \cdots \tag{1}$$

The monopole moment is given by

$$q = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \left(\cos(2\theta) + \frac{1}{3}\right) = 0. \tag{2}$$

The dipole moment is given by

$$\vec{p} = \int_S \vec{r}\sigma(\theta) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\left(\cos(2\theta) + \frac{1}{3}\right) = \vec{0}. \tag{3}$$

Next, the quadrapole moment is given by

$$Q_{ij} = \int_A (3x_i x_j - R^2\delta_{ij})\sigma(\theta). \tag{4}$$

They are given by

$$Q = \begin{pmatrix} -\frac{16\pi R^2}{15} & 0 & 0 \\ 0 & -\frac{16\pi R^2}{15} & 0 \\ 0 & 0 & \frac{32\pi R^2}{15} \end{pmatrix}. \tag{5}$$

Therefore, quadrapole moment is the first non-vanishing contribution to the potential for generic point.

$$V(r,\theta) = \frac{4R^4}{15\epsilon_0 r^3} P_2(\cos\theta) = \frac{2R^4}{15\epsilon_0 r^3}(3\cos^2\theta - 1) \tag{6}$$

There are no higher multipole moments.

2. (a) At large negative x, the electric field will be almost uniform inside or outside of the slab. Inside the slab, the electric field is $\vec{E} = -\frac{P}{\epsilon_0}\hat{y}$, and the electric field outside is zero. The bound charge density on the top surface is P and the bottom surface is $-P$.

   (b) Consider another copy of polarized semi-infinite slab, which extends from $x = 0$ to $x = +\infty$. If we connect both slabs, then we get just a infinite single slab, and the electric field at the origin is simply $\vec{E} = -\frac{P}{\epsilon_0}\hat{y}$. By symmetry, if we divide the slab into half, the E-field at the origin should be $\vec{E} = -\frac{P}{2\epsilon_0}\hat{y}$, because the electric field of the infinite slab at the origin must be the sum of electric fields from both slabs at $x = 0$.

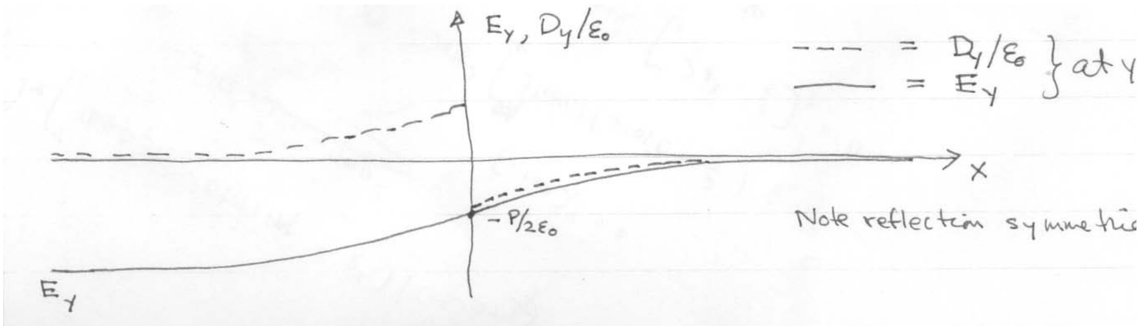

Figure 1: E and D field along the x-axis

(c) By the same superposition and symmetry argument as above,

$$\vec{E}(x, y = 0) + \vec{E}(-x, y = 0) = -\frac{P}{\epsilon_0}\hat{y}. \tag{7}$$

(d) Once we know the electric field, from $\vec{D} = \epsilon_0\vec{E} + P$, we can deduce the electric displacement. We know the following facts: $\vec{E}(x = \infty) = 0$, $\vec{E}(x = -\infty) = -P/\epsilon_0\hat{y}$, $\vec{E}(x = 0) = -P/2\epsilon_0\hat{y}$. Note that $\vec{E}$ field is continuous at $x = 0$, whereas $\vec{D}$ field can be discontinuous. The E and D fields are roughly given as the figure.